Using the Laplace Transform on a PDE PhL 4.11.11

Overview: After first showing how to do a Laplace Transform on a 2nd order ODE then a PDE with constant coefficients, I examine a simple application of the method to a heated half rod. I am able to fully solve this problem for u(x,t) in x-space, and am then able to obtain the correct U(s,t) in s-space. However, I show that the problem cannot be solved directly in s-space because a piece of information is missing. I can only obtain that piece of missing information if I solve the problem in x-space to find it. See Section 8 for conclusions.

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1. Laplace in a 2nd order ODE with constant coefficients

In doc "Laplace Transforms to solve simple ODE's.doc" we show that one can take a second order ODE like this

$$[\ A\partial_x^2+B\partial_x+C\ ]\ f(x) = g(x) \qquad f(0) = a \qquad f'(0) = b$$

and transform it into s-space to get

$$A[s^2F(s)-sf(0)-f'(0)] + B\ [sF(s)-f(0)]\ + C\ F(s) = G(s)$$

where of course *both f(0) and f'(0) must be known* in order to solve the s-space problem which is merely an algebraic problem.

2. Laplace in a 2nd order PDE with constant coefficients

Now suppose we have a PDE with two variables of this form

$$[\ A_x\partial_x^2+A_t\partial_t^2 + A_{xt}\partial_x\partial_t + B_x\partial_x + B_t\partial_t + C\ ]\ f(x,t) = g(x,t)$$

again all with constant coefficients. Suppose we transform x over to s. Rewrite as

$$[\ A_x\partial_x^2 + (A_{xt}\partial_t + B_x)\partial_x +(A_t\partial_t^2 + B_t\partial_t + C)\ ]\ f(x,t) = g(x,t)$$

Then we shall get

$$\{ A_x[s^2F(s,t)-sf(0,t)-f'(0,t)] + (A_{xt}\partial_t + B_x)[sF(s,t)-f(0,t)] + (A_t\partial_t^2 + B_t\partial_t + C)F(s,t) \} = G(s,t)$$

This can be rewritten with some effort in the following manner: (there are always 10 terms)

$$A_t\partial_t^2 F(s,t) + A_{xt}\partial_t[sF(s,t)] + B_t\partial_t F(s,t)$$

$$+ A_x[s^2F(s,t)-sf(0,t)-f'(0,t)] + B_x[sF(s,t)-f(0,t)] + CF(s,t) = G(s,t)$$

or

$$A_t\partial_t^2 F(s,t) + (A_{xt}s + B_t)\partial_t F(s,t) + A_x s^2F(s,t) + B_x sF(s,t) + CF(s,t)$$

$$+ A_x[-sf(0,t)-f'(0,t)] + B_x[-f(0,t)] = G(s,t)$$

or

$$A_t\partial_t^2 F(s,t) + (A_{xt}s + B_t)\partial_t F(s,t) + (A_x s^2 + B_x s + C)F(s,t)$$

$$= G(s,t) - A_x[-sf(0,t)-f'(0,t)] - B_x[-f(0,t)]$$

and we end up with an ODE in variable t having constant coefficients as shown, but the driving function now has extra stuff as shown. This equation can certainly be solve provided we have knowledge of these two functions

$$f(0,t) = a(t) \qquad f'(0,t) = b(t)$$

3. Special case: the heat PDE

Let's look now at a special case where operator L is the 1D heat conduction operator. Then we have

$$[\partial_t - \partial_x^2]\, u(x,t) = q(x,t)$$

and further, let's assume there are no distributed sources, so we have

$$[\partial_t - \partial_x^2]\, u(x,t) = 0$$

Applying our method of the previous section we have only $B_t = 1$ and $A_x = -1$ so we get

$$(\partial_t - s^2)U(s,t) = -su(0,t) - u'(0,t)$$

This is a *first* order linear ODE in t, but in order to have a solution, we need to know these *two* functions

$$u(0,t) = a(t) \qquad u'(0,t) = b(t) \qquad (u' = \partial_x u)$$

4. Constant heat on the half-rod problem: a problem with the s-space formulation.

This is the $\omega \to 0$ limit of the problem of Stakgold Exercise 7.16 on page 241 of volume 2. It is exactly the above problem where we start off with a "frozen rod" having u(x,t=0) = 0 and at t=0 we apply constant heat A at the left end. So here are the things we know:

