Transforms (integral and series) PhL 4.25.10
updated 12.8.15

To be added later: Laplace transforms, Mellin transforms

I guess the time has come to start gathering these things up in one place so I don't have to go foraging each time I want a transform. For each I will write out the expansion/projection pairs, along with completeness/orthogonality pairs, and a reference for derivation. Every Sturm-Liouville problem of course implies such a (1D) transform, so transforms are associated with ODE's and some of their solutions, as a general rule. I include comments on multi-dimensional transforms related to groups.

For me, the word "transform" implies the pair: "projection and expansion". In some cases the expansion is a series, and in other cases it is an integral. The projection is always an integral. I sometimes refer to the expansion as the "recovery" and to the projection as the "inversion".

Common usage, however, uses the words "series" and "transforms" to describe expansions which are sums and integrals, but I am going to stick to my own language, which agrees with Bateman.

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ET I, II: Bateman Tables of Integral Transforms Vol I,II (vol 4,5 of full series)

These tables only do the *integral* transforms, not the *series* transforms! The conventions used are slightly different from those I use below, but one does not care because one is just trying to look up an integral. I have a doc on this Bateman stuff, but here I just want to show what there is. Volume I contains these tables

Laplace transform

$$\int_0^\infty f(t)\, e^{-pt}\, dt.$$

Fourier sine transform

$$\int_0^\infty f(x) \sin(xy)\, dx.$$

Inverse Laplace transform

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(p)\, e^{pt}\, dp.$$

Fourier cosine transform

$$\int_0^\infty f(x) \cos(xy)\, dx.$$

Mellin transform

$$\int_0^\infty f(x)\, x^{s-1}\, dx.$$

Exponential Fourier transform

$$\int_{-\infty}^\infty f(x)\, e^{-ixy}\, dx.$$

Inverse Mellin transform

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s)\, x^{-s}\, ds.$$

Volume II contains these tables which include Bessel (5 tables) and "other" stuff (5 more tables):

Преобразование Ганкеля ($\mathfrak{H}_\nu$, глава VIII)

$$\int_0^\infty f(x)\, J_\nu(xy)\, (xy)^{1/2}\, dx.$$

Y-преобразование ($\mathfrak{Y}_\nu$, глава IX)

$$\int_0^\infty f(x)\, Y_\nu(xy)\, (xy)^{1/2}\, dx.$$

K-преобразование ($\mathfrak{K}_\nu$, глава X)

$$\int_0^\infty f(x)\, K_\nu(xy)\, (xy)^{1/2}\, dx.$$

H-преобразование (глава XI)

$$\int_0^\infty f(x)\, \mathbf{H}_\nu(xy)\, (xy)^{1/2}\, dx.$$

Преобразование Конторовича – Лебедева (глава XII)

$$\int_0^\infty f(x)\, K_{ix}(y)\, dx.$$

Интегралы дробного порядка в смысле Римана – Лиувилля ($\mathfrak{R}_\mu$, глава XIII)

$$\frac{1}{\Gamma(\mu)} \int_0^y f(x)\,(y-x)^{\mu-1}\,dx.$$

Интегралы дробного порядка в смысле Вейля ($\mathfrak{W}_\mu$, глава XIII)

$$\frac{1}{\Gamma(\mu)} \int_y^\infty f(x)\,(x-y)^{\mu-1}\,dx.$$

Преобразование Стилтьеса ($\mathfrak{S}$, глава XIV)

$$\int_0^\infty \frac{f(x)}{x+y}\,dx.$$

Обобщенное преобразование Стилтьеса ($\mathfrak{S}_\rho$, глава XIV)

$$\int_0^\infty \frac{f(x)}{(x+y)^\rho}\,dx.$$

Преобразование Гильберта (глава XV)

$$\frac{1}{\pi} \int_{-\infty}^\infty \frac{f(x)}{x-y}\,dx.$$

The last thing is called a **Hilbert** Transform, the previous two are **Stieltjes**, and the two before that are called **Fractional Integrals** where an integration endpoint appears also in the integrand. The K_{ix} thing is called **Kontorovich-Lebedev** and appears on page 317 of Stakgold. Before that are just the **Y,K,H** transforms (**H** is a **Struve** function), and the first **J** transform is called the **Hankel** transform and I show this one below. [The Fractional Integrals are involved in a theory called fractional calculus(wiki) wherein you can talk about $\partial^n f(x)$ and $\text{int}^n f(x)$ where n are non-integers. This is like continuing n! using $\Gamma(n+1)$.]

