Inverse Laplace Transforms PhL 4.10.11

This stuff was written while doing some Stakgold work (p 242 exercises), and it seems good to move the stuff here for possible future use, and also to declutter those Stak notes. See Maple guide for calls. The coordinate space variable below is x, the Laplace space variable is s, others are just parameters.

NOTE: These are just "ILT integrals". The function pair [g(s), G(x)] is only a Laplace transform pair if g(s) meets the requirement $g(s) \rightarrow 1/s^a$ for $a > 0$. Clearly this condition is violated in Cases 1, 2 and 5. In other cases with some unspecified f(s), it depends of course on f(s).

NOTE:

Case 1: g(s) = exp(s²t)/s // conjugate variable to s here is x; t is just a parameter 1
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Case 1: g(s) = exp(s²t)/s // conjugate variable to s here is x; t is just a parameter

Result: $G(x) \equiv (1/2\pi i) \int_C ds\, e^{xs} [\, e^{s^2 t}/s \,]$

$= (1/2) + (1/\pi) \int_0^\infty dy\, e^{-y^2 t} \sin(xy)/y$

$= (1/2) + (1/2\pi) \int_0^\infty d\rho\, e^{-\rho t} \sin(x\rho^{1/2})/\rho \qquad \rho = y^2$

$= (1/2) [\, 1 + \mathrm{erf}(x/[2\sqrt{t}\,])]$

Derivation #1:

In this situation, you cannot just drag the contour to the left and pick up the pole residue because the integrand only converges up and down.

$$G_3(x) = (1/2\pi i) \int_C ds\, e^{xs}\, g_3(s) = (1/2\pi i) \int_C ds\, e^{xs} \exp(s^2 t)/s$$

As always, the contour goes up to the right of the first order pole at s = 0. ET 1 has nothing. What do we do now? I think we get the vertical PP integral plus a right side CCW half turn around the pole. The integral part works like this

$$(1/2\pi i) \int_{-i\infty}^{i\infty} ds\, e^{xs} \exp(s^2 t)/s \qquad \text{// set } s = iy$$

$$= (1/2\pi i) \int_{-\infty}^{\infty} i dy \exp(-y^2 t)\, e^{ixy} / (iy)$$

$$= (1/2\pi i) \int_{-\infty}^{\infty} dy \exp(-y^2 t)\, e^{ixy} / y$$

I can handle perhaps everything at once by displacing the pole to s = -ε so that y = -is = iε meaning it goes up. So this pole factor becomes $1/(s+\varepsilon) = 1/(iy+\varepsilon) = (1/i)(1/(y-i\varepsilon)$, so our integral is really this

$$= (1/2\pi i) \int_{-\infty}^{\infty} dy \exp(-y^2 t)\, e^{ixy} /(y-i\varepsilon)$$

We now use our distribution theory result that

$1/(x+i\alpha) = -i\pi\delta(x) + pf(1/x)$ $\quad \alpha > 0$ $\quad$ Stak p 50 (1.27)
$1/(x+i\alpha) = +i\pi\delta(x) + pf(1/x)$ $\quad \alpha < 0$
$1/(y-i\varepsilon) = +i\pi\delta(y) + pf(1/y)$ $\quad \varepsilon > 0$

So our integral has two parts. The delta part gives this

$$(1/2\pi i) \int_{-\infty}^{\infty} dy \exp(-y^2 t)\, e^{ixy}\, i\pi\delta(y) = (1/2)$$

The other part gives this

$$= (1/2\pi i) \lim_{\varepsilon\to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{+\varepsilon}^{\infty} \right) dy \exp(-y^2 t)\, e^{ixy} / y$$

This is two integrals, In the first, let z = -y so it becomes

$$\int_{\infty}^{\varepsilon} (-dz) \exp(-z^2 t)\, e^{-ixz} /(-z) = \int_{+\varepsilon}^{\infty} dz \exp(-z^2 t)\, e^{-ixz} /(-z) = - \int_{+\varepsilon}^{\infty} dy \exp(-y^2 t)\, e^{-ixy} / y$$

Thus we have

$$= (1/2\pi i)\ \lim_{\varepsilon\to 0}\ \{\ -\ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ e^{-ixy}/y\ +\ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ e^{ixy}/y\ \}$$

$$= (1/2\pi i)\ \lim_{\varepsilon\to 0}\ \{\ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ (e^{ixy}\ -\ e^{-ixy})/y\ \}$$

$$= (1/2\pi i)\ \lim_{\varepsilon\to 0}\ \{\ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ 2i\sin(xy)/y\ \}$$

$$= (1/\pi)\ \lim_{\varepsilon\to 0}\ \{\ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ \sin(xy)/y\ \}$$

which we see converges at x = 0. So keep going

$$= (1/\pi)\ \int_{0}^{\infty} dy\ \exp(-y^2 t)\ \sin(xy)/y$$

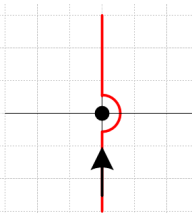
Now change to variable $x^2 = \rho$ and we have $y = \rho^{1/2} \Rightarrow dy = (1/2)\ \rho^{-1/2} d\rho$ and we get

$$= (1/\pi)\ \int_{0}^{\infty} (1/2)\ \rho^{-1/2} d\rho\ \exp(-\rho t)\ \sin(x\ \rho^{1/2})\ /\ \rho^{1/2}$$

$$= (1/2\pi)\ \int_{0}^{\infty} d\rho\ \exp(-\rho t)\ \sin(x\ \rho^{1/2})/\rho$$

Derivation #2:

$$I = (1/2\pi i)\ \int_{C} ds\ e^{xs}\ \exp(s^2 t)/s$$



$$I \equiv\ (1/2\pi i)\ \{\ \int_{-i\infty}^{-iR} + \int_{iR}^{i\infty} + \int_{halfturn}\ \}\ ds\ e^{xs}\ \exp(s^2 t)/s\ .$$

The first two terms are

$$I_1 = (1/2\pi i)\ \{\ \int_{-i\infty}^{-iR} + \int_{iR}^{i\infty}\ \}\ ds\ e^{xs}\ \exp(s^2 t)/s\ .$$