$u(x,0) = 0 \qquad x \geq 0$
$u(0,t) = A \qquad t \geq 0$

Thus we can write our ODE above as

$$(\partial_t - s^2)U(s,t) = -sA - u'(0,t) \qquad t \geq 0 \qquad U(s,0) = 0$$

We can even conclude that U(s,0) = LT [u(x,0)] = LT[0] = 0. BUT, we don't know u'(0,t), so it certainly seems that the problem cannot be solved in this manner!

5. Solve the rod problem in x-space.

First of all, consider our PDE

$$[\partial_t - \partial_x^2]\, u(x,t) = 0$$

What can we say about solutions to this equation in general? How many linearly independent solutions are there? What is the dimension of the nullspace? I don't really know much about PDE theory so I have no definitive answer.

Is this a candidate solution?

$$u(x,t) = \int_0^\infty d\rho\, e^{-\rho t} \sin(x\rho^{1/2})/\rho$$

$$\partial_t u(x,t) = -\int_0^\infty d\rho\, e^{-\rho t} \sin(x\rho^{1/2})$$

$$\partial_x^2 u(x,t) = -\int_0^\infty d\rho\, e^{-\rho t} \sin(x\rho^{1/2})$$

So yes, this object is in fact a candidate solution. But according to my ILT doc and Maple, we have

$$\int_0^\infty d\rho\, e^{-\rho t} \sin(x\rho^{1/2})/\rho = 2\int_0^\infty dy\, e^{-y^2 t} \sin(xy)/y = \pi\ \mathrm{erf}(x/[2\sqrt{t}\,])$$

so this integral is a solution we already know about. So let's just press forward.

We shall now *verify* what we already know is the correct x-space solution to our problem, namely,

$u(x,t) = A[1 - \mathrm{erf}(x/(2\sqrt{t}\,))]$ // this IS the solution in x-space!

In order to prove this is an answer, we first show that it satisfies [$\partial_t - \partial_x^2$] u(x,t) = 0 :

```
> restart;
> f := erf(x/(2*sqrt(t)));
```

$$f := \operatorname{erf}\left(\frac{1}{2}\frac{x}{\sqrt{t}}\right)$$

```
> diff(f,t)-diff(f,x,x);
```

$$0$$

Then we have to show that it satisfies the boundary conditions

$u(x,0) = A[1 - \text{erf}(\infty)] = A[1 - 1] = 0$

$u(0,t) = A[1 - \text{erf}(0] = A[1 - 0] = A$

And if we want we can compute

$u'(x,t) = \partial_x u(x,t) = -A \exp(-x^2/4t)/\sqrt{\pi t}$

since

```
diff(f,x);
```

$$\frac{e^{\left(-\frac{1}{4}\frac{x^2}{t}\right)}}{\sqrt{\pi}\sqrt{t}}$$

and therefore we know that

$u'(0,t) = -A/\sqrt{\pi t}$.

But we don't know this a priori, so how could we solve this problem doing Laplace in x? That is the Big Question of the moment. We see that the problem is easily solvable in x space, why is there so much trouble in s-space? We shall return to this question in a moment.

6. Solve the rod problem in s-space with this extra knowledge of u'(0,t) .

If we magically had this extra information, our s-space problem would be this:

$(\partial_t - s^2)U(s,t) = -sA + A/\sqrt{\pi t} \qquad t \geq 0 \qquad U(s,0) = 0$

We can set A = 1 and reinstall it later, so we have

$(\partial_t - s^2)U(s,t) = -s + 1/\sqrt{\pi t} \qquad t \geq 0 \qquad U(s,0) = 0$

