Those constant terms in Laplace Transforms PhL 10.2.14

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1. Those constant terms using Laplace Transforms.

Here is the famous Laplace transform and its inverse:

$$f(s) = \int_0^\infty dx\, e^{-xs}\, F(x)$$

$$F(x) = (1/2\pi i) \int ds\, e^{xs}\, f(s) \ .$$

Now suppose you are interested in the function $G(x) \equiv \partial_x F(x) = F'(x)$. What is the LT of this function? Well, write it out and then do parts integration,

$$g(s) = \int_0^\infty dx\, e^{-xs}\, G(x) = \int_0^\infty dx\, e^{-xs}\, \partial_x F(x)$$

$$= [\, e^{-xs}\, F(x)\,]_0^\infty - \int_0^\infty dx\, \partial_x e^{-xs} F(x)$$

$$= e^{-\infty s} F(\infty) - e^0\, F(0) - \int_0^\infty dx\, (-s) e^{-xs} F(x)$$

Now we must assume that $Re(s) > 0$ which is usually the case on the recovery contour, and we assume that $F(x)$ has only power divergence, not expo divergence for large ∞, so then $e^{-\infty s} F(\infty) = 0$. We then may continue

$$= 0 - F(0) + s \int_0^\infty dx\, e^{-xs}\, F(x)$$

$$= -F(0) + s\, f(s) \ .$$

We have then found this result:

$$G(x) \equiv \partial_x F(x) \qquad \leftrightarrow \qquad g(s) = s\, f(s) - F(0)$$

This result appears in Schaum p 162 item 32.7.

I refer in the doc title to this term -F(0) as one of "those constant terms".

If you were to go on to do a second derivative, you would get more of "those constant terms". Note here that we can write G(0) = F'(0) :

$$H(x) = \partial_x G(x) = \partial_x^2 F(x)$$

$$H(x) \equiv \partial_x G(x) \quad \leftrightarrow \quad h(s) = s\, g(s) - G(0)$$

$$H(x) \equiv \partial_x^2 F(x) \quad \leftrightarrow \quad h(s) = s\,[s\, f(s) - F(0)\,] - F'(0)$$

So the second derivative has two of "those extra terms",

$$H(x) \equiv \partial_x^2 F(x) \quad \leftrightarrow \quad h(s) = s^2 f(s) - s\, F(0) - F'(0)$$

and this result appears in Schaum p 162 item 32.8.

When you use Laplace Transforms to solve ODE's with constant coeffs (see separate doc), these "extra terms" are of course very important. They are the t = 0 boundary conditions!

Fact: With Laplace Transforms, you cannot just make the simple association

$$G(x) \equiv \partial_x F(x) \quad \leftrightarrow \quad g(s) = s\, f(s)$$

This is only true if it happens that F(0) = 0 !

2. Tricky Question

In my doc "questions Laplace transform.doc" I show that a valid Laplace transform f(s) must satisfy the requirement that, for large s, |f(s)| must decay faster than $1/|s|^a$ with a > 0. I call this the decay requirement on f(s). If we look at the above result g(s) = s f(s) - F(0), we can see that the second term along F(0) does *not* meet the decay requirement, so it seems that this g(s) violates our rule. What is going on here?

The explanation is given in the same doc just noted. Consider first (we assume F(x) is such that the integral converges),

$$f(s) = \int_0^\infty dx\, e^{-xs}\, F(x)\ .$$

As the doc shows, you can use the wonderful Stak Vol II Appendix B method to determine the large s behavior of f(s) as defined by this integral. Here is what you find:

$$f(s) \approx F(0)\, e^{-s0} / [\, s\ 1] \sim F(0)/s + \text{order}(1/s^2)$$

and this arises basically from the lower endpoint region. This meets the decay requirement, so f(s) and F(x) are a valid Laplace Transform pair. (The doc shows that, if it happens that F(0) = 0, the decay requirement is met even better.) So we have just shown that

$$s\,f(s) \;\rightarrow\; F(0) + \text{order}(1/s) \qquad \text{as} \qquad s\rightarrow\infty$$

Obviously then the function h(s) ≡ sf(s) does NOT meet our decay requirement since it → constant. But the function [sf(s) - F(0)] *does* meet the requirement because it behaves as 1/s or better.

That is why the functions { $\partial_{\mathbf{x}}F$, sf(s) - F(0) } form a valid Laplace Transform pair, whereas the pair of functions { $\partial_{\mathbf{x}}F$, sf(s) } do NOT form a valid Laplace Transform pair.

As a related issue, what is the "ILT integral" of a constant K?

$$\text{ILT}(K,x) = (1/2\pi i)\int ds\, e^{\mathbf{xs}}\, K = (1/2\pi i)\, K \int ds\, e^{\mathbf{xs}}\,.$$

If the contour is taken as just running up the imaginary axis, we have

$$\int ds\, e^{\mathbf{xs}} = 2\pi i\, \delta(x)$$

and then we find that

$$\text{ILT}(K,x) = K\,\delta(x)$$

which is a distribution. From this we conclude that

Fact: The ILT integrals of f(s) and f(s) + K differ by K δ(x).