In the first, take s→-s so that $\int_{-i\infty}^{-iR} ds \rightarrow \int_{iR}^{i\infty} ds$. Then the sum of the first two integrals is

$$I_1 = (1/2\pi i)\ \{\ \int_{iR}^{i\infty}\ \}\ ds\ \exp(s^2 t)\ [\ e^{xs}/s\ +\ e^{-xs}/(-s)]$$

$$= (1/2\pi i)\ \{\ \int_{iR}^{i\infty}\ \}\ ds\ \exp(s^2 t)\ [\ e^{xs}\ -\ e^{-xs}]\ /s$$

$$= (1/2\pi i)\ \{\ \int_{iR}^{i\infty}\ \}\ ds\ \exp(s^2 t)\ 2\ sh(xs)/s$$

$$= (1/\pi i)\ \{\ \int_{iR}^{i\infty}\ \}\ ds\ \exp(s^2 t)\ sh(xs)/s\ .$$

Now let $s = iy$ so that $\int_{iR}^{i\infty} ds = \int_{R}^{\infty} i\ dy$ so we continue along

$$= (1/\pi i)\ \int_{R}^{\infty} i\ dy\ \exp(-y^2 t)\ sh(xiy)/(iy)$$

$$= (1/\pi i)\ \int_{R}^{\infty} dy\ \exp(-y^2 t)\ sh(xiy)/(y)$$

$$= (1/\pi i)\ \int_{R}^{\infty} dy\ \exp(-y^2 t)\ i\sin(xy)/(y)$$

$$= (1/\pi)\ \int_{R}^{\infty} dy\ \exp(-y^2 t)\ \sin(xy)/(y)\ .$$

But this thing is non-singular so we can just take $R \to 0$ right now. We then have

$$I_1 = (1/\pi)\ \int_{0}^{\infty} dy\ \exp(-y^2 t)\ \sin(xy)/(y) \qquad // I_1 = (1/2)\ \mathrm{erf}(x/[2\sqrt{t}\,]),\ \text{see below}$$

Now what about the half turn?

$$(1/2\pi i)\ \int_{halfturn}\ ds\ e^{xs}\ \exp(s^2 t)/s \qquad s = Re^{i\theta} \qquad ds = is\ d\theta$$

$$= (1/2\pi i)\ \int_{-\pi/2}^{\pi/2}\ is d\theta\ e^{xs}\ \exp(s^2 t)/s$$

$$= (1/2\pi)\ \int_{-\pi/2}^{\pi/2}\ d\theta\ e^{xs}\ \exp(s^2 t)$$

$$= (1/2\pi)\ \int_{-\pi/2}^{\pi/2}\ d\theta\ \exp(xRe^{i\theta})\ \exp(R^2 e^{2i\theta}\ t)$$

But we take the limit $R \to 0$ right here and we get

$$= (1/2\pi)\ \int_{-\pi/2}^{\pi/2}\ d\theta\ =\ (1/2\pi)\ \pi\ = (1/2)$$

So we find that

$$I = (1/2\pi i)\ \int_{C}\ ds\ e^{xs}\ \exp(s^2 t)/s$$

$$= (1/\pi)\ \int_{0}^{\infty} dy\ \exp(-y^2 t)\ \sin(xy)/(y)\ + (1/2)$$

which agrees Derivation #1's first form.

Addendum :

We have shown that

$$G(x) = (1/2) + (1/\pi) \int_0^\infty dy\, e^{-y^2 t} \sin(xy)/y$$

It is easy to show that the integral is a solution of $(\partial_t - \partial_x^2)f = 0$. In fact, Maple tells us that

```
Int(exp(-y^2*t)*sin(x*y)/y, y=0..infinity);
```

$$\int_0^\infty \frac{e^{(-y^2 t)} \sin(x\,y)}{y}\, dy$$

```
assume(t>0);
int(exp(-y^2*t)*sin(x*y)/y, y=0..infinity);
```

$$\frac{1}{2}\pi\,\mathrm{erf}\left(\frac{1}{2}\frac{x}{\sqrt{t}}\right)$$

.

Thus we have

$$\int_0^\infty dy\, e^{-y^2 t} \sin(xy)/y = (\pi/2)\, \mathrm{erf}(x/[2\sqrt{t}\,])$$

and then

$$G(x) = (1/2) + (1/2)\, \mathrm{erf}(x/[2\sqrt{t}\,]) = (1/2)\,[\,1 + \mathrm{erf}(x/[2\sqrt{t}\,])]$$

<u>Verification of this result using Schaum</u>

Schaum tells us first that (page 170 32.127)

$$\mathrm{erf}(ax) \rightarrow \exp(s^2/4a^2)\, \mathrm{erfc}(s/2a)/s = \exp(s^2/4a^2)[\,1 - \mathrm{erf}(s/2a)]/s$$

and if we set $a = 1/[2\sqrt{t}\,]$ this becomes $a^2 = 1/(4t)$ $t = 1/(4a^2)$ $s/2a = s\sqrt{t}$

$$\mathrm{erf}(x/[2\sqrt{t}\,]) \rightarrow \exp(s^2 t)\,[\,1 - \mathrm{erf}(s\sqrt{t}\,)]/s$$

Thus we conclude that

$$G(x) \rightarrow (1/2)\,[\,1/s + \exp(s^2 t)\,[\,1 - \mathrm{erf}(s\sqrt{t}\,)]/s\,]$$

which does NOT agree with our starting point, horrors!!

Let's first find confirmation of Schaum. Here it is, big as day:

```
with(inttrans):
laplace(erf(a*x),x,s);
```

$$\frac{e^{\left(\frac{1}{4}\frac{s^2}{a^2}\right)}\,\mathrm{erfc}\left(\frac{1}{2}\frac{s}{a}\right)}{s}$$

.

So I now have a Major Problem do deal with. I seem to have two Laplace transforms for the same thing:

e^{s^2t}/s → $(1/2)\,[\,1 + \mathrm{erf}(x/[2\sqrt{t}\,])]$

$(1/2)\,[\,1/s + e^{s^2t}\,[\,1 - \mathrm{erf}(s\sqrt{t}\,)]/s\,]$ ← $(1/2)\,[\,1 + \mathrm{erf}(x/[2\sqrt{t}\,])]$

Rewrite the lower left hand side

$(1/2)\,1/s + (1/2)\,e^{s^2t}/s - e^{s^2t}\,\mathrm{erf}(s\sqrt{t}\,)]/s$

This leads to an obvious question: Is the Laplace Transform unique?

Case 2: $g(s) = e^{-sd}\,\exp(s^2t)/s^2$ // conjugate variable to s here is x; t,d are just parameters

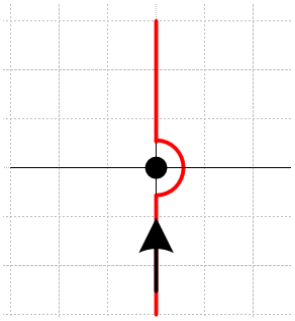
Result:
$$G(x) \equiv (1/2\pi i)\int_C ds\, e^{xs}\,[\,e^{-sd}\,e^{s^2t}/s^2\,]$$
$$= -(1/\pi)\int_0^\infty dy\ \{\cos[(x-d)y]\exp(-y^2t) - 1\}/y^2$$
$$= -(1/2\pi)\int_0^\infty d\rho\,\rho^{-3/2}\,\{\cos[(x-d)\,\rho^{1/2}]\,e^{-\rho t} - 1\}$$

Maple cannot do this, Schaum does not have it, Bateman ETI does not have it. This is another example of a case in which you cannot just drag the contour to the left since integrand only converges up and down. I suspect that for this reason people don't really regard this as an inverse Laplace transform, but instead just an integral that runs through a double pole. It provides an example of doing a "subtraction" in the integrand which explicitly shows that the integral is non-singular. An important point is that as long as the ends converge, a contour integral will be a finite thing. Thus, if you deform the contour in a way that creates diverging pieces, those pieces must cancel, and we see that in both derivations below. The second derivation maps the integral into the r-plane and then the contour integral becomes the real integral of a discontinuity. The fact that I get the same result both ways lends some validity to my result for which I have no outside confirmation.

