The Kantorovich-Lebedev Transform PhL 7.18.11

I first did this in my Stak Ch 7 raw2 notes, but will move those sections here now because things are pretty complicated. Stak does most of the work actually in Vol I Exercise 4.30 as we shall see here.

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1. Show that the correct Green's Function is that shown page 317 C.

Summary:

We have the following Green's function equation on interval $(0,\infty)$:

$-(xg')' + \mu x g - \lambda x^{-1} g = \delta(x-\xi)$ // Green's

We show that the free-space solution g is not unique, but that this is a good one to use:

$g_2 = I_{-i\gamma}(kx_<)\, K_{-i\gamma}(kx_>)$ where $\gamma \equiv \sqrt{\lambda}$, $\mu = k^2$

This has the right behavior at the two interval endpoints and has the right jump constant.

Details:

Our only task is to show that I as shown is finite at x = 0, and K is finite at $x = \infty$. I notice that AS2010 have a special redefinition of the I and K functions when ν is imaginary, and this makes the reflection formula for the new I be the same as it is for the new K, but let's NOT use these specially defined functions and stick with the traditional I and K functions. Then at x = 0 we have

10.30 Limiting Forms

10.30(i) $z \to 0$

When ν is fixed and $z \to 0$,

10.30.1 $$I_\nu(z) \sim (\tfrac{1}{2}z)^\nu / \Gamma(\nu+1), \quad \nu \neq -1, -2, -3, \ldots,$$

10.30.2 $$K_\nu(z) \sim \tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}, \qquad \Re\nu > 0,$$

10.30.3 $$K_0(z) \sim -\ln z.$$

In the case that $\nu = \pm i\gamma$ we get $I_{i\gamma}(z) \to z^{\pm i\gamma} = e^{\pm i\gamma \ln(z)}$. This limit is undefined really land the same for the K limit, so we have to make some assumption about ν. Suppose we assume that

$\nu = \pm i\gamma + \varepsilon$ where ε is real and small and we will later take $\varepsilon \to 0$

Then we have

$$z^\nu = z^{\pm i\gamma} z^\varepsilon = e^{\pm i\gamma \ln(z)} e^{\varepsilon \ln(z)}$$

Now if we first take $z \to 0$ and THEN take $\varepsilon \to 0$, we get $z^\nu \to 0$ in the first step. So with this in mind, we can conclude that $I_{\pm i\gamma}(z) \to 0$ as $z \to 0$ and this is appropriate for the "left" homo solution. If we change signs, we can then say

$$z^{-\nu} = e^{\pm i(-\gamma)\ln(z)} e^{-\varepsilon \ln(z)}$$

and then when $z \to \infty$, $K_{\pm i\gamma}(z) \to 0$.

To summarize, we have found appropriate candidate functions for the "left" and "right" side of our Green's function. Our Green's ODE is this

$-(xw')' + k^2xw - \gamma^2 w/x = \delta(x-\xi)$ $\mu = k^2$ $\lambda = \gamma^2$ // Green's

$-(xw')' + k^2xw - \gamma^2 w/x = 0$ $\mu = k^2$ $\lambda = \gamma^2$ // homo

and these are our candidate solutions

$$g = A\ I_{\pm i\gamma}(kx_<) K_{\pm i\gamma}(kx_>)$$

with either choice of sign. In my notes section (c) in Ch7raw2 I show that A = 1 provided, and I also show that the orders on the I and K function must be the same to use the Wronskian to show A = 1. So it would seem that we have at least two choices here:

$$g_1 = I_{i\gamma}(kx_<)\, K_{i\gamma}(kx_>)$$
$$g_2 = I_{-i\gamma}(kx_<)\, K_{-i\gamma}(kx_>)$$

Moreover, we know that $K_\alpha(z) = K_{-\alpha}(z)$ so we can add two more solutions:

$g_3 = I_{i\gamma}(kx_<) K_{-i\gamma}(kx_>)$
$g_4 = I_{-i\gamma}(kx_<) K_{+i\gamma}(kx_>)$

Now here is my question: We can see that $g_1 = g_3$ and $g_2 = g_4$ from the symmetry just noted. But the question is: are g_1 and g_2 the same, or are they different? I argue that they are different as follows:

$$g_2 = g_4 = I_{-i\gamma}(kx_<) K_{+i\gamma}(kx_>) = [\, I_{i\gamma}(kx_<) + (2/\pi)\sin(i\gamma\pi) K_{i\gamma}(kx_<) \,] K_{+i\gamma}(kx_>)$$

$$= g_1 + i(2/\pi)\mathrm{sh}(\gamma\pi) K_{i\gamma}(kx_<) K_{+i\gamma}(kx_>)$$

So g_1 and g_2 are not the same because they differ by this quantity:

$$g_2 - g_1 = i(2/\pi)\mathrm{sh}(\gamma\pi) K_{i\gamma}(kx_<) K_{+i\gamma}(kx_>)$$

