The Mellin Transform PhL 8.10.11

The Mellin Sine Transform

Consider this ODE on the interval (0,a)

$-(ru')' = \lambda\,(1/r)u \qquad s(r) = 1/r \qquad Lu = -(ru')'$

$-(ru')' - (\lambda/r)u = 0 \qquad -(ru')' - (\nu^2/r)u = 0$

This is the limit of a Bessel equation when k→0,

$-(xu')' + \nu^2 u/x - k^2 xu = 0 \qquad Z_\nu(kx) \qquad (*)$
$-(xu')' - \nu^2 u/x - k^2 xu = 0 \qquad Z_{\pm i\nu}(kx) \qquad (*)$
$-(xu')' - \nu^2 u/x + k^2 xu = 0 \qquad K_{\pm i\nu}(kx) \qquad (*)$

where in the last form (which will match ours for k=0) we see the real-valued KL function solutions.

Our limiting situation does not give of course a Bessel function. To avoid the need for ugly $\sqrt{\lambda}$ factors in exponents and elsewhere, let's use $\lambda = \nu^2$ as it appears in this last Bessel equation. The solutions are then of the form $r^{\pm i\nu}$ as we now confirm

$u = r^{\pm i\nu} \qquad u' = (\pm i)\nu\, r^{\pm i\nu - 1} \qquad (ru') = (\pm i)\nu\, r^{\pm i\nu}$

$(ru')' = (\pm i)\nu\,(\pm i)\nu\, r^{\pm i\nu - 1} = -\nu^2 r^{\pm i\nu}/r = -\nu^2 u/r \qquad \Rightarrow \qquad -(ru')' = +\nu^2 u/r \qquad \text{QED}$

However, if ν = 0, our solutions are $r^{\pm i0} = 1$, and then there must be a second independent solution which is ln(r), so in this case ν = 0 we get the famous A + B ln(r) solution.

A linear combination which vanishes at r=a is given by

$u_\lambda(r) = (1/2)[\,(r/a)^{i\nu} - (r/a)^{-i\nu}\,] = \sin[\,\nu \ln(r/a)\,] \qquad \nu = \sqrt{\lambda}$

and again if λ=ν=0 we would use $u_0(r) = \ln(r/a)$ to replace the above, which is in agreement with the limit of the sin function for small ν, ignoring the constant ν.

Probably r=0 is a limit-point situation [yes, p 316] and the limit of $u_\lambda(r=0)$ only exists in some distributional sense. But we know that our solution has finite s norm at r = 0 with our s(r) = 1/r because

$\{\int dr\,(1/r)\sin[\,b \ln(r/a)\,]\}|^{r=0} = \{\cos(b \ln(r/a))/b\}|^{r=0}$

which is no doubt 0 as a distribution, and in any event is bounded above and below by 1 and -1, so one cannot say that this endpoint diverges.

The functions $u_\lambda(r) = \sin[\sqrt{\lambda}\ \ln(r/a)]$ form a complete set on (0,a) as λ sweeps the + real axis. We know this because these are the EF's of a SL problem. It turns out that we don't have to worry about the λ=0 case in this transform, but I am not sure why. This is at the very end of the "cut" in the λ plane for the g integral. I will just ignore this issue for now.

We can establish normalization by requiring [remember that s(r) = 1/r]

$$u_\lambda(r) = C \sin[\sqrt{\lambda}\ \ln(r/a)]$$

$$\int_0^a dr(1/r)\, C^2 \sin[\sqrt{\lambda}\ \ln(r/a)] \sin[\sqrt{\lambda'}\ \ln(r/a)] = \delta(\lambda-\lambda')$$

Change variables to x = r/a to get

$$C^2 \int_0^1 dx(1/x) \sin[\sqrt{\lambda}\ \ln(x)] \sin[\sqrt{\lambda'}\ \ln(x)] = \delta(\lambda-\lambda')$$

Let $\sqrt{\lambda} = k$ (called ν above), change to y = lnx to get [dy = dx/x]

$$C^2 \int_{-\infty}^0 dy \sin(ky)\sin(k'y) = \delta(\lambda-\lambda')$$

$$C^2 \int_0^\infty dy \sin(ky)\sin(k'y) = \delta(\lambda-\lambda') \qquad (*)$$

But we know from our Fourier Sine Transform work that

$$\int_0^\infty dy \sin(ky)\sin(k'y) = (\pi/2)\delta(k-k')$$

And we also know that

$$\delta(\lambda-\lambda') = \delta(k^2-k'^2) = (1/2|k|)\delta(k-k') = (2\sqrt{\lambda})^{-1}\,\delta(k-k') \qquad \text{// k and k' both > 0}$$

so that $\delta(k-k') = (2\sqrt{\lambda})\,\delta(\lambda-\lambda')$ and putting this in (*) above we have

$$C^2\ (\pi/2)\delta(k-k') = \delta(\lambda-\lambda') = (2\sqrt{\lambda})^{-1}\,\delta(k-k')$$

$$C^2\,\pi = (1/\sqrt{\lambda}) \qquad C^2 = (\pi\sqrt{\lambda})^{-1} \quad C = (\pi\sqrt{\lambda})^{-1/2}$$