Derivation #1:

$$I \equiv (1/2\pi i) \int_{-i\infty}^{i\infty} ds\, e^{xs}\, e^{-sd} \exp(s^2 t)/s^2$$

and here is our picture



So we then have

$$I \equiv (1/2\pi i) \{ \int_{-i\infty}^{-iR} + \int_{iR}^{i\infty} + \int_{halfturn} \} ds\, e^{(x-d)s} \exp(s^2 t)/s^2$$

First, look at the first two integrals. In the first, take s→-s so that $\int_{-i\infty}^{-iR} ds \rightarrow \int_{iR}^{i\infty} ds$. Then the sum of the first two integrals is

$$I_1 \equiv (1/2\pi i) \{ \int_{-i\infty}^{-iR} + \int_{iR}^{i\infty} \} ds\, e^{(x-d)s} \exp(s^2 t)/s^2$$

$$= (1/2\pi i) \int_{iR}^{i\infty} ds\, [\; e^{-(x-d)s} \exp(s^2 t)/s^2 + e^{(x-d)s} \exp(s^2 t)/s^2 \;]$$

$$= (1/2\pi i) \int_{iR}^{i\infty} ds\, [\; e^{-(x-d)s} + e^{(x-d)s} \;] \exp(s^2 t)/s^2$$

$$= (1/2\pi i) \int_{iR}^{i\infty} ds\; 2 \cosh[(x-d)s]\; \exp(s^2 t)/s^2$$

$$= (1/\pi i) \int_{iR}^{i\infty} ds\; ch[(x-d)s]\; \exp(s^2 t)/s^2$$

Now let s = iy so that $\int_{iR}^{i\infty} ds = \int_{R}^{\infty} i\, dy$ so we continue along

$$= (1/\pi i) \int_{R}^{\infty} i dy\; ch[(x-d)iy]\; \exp(-y^2 t)/(-y^2)$$

$$= (1/\pi) \int_{R}^{\infty} dy\; \cos[(x-d)y]\; \exp(-y^2 t)/(-y^2)$$

$$= -(1/\pi) \int_{R}^{\infty} dy\; \cos[(x-d)y]\; \exp(-y^2 t)/\, y^2$$

Now we examine our half turn piece

$$(1/2\pi i)\int_{\mathtt{halfturn}} ds\, e^{(x-d)s} \exp(s^2t)/s^2 \qquad s = Re^{i\theta} \qquad ds = isd\theta$$

$$= (1/2\pi i) \int_{-\pi/2}^{\pi/2} isd\theta\, e^{(x-d)s} \exp(s^2t)/s^2$$

$$= (1/2\pi) \int_{-\pi/2}^{\pi/2} d\theta\, e^{(x-d)s} \exp(s^2t)/s$$

$$= (1/2\pi) \int_{-\pi/2}^{\pi/2} d\theta \exp([x-d]Re^{i\theta}) \exp(R^2e^{2i\theta}\, t)\, R^{-1}\, e^{-i\theta}$$

$$= (1/2\pi)\, R^{-1} \{ \int_{-\pi/2}^{\pi/2} d\theta \exp([x-d]Re^{i\theta}) \exp(R^2e^{2i\theta}\, t)\, e^{-i\theta} \}$$

Now we argue that we shall be taking $R\to 0$ soon, so we regard it as extremely small right now and therefore the exp objects are both 1 and we have

$$= (1/2\pi)\, R^{-1} \{ \int_{-\pi/2}^{\pi/2} d\theta\, e^{-i\theta} \} \qquad = (1/2\pi)\, R^{-1} \{ \int_{-\pi/2}^{\pi/2} d\theta \cos(\theta) \}$$

$$= (1/\pi)\, R^{-1} \{ \int_{0}^{\pi/2} d\theta \cos(\theta) \} \quad = (1/\pi)\, R^{-1}$$

So adding up our three terms, we get

$$I = -(1/\pi) \int_{R}^{\infty} dy\, \cos[(x-d)y]\, \exp(-y^2t)/\, y^2 + (1/\pi)\, R^{-1}$$

Now just to check we know that the singular part of the integral is this

$$-(1/\pi) \int_{R}^{\infty} dy\, /y^2 = -(1/\pi)R^{-1}$$

We can then write our integral this way

$$I = -(1/\pi) \int_{R}^{\infty} dy\, \cos[(x-d)y]\, \exp(-y^2t)/\, y^2 + (1/\pi) \int_{R}^{\infty} dy\, /y^2$$

$$= -(1/\pi) \int_{R}^{\infty} dy\, \{ \cos[(x-d)y] \exp(-y^2t) - 1\}/y^2$$

If we look now at the low end of this integrand we find

$$[...]/y^2 \approx [\, (\, 1 - y^2(x-d)^2/2 + ...\,)(1-y^2t + ...) - 1]/y^2 = [\, -(x-d)^2/2 - t\,]$$

so the integral does not diverge at the low end, and we can now take $R\to 0$ to get

$$I = -(1/\pi) \int_{0}^{\infty} dy\, \{ \cos[(x-d)y] \exp(-y^2t) - 1\}/y^2$$

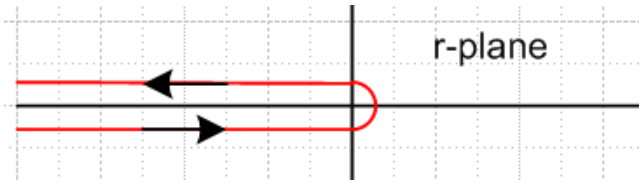
Now change variables to $y^2 = \rho$ so $y = \rho^{1/2}$ and $dy = (1/2)\,\rho^{-1/2}\,d\rho$ and then

$$I = -(1/\pi)\int_0^\infty (1/2)\,\rho^{-1/2}\,d\rho\ \{\cos[(x-d)\,\rho^{1/2}]\exp(-\rho t) - 1\}/\rho$$

$$= -(1/2\pi)\int_0^\infty \rho^{-3/2}\,d\rho\ \{[\cos[(x-d)\,\rho^{1/2}]\exp(-\rho t) - 1\}$$

Derivation #2:

I see two hours of work here where I change variables to $r = s^2$ and ponder the integral. Now think of the upper ds integral as being just less than $\pi/2$ in angle, so r^2 will be just less than π, etc. So the r plane integration looks like this