So if these two g's are really different, how can that be since I thought the Green's function was unique? But maybe a fundy Green's function is NOT unique! For example, $E = 1/4\pi r$ is the fundy for $\nabla^2 E = \delta(\mathbf{r})$ in 3D, but $E_1 = 1/4\pi r + 3.6$ is also a fundy solution. If we insist that $E \to 0$ on the boundary at $r=\infty$, then perhaps $1/4\pi r$ is the unique fundy solution.

Now what about a solution like g_1 above? We know from AS that

The defining property of the second standard solution $K_\nu(z)$ of (10.25.1) is

10.25.3 $$K_\nu(z) \sim \sqrt{\pi/(2z)}e^{-z},$$

so this decay is independent of the value of ν. So both g_1 and g_2 decay this way at $r \to \infty$. The difference also decays since it is KK.

If we look at the Green's ODE $-(xw')' + k^2xw - \gamma^2 w/x = \delta(x-\xi)$, it seems pretty clear that if you find a solution $f(x|\xi; k,\gamma)$, then $f(x|\xi; k,-\gamma)$ must also be a solution. Consider $w''+k^2w = 0$. If $w = e^{ikx}$ is a solution, then e^{-ikx} is also a solution by this same argument. That does not mean the two solutions thus obtained are the SAME solution. Certainly $e^{ikx} \neq e^{-ikx}$ in the same sense that $g_1 \neq g_2$.

So I arrive at a different conclusion than the one I reached earlier in the raw notes. My conclusion now is that

$g_1 = I_{i\gamma}(kx_<) K_{i\gamma}(kx_>)$
$g_2 = I_{-i\gamma}(kx_<) K_{-i\gamma}(kx_>)$

are each viable fundy Green's function solutions to $-(xw')' + k^2xw - \gamma^2 w/x = \delta(x-\xi)$, though they are not the same solution. In those other notes I conjectured that these were the same solution.

Side Question: you might try to argue that since everything in the ODE is real, then g_1 and g_2 must each be real. But that is a bogus argument. Consider again $w''+k^2w = 0$, an equation where everything is real. we can consider solutions $e^{\pm ikx}$ which are not real. So solutions need not be real (though real solutions might be found).

I turns out that $K_\nu(x)$ is real (for real x) when ν is real or when ν is pure imaginary. This follows from the following integral representation from AS p252, claimed valid for Re(ν) < 1 and x > 0:

$$K_\nu(x) = \sec\left(\tfrac{1}{2}\nu\pi\right) \int_0^\infty \cos(x \sinh t) \cosh(\nu t)\, dt$$

When $\nu = i\gamma$, we have $\sec(\pi\nu/2)\cosh(\nu t) = \sec(\pi i\gamma/2)\cosh(i\gamma\nu t) = \operatorname{sech}(\pi\gamma/2)\cos(\gamma\nu t)$ = stays real. This is a strange integral in this sense: for large t, factor ch(νt) is blowing up as $e^{\nu t}$, but the other factor is $\cos(xe^t)$ and is oscillating exponentially fast! Convergence of the integral is not obvious but possible.

I think $I_\nu(x)$ is in general complex for real x and imaginary ν. This is why AS on page 261 define a modified $I_{i\gamma}$ function $\mathcal{I}_\nu \equiv \mathrm{Re}(I_{i\nu})$ to go along with $\mathcal{K}_\nu \equiv K_{i\nu}$.

So my conclusion is that g_1 and g_2 as shown above are both complex.

So back to Exercise 4.30. If we take $g = g_2$ as shown p 317 C, I have shown that you end up with 4.137 and the "usual form" of the KL transform. See section (c) in the Ch7raw2 notes. So I think you could take the other sign setting $g = g_1$ and then you get a different form of the KL transform that is not normally used.

2. Verify the jump constant.

Summary: We use the Wronskian for I and K to show that the constant is simply A = 1. This has been assumed in the previous section, but here we show it is true.