This seems to be a well-defined first order ODE in variable t with an initial condition. I know that the homogeneous solution is $\exp(s^2 t)$ and then we need to find a particular solution. Let's cheat! We know this is the answer:

$$u(x,t) = 1 - \mathrm{erf}(x/(2\sqrt{t})) = 1 - \mathrm{erf}(\alpha x) = \mathrm{erfc}(\alpha x) \qquad \alpha = 1/(2\sqrt{t})$$

We ask Maple to find U(s,t) that goes with this solution:

```
with(inttrans):
alpha := 1/(2*sqrt(t));
```

$$\alpha := \frac{1}{2}\frac{1}{\sqrt{t}}$$

```
laplace(1-erf(alpha*x),x,s);
```

$$\frac{1}{s} - \frac{e^{(s^2 t)}\,\mathrm{erfc}(s\sqrt{t})}{s}$$

So this gives us a "candidate" particular solution that we can try out:

$$U(s,t) = [\,1 - \exp(s^2 t)\,\mathrm{erfc}(s\sqrt{t})]/s$$

Maple then tells us first that

```
diff(f,t) - s^2*f;
```

$$-s\,e^{(s^2 t)}\,\mathrm{erfc}(s\sqrt{t}) + \frac{e^{(s^2 t)}\,e^{(-s^2 t)}}{\sqrt{\pi}\sqrt{t}} - s^2\left(\frac{1}{s} - \frac{e^{(s^2 t)}\,\mathrm{erfc}(s\sqrt{t})}{s}\right)$$

```
simplify(%);
```

$$-\frac{-1 + s\sqrt{\pi}\sqrt{t}}{\sqrt{\pi}\sqrt{t}}$$

```
expand(%);
```

$$\frac{1}{\sqrt{\pi}\sqrt{t}} - s$$

and thus we have shown that this U(s,t) does in fact satisfy our ODE $(\partial_t - s^2)U(s,t) = -s + 1/\sqrt{\pi t}$!!

Now this solution is just a "particular solution" and we have to then find our full solution this way, by adding an unknown amount of the homo solution,

$$U(s,t) = [\,1 - \exp(s^2 t)\,\mathrm{erfc}(s\sqrt{t})]/s + \beta\exp(s^2 t)$$

Then we apply our BC to get

$$0 = U(s,0) = [\,1 - \exp(s^2 0)\,\mathrm{erfc}(s0)]/s + \beta\exp(s^2 0) = \beta \qquad \Rightarrow\ \beta = 0$$

Then our final solution *is* the one given above

$$U(s,t) = [\, 1 - \exp(s^2 t)\, \mathrm{erfc}(s\sqrt{t}\,)]/s$$

In passing, we use our fact that

$$\mathrm{erfc}\, z \sim \frac{e^{-z^2}}{\sqrt{\pi} z}$$

to show that for large s we have

$$U(s,t) \approx [\, 1 - \exp(s^2 t) \exp(-s^2 t)\, (\sqrt{\pi}\ s\sqrt{t}\ \ )^{-1}]/s = [\, 1 - 1/(\sqrt{\pi}\ s\sqrt{t}\ \ )]/s \sim A/s + B/s^2$$

and this meets our (recently realized) condition on U(s,t) (as we know it has to, since it came from a convergent Laplace transform integral).

7. Take the s-space solution back to x-space

Well there is nothing to do here really. We showed in the previous section that we have this "pair":

$$1 - \mathrm{erf}(x/(2\sqrt{t}\,)) \quad \leftrightarrow \quad U(s,t) = [\, 1 - \exp(s^2 t)\, \mathrm{erfc}(s\sqrt{t}\,)]/s$$

so if we could have solved our s-space problem to get this U(s,t), then we would have generated the known solution to our problem. So I have been able to show that, IF we had that extra data

$$u'(0,t) = -A/\sqrt{\pi t}$$

then our s-space problem would be

$$(\partial_t - s^2)U(s,t) = -s + 1/\sqrt{\pi t} \qquad t \geq 0 \qquad U(s,0) = 0$$

and we could perhaps have found U(s,t) as shown above, and in this way we could have solved the problem. BUT,

(1) we don't a priori know that $u'(0,t) = -A/\sqrt{\pi t}$
(2) this is a very clumsy method of solving this problem !! Much easier to solve in x-space.