So we then have

$$I = (1/2\pi i)\int_C (1/2)\,r^{-1/2}dr\,\exp([x-d]\,r^{1/2})\exp(rt)/r$$

$$= (1/4i\pi)\int_C dr\,r^{-3/2}\exp([x-d]\,r^{1/2})\exp(rt)$$

I am not happy with this thing. On scratch paper I have shown that

$$\text{disc}\,[r^{-3/2}\exp([x-d]\,r^{1/2})] = 2i\,|r|^{-3/2}\cos([x-d]|r|^{1/2}) \qquad \text{above - below}$$

So ignoring the half-turn part of the contour we get

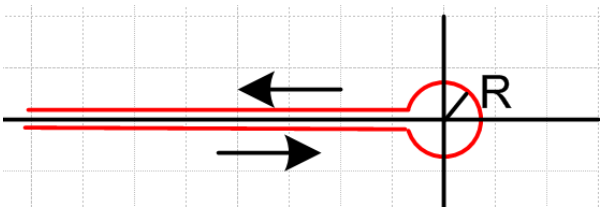
$$I = -(1/4i\pi)\int_{-\infty}^0 dr\,\text{disc}\{r^{-3/2}\exp([x-d]\,r^{1/2})\}\ \exp(rt)$$

$$-(1/4i\pi)\int_{-\infty}^0 dr\,\{2i\,|r|^{-3/2}\cos([x-d]|r|^{1/2})\}\exp(rt)$$

Now replace $r \to -r$ so that $\int_{-\infty}^0 dr \to \int_0^\infty dr$ and we then have

$$I_1 = -(1/2\pi)\int_0^\infty dr\ r^{-3/2}\cos([x-d]r^{1/2})\exp(-rt)$$

which diverges at the low end so needs some regulation there. Let's draw a better picture



so then the contribution shown above is really this:

$$I_1 = -(1/2\pi) \int_R^\infty dr\ r^{-3/2} \cos([x-d]\, r^{1/2}) \exp(-rt)$$

Then we have to compute the result of a full CCW turn around the origin:

$$I_2 = (1/4i\pi) \oint dr\, r^{-3/2} \exp([x-d]\, r^{1/2}) \exp(rt)$$

On scratch I find that if we let $R \to 0$ and throw out non contributing terms, we get

$$I_2 = +(1/\pi)R^{-1/2}$$

I wonder if it is possible that this cancels the divergent part of the other integral. I can get that part this way

$$I_1 \approx -(1/2\pi) \int_R^\infty dr\ r^{-3/2} = -(1/2\pi)\ 2\ R^{-1/2} = -(1/\pi)\ R^{-1/2}$$

and we do seem to get cancellation. It has to work this way because the contour in free space has to give a finite result since it certainly converges at the ends.

So how to I figure out the finite part of this integral, knowing that the divergent parts cancel? One idea then is to write

$$I = I_1 + I_2 = -(1/2\pi) \int_R^\infty dr\ r^{-3/2} \cos([x-d]\, r^{1/2}) \exp(-rt) + (1/\pi)R^{-1/2}$$

$$= -(1/2\pi) \int_R^\infty dr\ r^{-3/2} \cos([x-d]\, r^{1/2}) \exp(-rt) + (1/2\pi) \int_R^\infty dr\ r^{-3/2}$$

$$= -(1/2\pi) \int_R^\infty dr\ r^{-3/2} [\cos([x-d]\, r^{1/2}) \exp(-rt) - 1]$$

Then we can take the limit $R \to 0$.

$$= -(1/2\pi) \int_0^\infty dr\ r^{-3/2} [\cos([x-d]\, r^{1/2}) \exp(-rt) - 1]$$

and this is seen to be a convergent integral. Maple cannot do it. The good news is that this agrees with the result of my Derivation #1! (after bugs fixed in both derivations)

Case 3: g(s) = f(s) /(s^4- ω^2) where f(s) is analytic

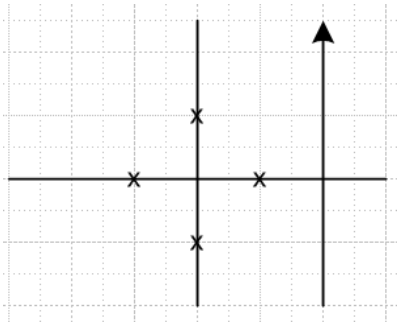
Result: $G(x) \equiv (1/2\pi i) \int_C ds\, e^{xs} [\, f(s)/(s^4-\omega^2)\,]$

$= (1/4a^3) \{ -i\, e^{-iat} f(-ia) + i\, e^{iat} f(ia) - e^{-at} f(-a) + e^{at} f(a) \}$ $\quad a = \sqrt{\omega}$

Derivation:

Note that:

$(s^4-\omega^2) = (s^2 + \omega)(s^2- \omega) = (s +i\sqrt{\omega}\,)\,(s -i\sqrt{\omega}\,)(s+\sqrt{\omega}\,)(s-\sqrt{\omega}\,)$



The general thing we want is this

$(1/2\pi i) \oint ds\, e^{st} f(s) / [(s +ia)(s -ia)(s+a)(s-a)]$ $\quad a = \sqrt{\omega}$

There will be four contributions to this integral.

Pole at s = - ia gives

$\{e^{st} f(s) / [(s -ia)(s+a)(s-a)]\}\,|^{s=-ia} = \{e^{-iat} f(-ia) / [(-2ia)(-ia+a)(-ia-a)]\}$
$= \{e^{-iat} f(-ia) / [(-i)(-i+1)(-i-1)2a^3]\} = \{e^{-iat} f(-ia) / [(2i)\, 2a^3]\}$ // Maple
$= e^{-iat} f(-ia) / (4ia^3) = -i\, e^{-iat} f(-ia) / (4a^3)$

=> Pole at s = -ia gives $-i\, e^{-iat} f(-ia) / (4a^3)$

Pole at s = +ia must give the same result but with a → -a

=> Pole at s = +ia gives = $i\, e^{iat} f(ia) / (4a^3)$

Pole at s = -a gives

$\{e^{st}f(s) / [(s +ia) (s -ia)(s-a)]\} |^{s=-a} = \{e^{-at}f(-a) / [(-a +ia) (-a -ia)(-2a)]\}$
$= \{e^{-at}f(-a) / [(-1 +i) (-1 -i)(-2)a^3]\} = \{e^{-at}f(-a) / [-4a^3]\}$ // Maple
$= - e^{-at}f(-a) / (4a^3)$