Details:

(c) Exercise 4.30: Derive p 317 C, since I don't know why p 317 C is the correct g. The big question is why we are using the negative sign in the K subscript. I think the signs have to be the same so that the jump condition will work. For an ODE of the form $a(x)\, g''(x) + \ldots. = \delta(x-\xi)$ we know the jump condition is this (just do it on scratch)

$$g'(x=\xi+) - g'(x=\xi-) = 1/a(\xi)$$

In our Green's ODE above we have $-(xg')' = -xg'' - g'$ so $a(x) = -x$ and we should then have

$$g'(x=\xi+) - g'(x=\xi-) = -1/\xi$$

Now suppose we conjecture that we can have a solution with unit coefficient as in p 317 C

$$g(x|\xi) = I_a(bx_<)\, K_a(bx_>)$$

Let's then check the jump condition. For the + case we have $x > \xi$ so

$g(x|\xi) = I_{\mathbf{a}}(b\xi)\, K_{\mathbf{a}}(bx)$ $\quad$ $g'(x|\xi) = I_{\mathbf{a}}(b\xi)\, b\, K_{\mathbf{a}}'(bx)$

and for the - case we have $x < \xi$ so

$g(x|\xi) = I_{\mathbf{a}}(bx)\, K_{\mathbf{a}}(b\xi)$ $\quad$ $g'(x|\xi) = b\, I_{\mathbf{a}}'(b\xi)\, K_{\mathbf{a}}(bx)$

so we end up with

$g'(x=\xi+) - g'(x=\xi-) = b\, \{\, I_{\mathbf{a}}(b\xi)\, K_{\mathbf{a}}'(bx) - I_{\mathbf{a}}'(b\xi)\, K_{\mathbf{a}}(bx)\, \} = b\, W\, [I_{\mathbf{a}}(bx), K_{\mathbf{a}}(bx)\,]$

A&S 2010 tells us that

10.28.2

$$\mathscr{W}\{K_\nu(z), I_\nu(z)\} = I_\nu(z)\, K_{\nu+1}(z) + I_{\nu+1}(z)\, K_\nu(z) = 1/z.$$

so our Wronskian is - 1/(bx) so we then get

$g'(x=\xi+) - g'(x=\xi-) = b\, [-\, 1/(bx)\,] = -1/x$

in exact agreement with what is required. So since the Wronskian is involved in the jump condition, we can see that both I and K subscripts have to be the same. [Except K has that little symmetry!]

So we then are faced with both positive or both negative. If μ and λ are real, then g has to be real, so I think if we complex conjugate both sides of C we get the same thing. So I conjecture that the RHS of page 317C is **the same** whether you take the - or + sign [wrong!]. I looked a bit at large and small x limits and think things are the same with either sign. We get an oscillating phase at x = 0 in either case (more AS).

So that concludes my "derivation" of page 317 C. I will accept p 317 D as just properties of the K and J functions.

3. Compute the KL completeness condition and state the KL transform

Summary: We identify λ as "the official λ" by considering various L operators. Then, using the above Green's function, we construct the completeness δ(x-ξ) using the generic completeness formula. The contour collapses about the real axis and we pick up the real axis integral of the discontinuity of the integrand and this gives the result

$x\, \delta(x-\xi) = (2/\pi^2) \int_0^\infty \kappa d\kappa\, \sinh(\pi\kappa) K_{i\kappa}(\sqrt{\mu}\, x)\, K_{i\kappa}(\sqrt{\mu}\, \xi)$

We then go on to interpret this as the two parts of the KL transform:

$f(x) = (2/\pi^2) \int_0^\infty \kappa d\kappa\, \sinh(\pi\kappa)\, K_{i\kappa}(\sqrt{\mu}\, x)\, F(\kappa)$ $\quad$ which is 4.139.

$$F(\kappa) \equiv \int_0^\infty dx\, f(x)\, K_{i\kappa}(\sqrt{\mu}\, x)\, /x$$

Details:

(d) (Volume I page 317) Exercise 4.30: Derive (4.137). Next, consider again from above

$$-(xg')' + \mu x g - \lambda x^{-1} g = \delta(x-\xi) \qquad \text{// Green's}$$

In the context of vol 1 p 268 the differential operators here are

$$Lg = -(xg')' + \mu x g = -\frac{d}{dx}\left[x \frac{d}{dx} \right] g + \mu x g \qquad L = -\frac{d}{dx}\left[x \frac{d}{dx} \right] + \mu x$$

$$\Rightarrow\ p(x) = x \qquad q(x) = \mu x$$

$$L_\lambda g = [L - \lambda s(x)]\, g \qquad \Rightarrow \qquad s(x) = 1/x$$

So with this p(x), q(x), s(x), our Green's equation above fits perfectly into the self-adjoint framework, and λ is *the usual* λ and so (4.44) page 274 applies and we then have [I quote from Ch 4 meta meta]