8. Comments on the failure of the Laplace method for this problem.

We noted above in Section 4 how our s-space problem for the heated rod becomes this

$$(\partial_t - s^2)U(s,t) = -sA - u'(0,t) \qquad t \geq 0 \qquad U(s,0) = 0$$

but the only boundary conditions we had for the problem in x-space were these on u(x,t) :

$u(x,0) = 0 \qquad x \geq 0$
$u(0,t) = A \qquad t \geq 0$

I don't see any way to conclude that $u'(0,t) = -A/\sqrt{\pi t}$ other than solving the entire problem in x-space, then finding this fact to be true. So I claim that the Laplace Transform method "fails" in its application to our problem here. Now it may happen that there is a way to do this, perhaps based on the method of characteristics of a PDE, but this is not something I know about right now, so for now I close the case!

9. Solve the rod problem in s-space assuming wrongly that u'(0,t) = 0.

Again with A = 1, our s-space system becomes

$$(\partial_t - s^2)U(s,t) = -s \qquad t \geq 0 \qquad U(s,0) = 0$$

The unique solution is found to be $U(s,t) = -(1/s)(\exp(s^2t)-1)$, which is not a valid f(s) Laplace Transform! That is to say, we have $U(s,t) \rightarrow -\exp(ts^2)/s$. If we blindly ignore this illegality, we can apply the ILT to our two terms (pretending thereby to come up with u(x,t)) and we find (see "ILT's doc" for the second integral)

$$U(s,t) = 1/s - \exp(s^2t)/s$$

$$u(x,t) = 1 - (1/2)\,[\,1 + \mathrm{erf}(x/[2\sqrt{t}\,])] = (1/2)\,[\,1 - \mathrm{erf}(x/[2\sqrt{t}\,])\,]$$

For some reason I don't understand, following this doubly erroneous path, we come up with exactly half of the correct answer! First of all, we have "the wrong problem" because we have thrown out the extra term $+1/\sqrt{\pi t}$ which should be on the RHS of our PDE. Second of all, after doing this, the f(s) we come up with violates the condition of the Laplace Transform method and so u(x,t) and U(s,t) are not really a Laplace Transform pair. But despite these two *major* issues, we come up with half the right answer. This same thing happens in the Stak problem where $\omega > 0$. I spent a lot of time looking for the missing factor of 2 before I realized the solution was invalid. Perhaps there is some sort of Modified Laplace Transform concept where you are allowed to violate the f(s) condition and in that framework maybe you could explain this seeming coincidence. Maybe Stak will talk about this later on.

10. General Comments about Appropriate Transforms(added later)

Some basic facts of the Laplace Transform are these:

$$L[\partial_x f(x), s] = s\ L[f(x), s] - f(0)$$

$$L[\partial_x^2 f(x), s] = s^2\ L[f(x), s] - s\ f(0) - f'(0)$$

If you want to use the Laplace transform to "transform out" a PDE term like $\partial_x f(x)$ or $\partial_x^2 f(x)$, you must have on hand the boundary conditions shown. We can compare the above ∂_x^2 "rule" to the corresponding rules for Fourier Sine and Fourier Cosine transforms.

$$F_c [\partial_x^2 f(x)] = -k^2 F_c[f(x)] - f'(0)$$

$$F_s [\partial_x^2 f(x)] = -k^2 F_s [f(x)] + k f(0)$$

In all these cases, we can imagine f(x) = u(x,t) where there are "other variables" involved and we are just transforming the x variable. Something important that we learn here is this: if you ONLY have f(0) as your BC, then your only option among the above three transforms is the F_s one (Fourier Sine). And, if you ONLY know f'(0), then your only option is the F_c one (Fourier Cosine). If you know BOTH boundary conditions, then you can use any of the three options (always assuming of course that the corresponding projection converges). So in our heat example above, we must use the Fourier Sine transform and I do that in Section (e) of the Stak Exercise 7.16 doc.