=> Pole at s = -a gives $- e^{-at}f(-a) / (4a^3)$

=> Pole at s = + a gives $+ e^{at}f(a) / (4a^3)$

So we can now add our four terms to get

$(1/2\pi i) \oint ds\, e^{st} f(s) / [(s+ia) (s-ia)(s+a)(s-a)]$
$= -i e^{-iat} f(-ia) / (4a^3) + i e^{iat} f(ia) / (4a^3)$
$- e^{-at}f(-a) / (4a^3) + e^{at}f(a) / (4a^3)$
$= (1/4a^3) \{ -i e^{-iat} f(-ia) + i e^{iat} f(ia) - e^{-at}f(-a) + e^{at}f(a) \}$

Case 3.1 special case where f(s) = s : g(s) = $s /(s^4 - \omega^2)$

f(s) = s : **Result:** $G(x) = (1/2a^2) \{ -\cos(at) + \cosh(at) \}$ $a = \sqrt{\omega}$

Derivation:

$G(x)= (1/4a^3) \{ -i e^{-iat} f(-ia) + i e^{iat} f(ia) - e^{-at}f(-a) + e^{at}f(a) \}$
$= (1/4a^3) \{ -i e^{-iat} (-ia) + i e^{iat} (ia) - e^{-at}(-a) + e^{at}(a) \}$
$= (1/4a^2) \{ -i e^{-iat} (-i) + i e^{iat} (i) - e^{-at}(-1) + e^{at} \}$
$= (1/4a^2) \{ -e^{-iat} - e^{iat} + e^{-at} + e^{at} \}$
$= (1/2a^2) \{ -(e^{-iat} + e^{iat})/2 + (e^{-at} + e^{at})/2 \}$
$= (1/2a^2) \{ -\cos(at) + \cosh(at) \}$

Verification from page 167 Schaum

$\frac{s}{s^4 - a^4}$	$\frac{1}{2a^2}(\cosh at - \cos at)$

Verification from Maple

```
restart;
with(inttrans):
assume(a>0);
invlaplace(s/(s^4-a^4), s, t);
```

$$\frac{\sin\left(\frac{1}{2}\sqrt{2}\,(-a^4)^{\left(\frac{1}{4}\right)}\,t\right)\sinh\left(\frac{1}{2}\sqrt{2}\,(-a^4)^{\left(\frac{1}{4}\right)}\,t\right)}{\sqrt{-a^4}}$$

```
simplify(%);
```

$$-\frac{I\sin\left(\left(\frac{1}{2}+\frac{1}{2}I\right)a\,t\right)\sinh\left(\left(\frac{1}{2}+\frac{1}{2}I\right)a\,t\right)}{a^2}$$

```
expand(%);
```

$$\frac{\sin\left(\frac{1}{2}a\,t\right)^2\cosh\left(\frac{1}{2}a\,t\right)^2}{a^2}+\frac{\cos\left(\frac{1}{2}a\,t\right)^2\sinh\left(\frac{1}{2}a\,t\right)^2}{a^2}$$

```
simplify(%);
```

$$\frac{\cosh\left(\frac{1}{2}a\,t\right)^2-\cos\left(\frac{1}{2}a\,t\right)^2}{a^2}$$

```
combine(%);
```

$$\frac{\frac{1}{2}\cosh(a\,t)-\frac{1}{2}\cos(a\,t)}{a^2}$$

Case 3.3 special case where f(s) = s^3 : g(s) = $s^3/(s^4-\omega^2)$

f(s) = s^3 : **Result:** G(x) = (1/2) { cos(at)+ ch(at) } a = $\sqrt{\omega}$

Derivation:

$g_2(s) = s^3/(s^4-\omega^2)$ => f(s) = s^3

Int = $(1/4a^3)$ { $-i\,e^{-iat}$ f(-ia) + $i\,e^{iat}$ f(ia) - e^{-at}f(-a) + e^{at}f(a) }
= $(1/4a^3)$ { $-i\,e^{-iat}(-ia)^3 + i\,e^{iat}(ia)^3 - e^{-at}(-a)^3 + e^{at}(a)^3$ }
= (1/4) { $-i\,e^{-iat}(-i)^3 + i\,e^{iat}(i)^3 - e^{-at}(-1)^3 + e^{at}$ }
= (1/4) { $e^{-iat}(-i)^4 + e^{iat}(i)^4 + e^{-at} + e^{at}$ }
= (1/2) { $(e^{-iat}+ e^{iat})/2 + (e^{-at} + e^{at})/2$ } // use this line below!
= (1/2) { cos(at)+ ch(at) }

Verification from page 167 Schaum

$\frac{s^3}{s^4 - a^4}$	$\frac{1}{2}(\cosh at + \cos at)$

= (1/2) { cos(at)+ ch(at) }

Verification from Maple

```
restart;
with(inttrans):
assume(a>0);
invlaplace(s^3/(s^4-a^4), s, t);
```

$$\cos\left(\frac{1}{2}\sqrt{2}\,(-a^4)^{\left(\frac{1}{4}\right)}\,t\right)\cosh\left(\frac{1}{2}\sqrt{2}\,(-a^4)^{\left(\frac{1}{4}\right)}\,t\right)$$

```
simplify(%);
```

$$\cos\left(\left(\frac{1}{2}+\frac{1}{2}I\right)a\,t\right)\cosh\left(\left(\frac{1}{2}+\frac{1}{2}I\right)a\,t\right)$$

```
expand(%);
```

$$\cos\left(\frac{1}{2}a\,t\right)^2\cosh\left(\frac{1}{2}a\,t\right)^2+\sin\left(\frac{1}{2}a\,t\right)^2\sinh\left(\frac{1}{2}a\,t\right)^2$$

```
simplify(%);
```

$$\cosh\left(\frac{1}{2}a\,t\right)^2-1+\cos\left(\frac{1}{2}a\,t\right)^2$$

```
combine(%);
```

$$\frac{1}{2}\cosh(a\,t)+\frac{1}{2}\cos(a\,t)$$

Case 4: g(s) = f(s) /($s^4 + \omega^2$) where f(s) is analytic

Result: G(x) ≡ $(1/2\pi i)\int_C ds\, e^{xs}\,[\, f(s)/(s^4+\omega^2)\,]$

= $(1/4a^3)\,\{\, -i\, e^{-iat}\, f(-ia) + i\, e^{iat}\, f(ia) - e^{-at}f(-a) + e^{at}f(a)\,\}$ $\qquad a = i^{1/2}\sqrt{\omega}$

Derivation:

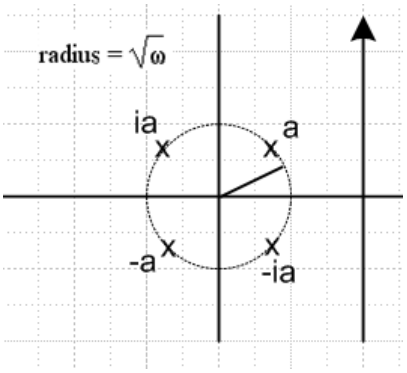
The results here may be obtained from those of Case 3 in the following manner. First, write

g(s) = f(s) /($s^4 + \omega^2$)