Completeness: $-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)/s(x) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi)$

Orthogonality: $\langle \sqrt{s}\, \varphi_n, \sqrt{s}\, \varphi_m \rangle = K_n\, \delta_{n,m}$

And I now provide a ***better quote*** from "a review of the generic singular BC problem.doc"

$$-(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_{\lambda n}\, \varphi_{\lambda n}{}^*(x) \varphi_{\lambda n}(\xi) + \int d\lambda' \varphi_\lambda(x)^*\, \varphi_{\lambda'}(\xi) = \delta(x-\xi)/s(x)$$

$$\langle \varphi_{\lambda m}, s\varphi_{\lambda n} \rangle = \delta_{n,m} \qquad \langle \varphi_\lambda, s\varphi_{\lambda'} \rangle = \delta(\lambda-\lambda') \qquad \langle \varphi_\lambda, s\varphi_{\lambda n} \rangle = 0$$

In the current situation, we have only a continuous spectrum, so this becomes

$$-(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \int d\lambda' \varphi_\lambda(x)^*\, \varphi_{\lambda'}(\xi) = \delta(x-\xi)/s(x)$$

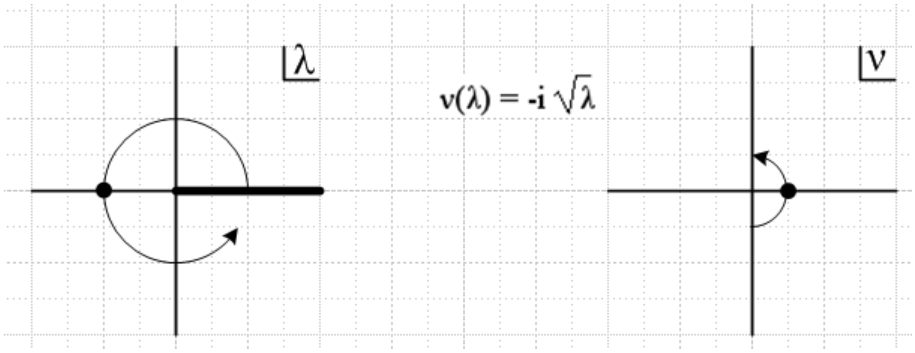
$$\langle \varphi_\lambda, s\varphi_{\lambda'} \rangle = \delta(\lambda-\lambda')$$

So what happens when we jam g as in C into our completeness formula?

$$g = I_{-i\sqrt{\lambda}}(\sqrt{\mu}\, x_<)\ K_{-i\sqrt{\lambda}}(\sqrt{\mu}\, x_>)$$

So we then need to know about the singularities of things like $K_\nu(x)$ in the ν plane. This is something I have never even considered. I think, having stared for example at integral reps on GR4 p 958, that both I and K are analytic in ν. So the only singularity comes from the fact that $\nu = \nu(\lambda) = -i\sqrt{\lambda}$. The trick here

is that we want ν to be real and positive when λ lies opposite the cut in the λ plane, so this is different from considering say $\rho(\lambda) = \sqrt{\lambda}$. So with $\nu(\lambda) = -i\sqrt{\lambda}$, in order to have $\nu > 0$ we need $\sqrt{\lambda} = i$ and that means we need $\lambda = -1$, and that means we need to take the cut on the *positive* λ axis. Here then is our picture set:



Our "sheet" in ν space is the right half ν plane, which I think is healthy for $K_\nu(z)$. So I claim then that the g function has a discontinuity across a cut on the positive real axis. Therefore:

$$\text{disc}[g] = g(\lambda = \lambda e^{i0}) - g(\lambda = \lambda e^{2i\pi})$$

$$= g(\sqrt{\lambda} = \sqrt{\lambda}) - g(\sqrt{\lambda} = -\sqrt{\lambda})$$

$$= I_{-i\sqrt{\lambda}}(\sqrt{\mu}\, x_<)\; K_{-i\sqrt{\lambda}}(\sqrt{\mu}\, x_>) - I_{i\sqrt{\lambda}}(\sqrt{\mu}\, x_<)\; K_{i\sqrt{\lambda}}(\sqrt{\mu}\, x_>)$$