Earlier today I was claiming that ILT(K,x) = 0 because you could close the contour to the left and there are no singularities there. But if x = 0, you cannot do this because there is no decay on the left side great circle.

3. What happens if you just run ∂_x through the ILT integral?

Consider the inversion formula

$$F(x) = (1/2\pi i)\int ds\, e^{\mathbf{xs}}\, f(s)$$

where the contour integral is a vertical line in the s-plane to the right of all singularities of f(s). It seems that we can do this

$$\partial_{\mathbf{x}}F(x) = (1/2\pi i)\int ds\, \partial_{\mathbf{x}}\, e^{\mathbf{xs}}\, f(s) = (1/2\pi i)\int ds\, s\, e^{\mathbf{xs}}\, f(s) = (1/2\pi i)\int ds\, e^{\mathbf{xs}}\, [sf(s)] \qquad (*)$$

and then we conclude that

$$G(x) = \partial_x F(x) \qquad \leftrightarrow \qquad g(s) = sf(s)$$

But what happened to our extra term??? We are supposed to get g(s) = s f(s) - F(0) !!

Resolution: The fact that $\partial_x F(x) = (1/2\pi i) \int ds\, e^{xs}\, [sf(s)]$ is probably just fine, and the integral up the vertical axis probably converges just fine. That is to say, if f(s) is the valid LT of F(x), then the integral shown here is a valid integral representation for the function $\partial_x F(x)$. But since $sf(s) \to F(0)$ for large s, we cannot close the contour to the left. But this then causes our Laplace Transform verification proof to fail (see doc) and we then conclude that { $\partial_x F(x)$, sf(s) } do not form a valid Laplace Transform pair. On the other hand, { $\partial_x F(x)$, sf(s) - F(0) } is a valid pair because for sf(s) - F(0) we *can* close the contour to the left, and then the transform verification works. [see doc for that verification proof]

The upshot is this: we can apply ∂_x to both sides of $F(x) = (1/2\pi i) \int ds\, e^{xs}\, f(s)$ and in doing so we likely get a valid equation $\partial_x F(x) = (1/2\pi i) \int ds\, e^{xs}\, [sf(s)]$ (if the integral converges). But this fact does not imply that [sf(s)] is the Laplace Transform of $\partial_x F(x)$!

Earlier, I thought that maybe the problem was an order-interchange issue with two limits, one being the limit which defines a derivative ∂_x and the other being $s \to \pm i\infty$. I no longer thing order interchange is relevant, but I keep my earlier notes in Appendix A below.

4. How is this reflected in the world of Fourier Transforms?

Here is the Fourier Transform and its inverse: (from my doc on same, but changed letters)

$$f(\omega) = \int_{-\infty}^{\infty} dx\, F(x)\, e^{-i\omega x} \qquad \text{projection = transform} \qquad (1.1)$$

$$F(x) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, f(\omega)\, e^{+i\omega x}\,. \qquad \text{expansion = inverse transform} \qquad (1.2)$$

Now suppose we try the same path as above. We have $G(x) \equiv \partial_x F(x) = F'(x)$ so then

$$g(\omega) = \int_{-\infty}^{\infty} dx\, G(x)\, e^{-i\omega x} = \int_{-\infty}^{\infty} dx\, \partial_x F(x)\, e^{-i\omega x}$$

and we continue with the parts integration

$$= [\, F(x)\, e^{-i\omega x}]\,{}^{\infty}_{-\infty} - \int_{-\infty}^{\infty} dx\, F(x)\, \partial_x\, e^{-i\omega x}$$

$$= F(\infty)\, e^{-i\omega\infty} \; - \; F(-\infty)\, e^{i\omega\infty} \; - \; \int_{-\infty}^{\infty} dx\, F(x)\, (-i\omega)\, e^{-i\omega x}$$

Now we must assume a stricter condition on F(x) than in the Laplace. There F(x) could be power divergent, but here (real axis Fourier) as have to assume F(x) → 0 as x goes to either +∞ or -∞. Then we may continue

$$= 0 - \; 0 - (-i\omega) \int_{-\infty}^{\infty} dx\, F(x)\, e^{-i\omega x} \; = i\omega\, f(\omega)$$

We then end up with the famous rule,

$$G(x) \equiv \partial_{\mathbf{x}} F(x) \qquad \leftrightarrow \qquad g(\omega) = i\omega\, f(\omega)$$

and there are none of "those extra terms".

The big difference is that we have to assume F(±∞) = 0 for our underlying function F(x).

So here is a conclusion:

Fact: Let F(x) have a Fourier transform f(ω).
Let G(x) = $\partial_{\mathbf{x}}$F(x).
Then: g(ω) = iω f(ω) *provided that* F(±∞) = 0.