If we define a for this case by $a = i^{1/2}\sqrt{\omega} = \sqrt{i\omega}$ we find that

$(s^4+\omega^2) = (s+i\sqrt{i\omega})(s-i\sqrt{i\omega})(s+\sqrt{i\omega})(s-\sqrt{i\omega}) = (s+ia)(s-ia)(s+a)(s-a)$ $\quad a = i^{1/2}\sqrt{\omega}$

and the poles lie symmetrically this way:



so compared to the Case 3, all the poles have been rotated CCW by $\pi/4$ around the circle. We obtain formulas for this case by making the replacement $a \rightarrow a = i^{1/2}\sqrt{\omega}$ in Case 3. Thus we find

Result: $G(x) \equiv (1/2\pi i) \int_C ds\, e^{xs} [\, f(s)/(s^4+\omega^2)\,]$

$= (1/4a^3) \{ -i\, e^{-iat} f(-ia) + i\, e^{iat} f(ia) - e^{-at} f(-a) + e^{at} f(a) \}$ $\quad a = i^{1/2}\sqrt{\omega}$

where of course $i^{1/2} = e^{i\pi/4}$. This is not a very convenient form for the result, but we leave it as is.

Case 4.1 special case where f(s) = s : $\quad$ **g(s) = $s/(s^4+\omega^2)$**

Result: $G(x) = = (1/\omega) \sin(ct) \sinh(ct)$ $\quad c \equiv \sqrt{\omega/2}$

Derivation:

For starters, from Case 3.1 we get

f(s) = s : $\quad$ Result: $\quad G(x) = (1/2a^2) \{ -\cos(at) + \cosh(at) \}$ $\quad a \equiv i^{1/2}\sqrt{\omega}$

We then have lots of algebra to get this into a simpler form. It is convenient to write

$a = i^{1/2}\sqrt{\omega} = (i+1)\sqrt{\omega/2} = (i+1)c$ $\quad c \equiv \sqrt{\omega/2}$ $\quad a^2 = i\omega$ $\quad a^4 = -\omega^2$

Then

$G(x) = (1/2i\omega) \{ -\cos(at) + \cosh(at) \}$ $\quad (i+1)c$

$\cos(at) = (1/2) [\, e^{iat} + e^{-iat}] = (1/2) [\, e^{i(i+1)c\,t} + e^{-i(i+1)c\,t}]$

$= (1/2)\,[\, e^{(i-1)c\,t} + e^{(1-i)c\,t}]$

$\cosh(at) = (1/2)\,[\, e^{at} + e^{-at}] = (1/2)\,[\, e^{(i+1)c\,t} + e^{-(i+1)c\,t}]$

Then

$G(x) = (1/2i\omega)\,\{ -\cos(at) + \cosh(at) \}$
$= (1/4i\omega)\,[\, - e^{(i-1)c\,t} - e^{(1-i)c\,t} + e^{(i+1)c\,t} + e^{-(i+1)c\,t}\,]$
$= (1/4i\omega)\,[\, - e^{-ct}\, e^{ict} - e^{ct}\, e^{-ict} + e^{ct}\, e^{ict} + e^{-ct}\, e^{-ict}\,]$
$= (1/4i\omega)\,[\, - (\, e^{-ct}\, e^{ict} - e^{-ct}\, e^{-ict}\,) + e^{ct}\, e^{ict} - e^{ct}\, e^{-ict}\,]$
$= (1/2i\omega)\,[\, - e^{-ct}\,(\, e^{ict} - e^{-ict}\,)/2 + e^{ct}\,(e^{ict} - e^{-ict})/2\,]$
$= (1/2i\omega)\,[\, - e^{-ct}\, i\sin(ct) + e^{ct}\, i\sin(ct)\,]$
$= (1/2\omega)\,\sin(ct)\,[\, - e^{-ct} + e^{ct}\,]$
$= (1/\omega)\,\sin(ct)\,\mathrm{sh}(ct)$ $\qquad c \equiv \sqrt{\omega/2}$

<u>Verification from Schaum page 167</u>

Schaum claims that $\qquad c^2 = \omega/2$

$s/(s^4 + 4c^4) \quad \rightarrow \quad \sin(ct)\mathrm{sh}(ct)/(2c^2)$

Then set $2c^2 = \omega$ to get $s/(s^4 + \omega^2)$ so $c = \sqrt{\omega/2}$ our same c, so and Schaum's result is then

$\sin(ct)\mathrm{sh}(ct)/(2c^2) = \sin(ct)\mathrm{sh}(ct)/(\omega)$ $\qquad$ // agrees with derivation above

---

Case 4.3 special case where $f(s) = s^3$: $\qquad g(s) = s^3/(s^4 + \omega^2)$

Result: $\qquad G(x) = (1/4)\,[\,\sin(ct)\,\mathrm{ch}(ct)]$ $\qquad c \equiv \sqrt{\omega/2}$

Derivation:

For starters, from Case 3.3 we get

$G(x) = (1/2)\,\{\ \cos(at) + \mathrm{ch}(at)\ \}$ $\qquad a = \sqrt{\omega}$

so as in the previous case our result becomes

$G(x) = (1/2)\,\{\ \cos(at) + \mathrm{ch}(at)\ \}$ $\qquad a \equiv i^{1/2}\sqrt{\omega}$

We already know from the previous case that

$\cos(at) = (1/2)\,[\, e^{(i-1)c\,t} + e^{(1-i)c\,t}\,]$

$\cosh(at) = (1/2)\,[\, e^{(i+1)c\,t} + e^{-(i+1)c\,t}]$

So we then have

$G(x) = (1/2)\,\{\ \cos(at) + ch(at)\ \}$

$= (1/4)\,\{\ e^{(i-1)c\,t} + e^{(1-i)c\,t} + e^{(i+1)c\,t} + e^{-(i+1)c\,t}]\ \}$

$= (1/4)\,[\ + e^{-ct}\ e^{ict} + e^{ct}\ e^{-ict} + e^{ct}\ e^{ict} + e^{-ct}\ e^{-ict}\ ]$

$= (1/4)\,[\ + e^{-ct}\ (e^{ict} + e^{-ict})\ + e^{ct}\ (e^{ict} + e^{-ict}\ )$

$= (1/4)\,[\ (e^{ict} + e^{-ict})\ (e^{-ct} + e^{ct})$

$= \cos(ct)\ ch(ct)$

Verification from Schaum page 167

Schaum claims that $c^2 = \omega/2$

$s^3/(s^4 + 4c^4) \quad \rightarrow \quad \sin(ct)sh(ct)$

Then set $2c^2 = \omega$ to get $s/(s^4 + \omega^2)$ so $c = \sqrt{\omega/2}$ our same c and Schaum's result is then

$\sin(ct)sh(ct)$ // agrees with derivation above

Case 5: $g(s) = \exp(s^2t)\ s^3/(s^4+\omega^2)$

Result: $G(x) = (1/2)\exp(\sqrt{\omega/2}\ x)\ \cos(\sqrt{\omega/2}\ x + \omega t)$

$+ (1/2\pi)\ \int_0^\infty d\rho\ e^{-\rho t} \sin(x\rho^{1/2})\rho\ /(\rho^2+\omega^2)$

Derivation:

As shown below, two of our poles are to the right of the imaginary axis, so when we slide the contour left to the imaginary axis we pick up those two pole residues plus the residual integral. Here are the details.