Now for shorthand, let $\alpha = i\sqrt{\lambda}$ so we have [so then $\alpha = -\nu$]

$$= I_{-\alpha} K_{-\alpha} - I_\alpha K_\alpha$$

$$= [I_\alpha + (2/\pi)\sin(\pi\alpha)K_\alpha]\, K_\alpha - I_\alpha K_\alpha \qquad \text{// using p 317 D which I assume is correct}$$

$$= I_\alpha K_\alpha + (2/\pi)\sin(\pi\alpha)K_\alpha K_\alpha - I_\alpha K_\alpha$$

$$= (2/\pi)\sin(\pi\alpha)K_\alpha K_\alpha$$

So we have then shown that

$$g = I_{-i\sqrt{\lambda}}(\sqrt{\mu}\, x_<)\; K_{-i\sqrt{\lambda}}(\sqrt{\mu}\, x_>) = I_\nu(\sqrt{\mu}\, x_<)\, K_\nu(\sqrt{\mu}\, x_<) = I_{-i\kappa}(\sqrt{\mu}\, x_<)\, K_{-i\kappa}(\sqrt{\mu}\, x_<)$$

$\text{disc}[g] = (2/\pi)\sin(\pi\alpha)K_\alpha K_\alpha$ where $\alpha = i\sqrt{\lambda} = -\nu = i\kappa$

or

$\text{disc}[g] = -(2/\pi)\sin(\pi\nu)K_\nu K_\nu$ where $\nu = -i\sqrt{\lambda}$

We now compute [usual CCW sense]

$$\oint d\lambda\, g(x|\xi;\lambda) = -\int_0^\infty d\lambda\, \mathrm{disc}[g] = -\int_0^\infty d\lambda\, (2/\pi)\sin(\pi\alpha)K_\alpha K_\alpha$$

Now let $\kappa = \sqrt{\lambda}$ so that $2\kappa d\kappa = d\lambda$ and also $\alpha = i\kappa$ so

$$= -\int_0^\infty 2\kappa d\kappa\, (2/\pi)\sin(\pi i\kappa)K_{i\kappa}K_{i\kappa}$$

$$= -(4i/\pi)\int_0^\infty \kappa d\kappa\, \sinh(\pi\kappa)K_{i\kappa}K_{i\kappa}$$

Now we apply our δ rule to get

Completeness: $$-(1/2\pi i)\oint d\lambda\, g(x|\xi;\lambda) = \delta(x-\xi)/s(x)$$

$$-(1/2\pi i)\{-(4i/\pi)\int_0^\infty \kappa d\kappa\, \sinh(\pi\kappa)K_{i\kappa}K_{i\kappa}\} = \delta(x-\xi)\, x$$

$$(2/\pi^2)\int_0^\infty \kappa d\kappa\, \sinh(\pi\kappa)K_{i\kappa}K_{i\kappa} = \delta(x-\xi)\, x$$

or

$$x\,\delta(x-\xi) = (2/\pi^2)\int_0^\infty \nu d\nu\, \sinh(\pi\nu)K_{i\nu}(\sqrt{\mu}\, x_<)\, K_{i\nu}(\sqrt{\mu}\, x_>)$$

or

$$x\,\delta(x-\xi) = (2/\pi^2)\int_0^\infty \nu d\nu\, \sinh(\pi\nu)K_{i\nu}(\sqrt{\mu}\, x)\, K_{i\nu}(\sqrt{\mu}\, \xi) \qquad \text{// } \mathbf{agrees\ with\ 4.137}$$

where in the last step we use the K symmetry of the integrand. Warning: By using dummy variable ν, we have added confusion since $\nu = -i\sqrt{\lambda}$ is a constant appearing in the ODE. So better to write the above as

$$x\,\delta(x-\xi) = (2/\pi^2)\int_0^\infty \kappa d\kappa\, \sinh(\pi\kappa)K_{i\kappa}(\sqrt{\mu}\, x)\, K_{i\kappa}(\sqrt{\mu}\, \xi)$$

In the next section, however, I will continue to use confusing dummy variable ν since Stak does.

(e) Exercise 4.30: Derive (4.138,9). Now let's assume the projection of f(x) defined in 4.138

$$F_k(\nu) \equiv \int_0^\infty dx\, f(x)\, K_{i\nu}(\sqrt{\mu}\, x)/x$$

We apply the operator $(2/\pi^2)\int_0^\infty \nu d\nu\, \sinh(\pi\nu)\, K_{i\nu}(\sqrt{\mu}\, x)$ to both sides and we get

$$(2/\pi^2) \int_0^\infty \nu d\nu \sinh(\pi\nu) K_{i\nu}(\sqrt{\mu}\, x) F_k(\nu)$$

$$= (2/\pi^2) \int_0^\infty \nu d\nu \sinh(\pi\nu) K_{i\nu}(\sqrt{\mu}\, x) \int_0^\infty dx' f(x') K_{i\nu}(\sqrt{\mu}\, x')/x'$$

$$= \int_0^\infty dx' f(x')/x' * (2/\pi^2) \int_0^\infty \nu d\nu \sinh(\pi\nu) K_{i\nu}(\sqrt{\mu}\, x) K_{i\nu}(\sqrt{\mu}\, x')$$

$$= \int_0^\infty dx' f(x')/x' * x\,\delta(x-x') = \int_0^\infty dx' f(x')\,\delta(x-x') = f(x)$$