Thus, our properly normalized eigenfunctions are

$$u_\lambda(r) = (\pi\sqrt{\lambda})^{-1/2} \sin[\sqrt{\lambda}\ \ln(r/a)] \qquad ****$$

We can then call upon our Big Result from "review of generic singular...doc" which says

$$- (1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_{\lambda n}\, \varphi_{\lambda n}{}^*(x)\varphi_{\lambda n}(\xi) + \int d\lambda' \varphi_{\lambda}(x)^*\, \varphi_{\lambda'}(\xi) = \delta(x-\xi)/s(x)$$

and since we have only a continuous spectrum

$$- (1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \int d\lambda' \varphi_{\lambda'}(x)^*\, \varphi_{\lambda'}(\xi) = \delta(x-\xi)/s(x)$$

So we then find that

$$\delta(r-r')r = \int d\lambda'\, (\pi\sqrt{\lambda'}\,)^{-1} \sin[\, \sqrt{\lambda'}\ \ln(r/a)\,] \sin[\, \sqrt{\lambda'}\ \ln(r'/a)\,]$$

$$\delta(r-r')r = (1/\pi) \int d\lambda'\, (1/\sqrt{\lambda'}\,) \sin[\, \sqrt{\lambda'}\ \ln(r/a)\,] \sin[\, \sqrt{\lambda'}\ \ln(r'/a)\,]$$

If we now change to $\sqrt{\lambda'} = k'$ then $dk' = (1/2)\, \lambda'^{-1/2} d\lambda' = (2\sqrt{\lambda'}\,)^{-1} d\lambda'$ then

$d\lambda' = (2\sqrt{\lambda'}\,)\, dk'$ so that $d\lambda'\, (1/\sqrt{\lambda'}\,) = 2\, dk'$ and we find

$$\delta(r-r')r = (2/\pi) \int_0^\infty dk' \sin[\, k' \ln(r/a)\,] \sin[\, k' \ln(r'/a)\,]$$

and this agrees with Stak 4.134 on page 316 vol I. I will rewrite this using k

$$\delta(r-r')r = (2/\pi) \int_0^\infty dk \sin[\, k \ln(r/a)\,] \sin[\, k \ln(r'/a)\,] \qquad ****$$

Then suppose we try this for a transform

$$F(k) = \int_0^a dr \sin[\, k \ln(r/a)\,]\, f(r) \qquad ****$$

$$f(r) = (2/\pi) \int_0^\infty dk \sin[\, k \ln(r/a)\,]\, F(k)$$

Stak calls this the Mellin Sine Transform "for want of a better term". And the conjugate to r is here called k, but we know it is really $k = \sqrt{\lambda}$ where λ is the true continuous eigenvalue.

On page 316 Stak asks the reader to obtain the completeness statement using the other method of first finding the Green's Function on (0,a) and then stuffing that into the g integral formula. You see that his Green's has a λ cut, and this creates a discontinuity which then gives the formula. Notice that his Green's function vanishes at r = 0 cleanly if you set λ to $\lambda + i\varepsilon$, take the limit $r \to 0$, then set $\varepsilon = 0$. This is his method of limited absorption. I never did this Exercise 4.28 in my raw Chapter 4 notes. Stak's method shows that there is no "special case" to worry about at $\lambda = 0$

If you replace the BC that u(a) = 0 with u'(a) = 0, you get the **Mellin Cosine Transform** as commented on in Exercise 4.29.

Summary of the Mellin Sine Transform

$u_\lambda(r) = (\pi\sqrt{\lambda})^{-1/2} \sin[\sqrt{\lambda}\ \ln(r/a)]$ $\quad k = \sqrt{\lambda}$ // eigenfunctions

$\int_0^\infty dr \sin(kr)\sin(k'r) = (\pi/2)\delta(k-k')$ // orthogonality

$\int_0^a dr(1/r)\, u_\lambda(r)\, u_{\lambda'}(r) = \delta(\lambda-\lambda')$ $\quad s(r) = 1/r$

$(2/\pi) \int_0^\infty dk \sin[k \ln(r/a)] \sin[k \ln(r'/a)] = \delta(r-r')r$ // completeness

$\int_0^\infty d\lambda\, \varphi_\lambda(r)^* \varphi_\lambda(r') = \delta(r-r')\, r$

$F(k) = \int_0^a dr \sin[k \ln(r/a)]\, f(r)$ // projection

$f(r) = (2/\pi) \int_0^\infty dk \sin[k \ln(r/a)]\, F(k)$ // expansion