The denominator may be written (repeating a bit of the above)

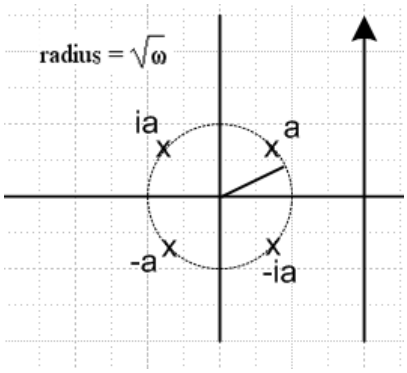
$(s^4+\omega^2) = (s+i\sqrt{i\omega}\ )(s-i\sqrt{i\omega}\ )(s+\sqrt{i\omega}\ )(s-\sqrt{i\omega}\ )$

$$= (s+i^{3/2}\sqrt{\omega}\,)(s-\,i^{3/2}\sqrt{\omega}\,)(s+i^{1/2}\sqrt{\omega}\,)(s-\,i^{1/2}\sqrt{\omega}\,)$$

Define $a = i^{1/2}\sqrt{\omega}$ and then

$$(s^4+\omega^2) = (s+ia)(s-ia)(s+a)(s-a) \qquad\qquad a = i^{1/2}\sqrt{\omega}$$

These poles lie symmetrically this way:



The plan is to shift the contour to the imaginary axis, picking up two pole residues. The imaginary axis integral would then be this

$$I_1 = (1/2\pi i)\ \int_{-i\infty}^{i\infty} ds\ \exp(s^2t)\ e^{xs}\ s^3/(s^4+\omega^2)$$

We then rotate this thing to the real axis using **s = iy** and we get

$$I_1 = (1/2\pi i)\ \int_{-\infty}^{\infty} idy\ \exp(-y^2t)\ e^{xiy}\ (iy)^3/((iy)^4+\omega^2)$$

$$= (1/2\pi i)\ \int_{-\infty}^{\infty} i\ dy\ \exp(-y^2t)\ e^{ixy}\ (-i)y^3/(y^4+\omega^2)$$

$$= (1/2\pi i)\ \int_{-\infty}^{\infty} dy\ \exp(-y^2t)\ e^{ixy}\ y^3/(y^4+\omega^2) \qquad\qquad e^{ixy} = \cos(xy) + i\,\sin(xy)$$

but only the sine part survives and we then reflect the negative side gaining a factor of 2

$$= (1/\pi i)\ \int_{0}^{\infty} dy\ \exp(-y^2t)\ i\,\sin(xy)\ y^3\ /(y^4+\omega^2)$$

$$= (1/\pi)\ \int_{0}^{\infty} dy\ \exp(-y^2t)\ \sin(xy)\ y^3\ /(y^4+\omega^2)$$

Now change to variable $y^2 = \rho$ and we have $y = \rho^{1/2} \Rightarrow dy = (1/2)\ \rho^{-1/2}d\rho$

$$= (1/\pi)\ \int_{0}^{\infty} (1/2)\ \rho^{-1/2}d\rho\ \ \exp(-\rho t)\ \sin(x\rho^{1/2})\rho^{3/2}\ /(\rho^2+\omega^2)$$

$$= (1/2\pi) \int_0^\infty d\rho \; e^{-\rho t} \sin(x\rho^{1/2})\rho /(\rho^2+\omega^2) \qquad \text{// residual integral part}$$

which we happily recognize as the integral appearing in p 242 C (apart from the overall constant)

Let's now do the two pole residues and see what happens there.

$$(1/2\pi i) \oint ds \; e^{sx} f(s) / [(s+ia)(s-ia)(s+a)(s-a)]$$

There will be two contributions of interest.

Pole at s = a gives

$$\{e^{sx}f(s)/ [(s+ia)(s-ia)(s+a)] \}^{s=a} = \{e^{ax}f(a)/ [(a+ia)(a-ia)(2a)] \}$$
$$= \{e^{ax}f(a)/ [(1+i)(1-i)(2)a^3] \} = \{e^{ax}f(a)/ [(1-i^2)(2)a^3] \} = e^{ax}f(a)/ 4a^3$$

Pole at s = -ia gives

$$\{e^{sx}f(s)/ [(s-ia)(s-a)(s+a)] \}^{s=-ia} = \{e^{-iax}f(-ia)/ [(-2ia)(-ia-a)(-ia+a)] \}$$
$$= \{e^{-iax}f(-ia)/ [(-2i)(-i-1)(-i+1)a^3]\} = \{e^{-iax}f(-ia)/ [(2i)(1+i)(1-i)a^3] \}$$
$$= \{e^{-iax}f(-ia)/ [(2i)(1-i^2)a^3] \} = \{e^{-iax}f(-ia)/ [(2i)2a^3] \} = e^{-iax}f(-ia) / (4ia^3)$$

So the contributions of these two poles add up to

$$\text{both poles} = e^{ax}f(a)/ 4a^3 + e^{-iax}f(-ia) / (4ia^3) = (1/4a^3) [e^{ax}f(a) - i\, e^{-iax}f(-ia)]$$

Now our interest is with $f(s) = \exp(s^2 t)\, s^3$. We then have

$$\text{both poles} = (1/4a^3) [e^{ax}f(a) - i\, e^{-iax}f(-ia)]$$
$$= (1/4a^3) [e^{ax} \exp(a^2 t)\, a^3 - i\, e^{-iax} \exp([-ia]^2 t) [-ia]^3]$$
$$= (1/4a^3) [e^{ax} \exp(a^2 t)\, a^3 - i\, e^{-iax} \exp(-a^2 t) (i)a^3]$$
$$= (1/4) [e^{ax} \exp(a^2 t) + e^{-iax} \exp(-a^2 t)]$$

Now

$$a = i^{1/2}\sqrt{\omega} = (1+i)\sqrt{\omega/2} \qquad a^2 = i\omega \qquad \exp(a^2 t) = e^{i\omega t}$$

So we then have

$$\text{both poles} = (1/4) [e^{ax}e^{i\omega t} + e^{-iax}\, e^{-i\omega t}]$$
$$= (1/4) [\exp(ax+i\omega t) + \exp(-iax-i\omega t)]$$