Thus we have shown that

$f(x) = (2/\pi^2) \int_0^\infty \nu d\nu \sinh(\pi\nu) K_{i\nu}(\sqrt{\mu}\, x) F_k(\nu)$ which is 4.139.

$F_k(\nu) \equiv \int_0^\infty dx\, f(x) K_{i\nu}(\sqrt{\mu}\, x)/x$ // k stands for Kantorovich.

and thus we have (finally) derived the Kantorovich-Lebedev Transform shown vol 1 p 317. Notice once again that λ was the great circle Green's function integration variable and appeared like so in our Bessel ODE

$$-(xg')' + \mu x g - \lambda x^{-1} g = \delta(x-\xi)$$

and of course μ is the other parameter.

4. Sturm-Liouville comments on the above

(e1) Comment on KL: The KL transform is the transform associated with a certain Sturm-Liouville problem. If we define $Lg = -(xg')' + \mu x g$, and then define $L_\lambda g = [L - \lambda s(x)]\, g$ where $\lambda = -\nu^2$ and $s(x) = 1/x$, then $L_\lambda g = 0$ is a Bessel ODE with constants are μ and ν. The operator L is self-adjoint with $p(x) = x$ and $q(x) = \mu x$ and thus fits into our "Sturm Liouville theory" framework. The interval of interest is x in (0,∞). So the KL transform is associated with completeness of the eigenfunctions of this SL problem:

$-(xu')' + \mu x u + \nu^2 u/x = 0$ x in (0,∞) $s(x) = 1/x$ $\lambda = -\nu^2 = \kappa^2$ $\nu = -i\sqrt{\lambda} = -i\kappa$

To find the completeness relation, we have to first find the Green's Function g, and then integrate it around the great circle to get $\delta(x-\xi)$. Since the generic $\delta(x-\xi)/s(x) = \Sigma_n \varphi_n(x) \overline{\varphi}_n(\xi)$, looking at 4.137 we can conclude that the normalized complete set of eigenfunctions for this SL problem are these:

$\varphi_\kappa(x) = (\sqrt{2}/\pi) \sqrt{\kappa \sinh(\pi\kappa)}\; K_{i\kappa}(\sqrt{\mu}\, x)$ orthog: $\int_0^\infty dx/x\; \varphi_\kappa(x)^* \varphi_{\kappa'}(x) = \delta(\kappa-\kappa')$

As a reminder from above, we have

$g = I_{-i\kappa}(\sqrt{\mu}\, x_<)\, K_{-i\kappa}(\sqrt{\mu}\, x_<)$ $\qquad\qquad$ $\kappa \equiv i\nu = \sqrt{\lambda}$

$x\,\delta(x-\xi) = (2/\pi^2) \int_0^\infty \kappa d\kappa \sinh(\pi\kappa) K_{i\kappa}(\sqrt{\mu}\, x)\, K_{i\kappa}(\sqrt{\mu}\, \xi)$

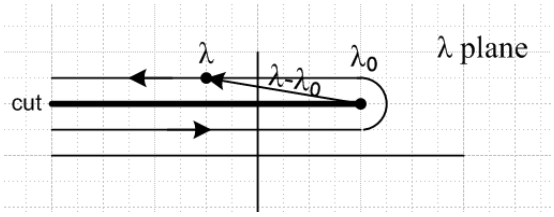
5. Technical topic on contour integration.

Here is the motivation for this section. Suppose we have an integrand F(ν) which is analytic in ν, and suppose we have $\nu = \sqrt{\lambda}$, a slight pedantic variation from our situation above. Our λ great circle integral then has as its integrand $f(\lambda) = F(\sqrt{\lambda})$. Although F is analytic, it certainly could have zeros. So near such a zero, we might have $F(\sqrt{\lambda}) \approx (\sqrt{\lambda} - \sqrt{\lambda_0})\, F(\sqrt{\lambda_0})$. In fact, to be specific, suppose $f(\lambda) = F(\sqrt{\lambda}) = (\sqrt{\lambda} - \sqrt{\lambda_0})g(\lambda)$, where λ_0 is some point out in the complex λ plane OTHER THAN the origin, and we add an *analytic* in λ factor g(λ) to provide GC convergence, say. We would be inclined to say that this function f(λ) has a branch point at $\lambda=\lambda_0$ and a cut that we might for example pull off to the left. Our great circle integral then gets wrapped around this cut, and we then have a discontinuity integral to worry about. What do we have to say about this situation? Can we ignore it for some reason?