Now what about this alternative derivation:

$$F(x) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, f(\omega)\, e^{+i\omega x}$$

$$\partial_{\mathbf{x}} F(x) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, f(\omega)\, \partial_{\mathbf{x}} e^{+i\omega x} \; = \; (1/2\pi) \int_{-\infty}^{\infty} d\omega\, [f(\omega)\, i\omega]\, e^{+i\omega x}$$

This seems to say that

$$G(x) = \partial_{\mathbf{x}} F(x) \qquad \leftrightarrow \qquad g(\omega) = i\omega\, f(\omega)$$

without any conditions! But the FT is symmetric in both directions, and for it to be valid and for us to have a valid transform pair { G(x), iωf(ω) } we would need to have ωf(ω) → 0 for large ω. If this is not true, then I think $\int_{-\infty}^{\infty} d\omega\, [f(\omega)\, i\omega]\, e^{+i\omega x}$ does not converge, and then it cannot be a reasonable expression for $\partial_{\mathbf{x}}$F(x) .

Now Stakgold does show how the *generalized* Fourier Transform works which then allows F(x) to have power growth and you still obtain f(ω). You shift the ω contour vertically to make this go. But then in the recovery integral the variable ω takes only complex values.

Appendix A: An example of order interchange failure?

These notes I think are irrelevant to the problem at hand but I keep them anyway.

Maybe I can answer this question in terms of my "order interchange for limits" doc located in math top level. We have two limits here which I will try to write out:

$$\partial_x F(x) = \lim_{dx\to 0} [\, F(x+dx)-F(x)]/dx$$

$$F(x) = (1/2\pi i) \int ds\, e^{xs} f(s) = \lim_{a\to\infty} \int_{-ia}^{ia} ds\, e^{xs} f(s)$$

Then "moving ∂_x through the integral" is really this in more detail :

$$F(x) = \lim_{a\to\infty} \int_{-ia}^{ia} ds\, e^{xs} f(s)$$

$$\partial_x F(x) = \lim_{dx\to 0} [\, F(x+dx)-F(x)]/dx$$

$$= \lim_{dx\to 0} [\lim_{a\to\infty} (1/2\pi i) \int_{-ia}^{ia} ds\, e^{(x+dx)s} f(s) - \lim_{a\to\infty} (1/2\pi i) \int_{-ia}^{ia} ds\, e^{xs} f(s)] /dx$$

$$= (1/2\pi i) \lim_{dx\to 0} [\lim_{a\to\infty} \int_{-ia}^{ia} ds\, e^{(x+dx)s} f(s) - \lim_{a\to\infty} \int_{-ia}^{ia} ds\, e^{xs} f(s)] /dx$$

Now we are allowed to say

$$\lim_{a\to\infty} q(a) - \lim_{a\to\infty} r(a) = \lim_{a\to\infty} [\, q(a) - r(a) \,] = q(\infty) - r(\infty)$$

because I think a limit and a finite sum always commute. So we then continue the above

$$= (1/2\pi i) \lim_{dx\to 0} \{ \lim_{a\to\infty} [\int_{-ia}^{ia} ds\, e^{(x+dx)s} f(s) - \int_{-ia}^{ia} ds\, e^{xs} f(s)] /dx \}$$

Now suppose you were allowed to interchange the limit orderings. Then we would get

$$= (1/2\pi i) \lim_{a\to\infty} \lim_{dx\to 0} [\int_{-ia}^{ia} ds\, e^{(x+dx)s} f(s) - \int_{-ia}^{ia} ds\, e^{xs} f(s)] /dx$$

Now since integral to ia is finite, we can move the dx limit inside to get

$$= (1/2\pi i) \lim_{a\to\infty} [\int_{-ia}^{ia} ds \lim_{dx\to 0} \{ e^{(x+dx)s} - e^{xs} \}/dx \} f(s)$$

$= (1/2\pi i) \lim_{a\to\infty} [\int_{-ia}^{ia} ds\ \partial_x e^{xs} f(s) \quad = \quad = (1/2\pi i) \lim_{a\to\infty} [\int_{-ia}^{ia} ds\ s\, e^{xs} f(s)$

$= (1/2\pi i) \int_{-i\infty}^{i\infty} ds\ e^{xs} [sf(s)]$

So this shows how, if order interchange of limits is allowed, we end up with the claim made in this section, where F(0) is missing.

Comments: I think the order interchange shown above really is valid. I got all sidetracked on uniform convergence and Moore's Theorem and made some new notes on that subject stored now in my little order interchange math folder. Perhaps these notes apply if the integral obtained by ∂_x applied to F(x) does not converge, but I doubt even that. I think it was just a red herring, but one worth checking out. The reason the direct application of ∂_x "does not work" is explained in item 3 above, which in turn is based on the results of item 2.