The expo arguments are

$$\exp(ax+i\omega t) = \exp((1+i)\sqrt{\omega/2}\, x+i\omega t) = \exp(\sqrt{\omega/2}\, x) \exp[i(\sqrt{\omega/2}\, x + \omega t)]$$
$$\exp(-iax-i\omega t) = \exp(-i(1+i)\sqrt{\omega/2}\, x-i\omega t) = \exp((1-i)\sqrt{\omega/2}\, x-i\omega t) = \exp(\sqrt{\omega/2}\, x) \exp[i(-\sqrt{\omega/2}\, x- \omega t)$$

Then we have

both poles = (1/4) [exp(ax+iωt) + exp(-iax-iωt)]
= (1/4) [exp($\sqrt{\omega/2}$ x) exp[i($\sqrt{\omega/2}$ x + ωt)] + exp($\sqrt{\omega/2}$ x) exp[i(-$\sqrt{\omega/2}$ x- ωt)]
= (1/4) exp($\sqrt{\omega/2}$ x) [exp[i($\sqrt{\omega/2}$ x + ωt)] + exp[i(-$\sqrt{\omega/2}$ x- ωt)]
= (1/2) exp($\sqrt{\omega/2}$ x) cos($\sqrt{\omega/2}$ x + ωt) // two pole residues part

Therefore, our final result including both terms is:

$G(x) = (1/2)\exp(\sqrt{\omega/2}\,x)\cos(\sqrt{\omega/2}\,x + \omega t) + (1/2\pi)\int_0^\infty d\rho\; e^{-\rho t}\sin(x\rho^{1/2})\rho/(\rho^2+\omega^2)$

Case 6: g(s) = f(s) exp(s²t)/s // conjugate variable to s here is x; t is just a parameter

Result: $G(x) \equiv (1/2\pi i)\int_C ds\, e^{xs}[\, f(s)\, e^{s^2 t}/s\,]$

$= (f(0)/2) + (1/\pi)\int_0^\infty dy \exp(-y^2 t)\ \mathrm{Im}(f(iy)e^{ixy})/y$

$= (f(0)/2) + ((1/2\pi)\int_0^\infty d\rho\ \exp(-\rho t)\ \mathrm{Im}\{\, f(i\,\rho^{1/2})\exp(ix\,\rho^{1/2})\}/\rho \qquad \rho = y^2$

Derivation #1: // just editing work done above in Case 1 !!

In this situation, you cannot just drag the contour to the left and pick up the pole residue because the integrand only converges up and down.

$G_3(x) = (1/2\pi i)\int_C ds\, e^{xs}\, g_3(s) = (1/2\pi i)\int_C ds\, f(s)\, e^{xs}\exp(s^2 t)/s$

As always, the contour goes up to the right of the first order pole at s = 0. ET 1 has nothing. What do we do now? I think we get the vertical PP integral plus a right side CCW half turn around the pole. The integral part works like this

$(1/2\pi i)\int_{-i\infty}^{i\infty} ds\, e^{xs} f(s)\exp(s^2 t)/s$ // set s = iy

$= (1/2\pi i)\int_{-\infty}^{\infty} i\,dy\, f(iy)\exp(-y^2 t)\, e^{ixy}/(iy)$

$= (1/2\pi i)\int_{-\infty}^{\infty} dy\, f(iy)\exp(-y^2 t)\, e^{ixy}/y$

I can handle perhaps everything at once by displacing the pole to s = -ε so that y = -is = iε meaning it goes up. So this pole factor becomes 1/(s+ε) = 1/(iy+ε) = (1/i)(1/(y-iε) , so our integral is really this

$$= (1/2\pi i) \int_{-\infty}^{\infty} dy\ f(iy)\exp(-y^2 t)\ e^{ixy}/(y-i\varepsilon)$$

We now use our distribution theory result that

$$1/(x+i\alpha) = -i\pi\delta(x) + pf(1/x) \qquad \alpha > 0 \qquad \text{Stak p 50 (1.27)}$$
$$1/(x+i\alpha) = +i\pi\delta(x) + pf(1/x) \qquad \alpha < 0$$
$$1/(y-i\varepsilon) = +i\pi\delta(y) + pf(1/y) \qquad \varepsilon > 0$$

So our integral has two parts. The delta part gives this

$$(1/2\pi i) \int_{-\infty}^{\infty} dy\ f(iy)\exp(-y^2 t)\ e^{ixy}\ i\pi\delta(y) = (1/2)\ f(0)$$

The other part gives this

$$= (1/2\pi i) \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{+\varepsilon}^{\infty} \right) dy\ f(iy)\exp(-y^2 t)\ e^{ixy}/y$$

This is two integrals, In the first, let z = -y so it becomes

$$\int_{\infty}^{\varepsilon} (-dz)\ \exp(-z^2 t)\ f(-iz)e^{-ixz}/(-z) = \int_{+\varepsilon}^{\infty} dz\ \exp(-z^2 t)\ f(-iz)e^{-ixz}/(-z)$$
$$= -\int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ f(-iy)e^{-ixy}/y$$

Thus we have

$$= (1/2\pi i) \lim_{\varepsilon\to 0} \{ -\int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ f(-iy)e^{-ixy}/y + \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ f(iy)e^{ixy}/y \}$$

$$= (1/2\pi i) \lim_{\varepsilon\to 0} \{ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ (f(iy)e^{ixy} - f(-iy)e^{-ixy})/y \}$$

$$= (1/2\pi i) \lim_{\varepsilon\to 0} \{ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ (f(iy)e^{ixy} - cc\ )/y \}$$

$$= (1/2\pi i) \lim_{\varepsilon\to 0} \{ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ 2i\ \mathrm{Im}(f(iy)e^{ixy})/y \}$$

$$= (1/\pi) \lim_{\varepsilon\to 0} \{ \int_{+\varepsilon}^{\infty} dy\ \exp(-y^2 t)\ \mathrm{Im}(f(iy)e^{ixy})/y \}$$

At this point, I will just assume that f(s) is such that the lower endpoint of this integral converges, so

$$= (1/\pi) \int_{0}^{\infty} dy\ \exp(-y^2 t)\ \mathrm{Im}(f(iy)e^{ixy})/y$$

Now change to variable $x^2 = \rho$ and we have $y = \rho^{1/2} \Rightarrow dy = (1/2)\ \rho^{-1/2} d\rho$ and we get

$$= (1/\pi) \int_0^\infty (1/2)\, \rho^{-1/2} d\rho \;\; \exp(-\rho t) \;\; \mathrm{Im}\{ f(i\, \rho^{1/2}) \exp(ix\, \rho^{1/2}) \} / \rho^{1/2}$$

$$= (1/2\pi) \int_0^\infty d\rho \;\; \exp(-\rho t) \;\; \mathrm{Im}\{ f(i\, \rho^{1/2}) \exp(ix\, \rho^{1/2}) \} / \rho$$

And if f(s) = 1, we recover our earlier result for the residual integral part

$$= (1/2\pi) \int_0^\infty d\rho \, \exp(-\rho t) \, \sin(x\, \rho^{1/2}) / \rho$$