Before answering that question, we can get the answer a different way. Suppose we replace the great circle integral with an integral in the ν plane. This integral would be a half circle around the upper half plane in ν. If λ swings all the way around, then ν swings halfway around. We then have $d\nu = (1/2)\lambda^{-1/2} d\lambda = (1/2\nu) d\lambda$ so that $d\lambda = 2\nu d\nu$. Our integrand is then $d\lambda\, F(\sqrt{\lambda}) = 2\nu d\nu F(\nu) = d\nu\, [2\nu F(\nu)]$ which is analytic in ν, since ν and F(ν) are both analytic, so we can deform the ν contour so it runs right to left just above the real axis. No ν plane cuts or poles were picked up which is why we could deform. Then we go back to λ space and we find that we have a λ contour that wraps the real axis. Thus, there must have been no strange contributions in the λ plane of the type noted in the previous paragraph!

But now how do we explain this directly in the λ plane! The answer is that there might be a cut of the kind noted above, but its discontinuity will be zero! I will show this with the example:

$f(\lambda) = (\sqrt{\lambda} - \sqrt{\lambda_0})g(\lambda)$ $\qquad$ g(λ) analytic, $\qquad$ λ_0 not at the origin $\quad \lambda-\lambda_0$



The picture shows our branch point and cut and our induced discontinuity integral. Notice the little vector $\lambda-\lambda_0$. For λ on the wrapping contour we can say (think of π as really π-ε here)

$(\lambda-\lambda_0)^{\lambda \text{ above}} = |\lambda-\lambda_0|\, e^{+i\pi}$
$(\lambda-\lambda_0)^{\lambda \text{ below}} = |\lambda-\lambda_0|\, e^{-i\pi}$

Now write f(λ) in a manner that shows this vector

$f(\lambda) = (\sqrt{(\lambda-\lambda_0)+\lambda_0} - \sqrt{\lambda_0})g(\lambda)$

If we evaluate f(λ) above and below the cut, we get exactly the same thing so there is no discontinuity! Specifically: [since g is analytic, it is the same above or below the cut]

$$f(\lambda+) = \left(\sqrt{|\lambda-\lambda_0|\, e^{+i\pi} + \lambda_0} - \sqrt{\lambda_0}\right) g(\lambda)$$
$$f(\lambda-) = \left(\sqrt{|\lambda-\lambda_0|\, e^{-i\pi} + \lambda_0} - \sqrt{\lambda_0}\right) g(\lambda)$$

But for λ on the cut we have $|\lambda-\lambda_0| = \lambda_0-\lambda > 0$ = real, so we can process the above two lines

$$f(\lambda+) = \left(\sqrt{-|\lambda-\lambda_0| + \lambda_0} - \sqrt{\lambda_0}\right) g(\lambda) = \left(\sqrt{-(\lambda_0-\lambda) + \lambda_0} - \sqrt{\lambda_0}\right) g(\lambda) = \left(\sqrt{\lambda} - \sqrt{\lambda_0}\right) g(\lambda)$$
$$f(\lambda-) = \left(\sqrt{-|\lambda-\lambda_0| + \lambda_0} - \sqrt{\lambda_0}\right) g(\lambda) = \left(\sqrt{-(\lambda_0-\lambda) + \lambda_0} - \sqrt{\lambda_0}\right) g(\lambda) = \left(\sqrt{\lambda} - \sqrt{\lambda_0}\right) g(\lambda)$$

Because $\lambda_0 \neq 0$, the vector phase of $\pm\pi$ makes no difference, and f(λ+) = f(λ-) so there is no discontinuity! Now let's rewrite the above a little more carefully, maintaining our small ε. Then

$$f(\lambda+) = \left(\sqrt{\lambda_+} - \sqrt{\lambda_0}\right) g(\lambda)$$
$$f(\lambda+) = \left(\sqrt{\lambda_-} - \sqrt{\lambda_0}\right) g(\lambda)$$

We know that the function $\sqrt{\lambda}$ has a branch point at the origin with cut to the left, but this is far away from our contour in question above. This, near our contour, we know that $\sqrt{\lambda}$ is analytic, so we know then that $\sqrt{\lambda_+} = \sqrt{\lambda_-}$ where of course we mean by $\lambda_\pm$ points just above and below our contour.

Now let's do this more generally. Suppose we have f(λ) = F($\sqrt{\lambda}$) where F(ν) is analytic. Suppose f(λ) were to have a branch point at some $\lambda_0 \neq 0$ due to a zero (or a power of a zero). We imagine then a cut to the left, and we draw the above picture, and we compute the discontinuity across the cut. We first write

$$f(\lambda) = F\left(\sqrt{(\lambda-\lambda_0)+ \lambda_0}\right)$$

So don't know the nature of the cut or the function near the cut as we did in our example, but we know that the branch point is at some non-zero λ_0 and we can still consider our vector $(\lambda-\lambda_0)$. We can evaluate f(λ) above and below the cut and as in our example we get f(λ+) = f(λ-). That is

$$f(\lambda+) = F\left(\sqrt{|\lambda-\lambda_0|\, e^{+i\pi} + \lambda_0}\right) = F\left(\sqrt{\lambda_+}\right)$$
$$f(\lambda-) = F\left(\sqrt{|\lambda-\lambda_0|\, e^{-i\pi} + \lambda_0}\right) = F\left(\sqrt{\lambda_-}\right)$$

Now, since λ_+ and λ_- are very close together in the λ plane, we know that $\sqrt{\lambda_+}$ and $\sqrt{\lambda_-}$ are also very close together. If F(ν) were NOT analytic, there might be a possibility of it having a cut passing between the two points $\sqrt{\lambda_+}$ and $\sqrt{\lambda_-}$, or perhaps a pole right on our contour, or something not good. But if F(ν) is analytic, then we know it is smooth in the neighborhood of $\sqrt{\lambda_+}$ and $\sqrt{\lambda_-}$ and so $F(\sqrt{\lambda_+}) = F(\sqrt{\lambda_-})$.

We have now proven the following theorem:

Theorem: If $F(\nu)$ is analytic in ν, then if the function $f(\lambda) = F(\sqrt{\lambda})$ has a branch point at some non-zero point λ_0, due to a zero in $F(\nu)$ at $\nu_0 = \sqrt{\lambda_0}$, the discontinuity across that branch cut is 0, and it will make no contribution to a great circle contour integral as discussed above.

Now, suppose $F(\nu)$ has a pole at some ν_0 in the upper half ν plane. We know that our $d\nu$ integral will pick up the residue of this pole, and we will get some contribution. But how do we understand this in the λ plane? For example, suppose we have $d\lambda\, f(\lambda) = d\lambda\, g(\lambda)/(\sqrt{\lambda} - \sqrt{\lambda_0})$. Clearly this $f(\lambda)$ has a branch point at λ_0 and a cut going off to the left. (we are ignoring the $\lambda=0$ branch point since it is "far away"). Our argument above tells us that the cut has no discontinuity, but right at the branch point things are hazy since we have $f(\lambda_0) = \infty$ there. We would have to carefully include a right-side half turn around this point, and that would then duplicate our ν space pole residue! In the case of the zero, such a half turn contributed 0 since the branch point was, after all, a "zero".

Summary of Results: I did this in "a review of the generic singular BC problem" which refers back to earlier sections of THIS doc. Here was my last summary

$$-(xg')' + \alpha^2 x g - \beta^2 x^{-1} g = 0$$

$$\varphi_\beta(x) = (1/\pi)\sqrt{\mathrm{sh}(\pi\beta)}\; K_{i\beta}(\alpha x) \qquad \lambda = \text{eigenvalue} = \beta^2$$

$$f(x) = (2/\pi^2) \int_0^\infty d\beta\, \beta\, \mathrm{sh}(\pi\beta)\, K_{i\beta}(\alpha x)\, F_\alpha(\beta) \qquad \text{// expansion}$$

$$F_\alpha(\beta) \equiv \int_0^\infty dx\, x^{-1} f(x)\, K_{i\beta}(\alpha x) \qquad \text{// projection}$$

$$\int_0^\infty dx\, x^{-1} K_{i\beta}(\alpha x)\, K_{i\beta'}(\alpha x) = \delta(\beta-\beta')\, \pi^2/[2\beta \mathrm{sh}(\pi\beta)] \qquad \text{// orthogonality}$$

$$x\,\delta(x-x') = (2/\pi^2) \int_0^\infty d\beta\, \beta\, \mathrm{sh}(\pi\beta)\, K_{i\beta}(\alpha x)\, K_{i\beta}(\alpha x') \qquad \text{// completeness}$$