Fourier Bessel Series Transform on interval (0,b) **Jackson p 73-4**

expansion $f(x) = \Sigma_{k=1}^{\infty} A_{k,\nu}\ J_\nu(x_{\nu,k}x/b)$ (3.96)

projection: $A_{k,\nu} = 2/[b^2 J_{\nu+1}(x_{\nu,k})^2] \int_0^b dx\ x\ f(x)\ J_\nu(x_{\nu,k}x/b)$ (3.97)

where $x_{\nu,n}$ for n = 1,2,3... are the zeros of J_ν , meaning that $J_\nu(x_{\nu,n}) = 0$. Parameter ν is called the Bessel function **order** and is a **bystander** parameter for this series transform. Sometimes the norm factor appears in other ways due to the fact that $J_{\nu+1}(x_{\nu,k}) = -J_{\nu-1}(x_{\nu,k}) = -J_\nu'(x_{\nu,k})$.

The ODE is what I call the **scaled Bessel Equation** which I write in various forms in my curvilinear Jackson cylindricals doc. The eigenvalue problem can be written this way (λ is the official "eigenvalue")

$L\,J_\nu(ax) = a^2 s(x)\,J_\nu(ax)$ $\quad L = -\partial_x(x\partial_x) + \nu^2/x$

$L\,J_\nu(\sqrt{\lambda}\,x) = \lambda\ s(x)\,J_\nu(\sqrt{\lambda}\,x)$ $\quad L = -\partial_x(x\partial_x) + \nu^2/x$

$s(x) = x = \text{weight}$ $\quad \sqrt{\lambda} = a = (x_{\nu,k}/b)$ $\quad \lambda = a^2 = (x_{\nu,k}/b)^2$

Notice that the eigenfunctions are not $J_\nu(x)$ in the sense of $P_n^m(z)$ for Legendre. The argument is scaled, so the eigenfunctions are in fact $J_\nu(a_k x)$ so the quantization label k appears as part of the argument.

Sometimes the weight function x is inserted in different ways. For example, you could define $g(x) = x\, f(x)$ and write the transform for g, or you could write $h(x) = \sqrt{x}\, f(x)$, which puts then a $\sqrt{x}$ in each side of the transform. Obviously the constant factor could be distributed some other way as well.

If we insert the projection above into the expansion, we get this completeness result:

$$\delta(x-x')/x \;=\; \Sigma_{k=1}^{\infty} \left[b\, J_{\nu+1}(x_{\nu,k})/\sqrt{2}\,\right]^{-2} J_\nu(x_{\nu,k}x/b)\, J_\nu(x_{\nu,k}x'/b) \;=\; \Sigma_{k=1}^{\infty} \varphi_{\nu,k}(x)\, \varphi_{\nu,k}(x')$$

where we define these normalized eigenfunctions

$$\varphi_{\nu,k}(x) = \sqrt{2}\, J_\nu(x_{\nu,k}x/b)/\left[b\, J_{\nu+1}(x_{\nu,k})\right] \qquad \text{// see Stak p 307 J, sign is arbitrary, b=1}$$

This completeness result fits into "the general form of things" given by Stakgold,

$$-(1/2\pi i) \oint d\lambda\, g(x|\xi;\lambda) \;=\; \delta(x-\xi)/s(x) = \Sigma_n\, \varphi_n(x)\, \bar{\varphi}_n(\xi) \;+\; \int d\kappa\, \varphi_\kappa(x)\, \bar{\varphi}_\kappa(\xi) \qquad \text{// (4.95) Stak}$$

where in our current example the eigenfunctions are real, $s(x) = x$, and there is only a discrete spectrum. Notice that s(x) appears explicitly here with the $\delta(x-\xi)$. You could instead associated $\sqrt{s(x)}$ with each φ_k and redefine them so that completeness makes just the delta function.

Now we can insert the expansion into the projection to get orthogonality

$$\int_0^b dx\, x\, J_\nu(x_{\nu,k}x/b)\, J_\nu(x_{\nu,k'}x/b) \;= \left[b\, J_{\nu+1}(x_{\nu,k})/\sqrt{2}\,\right]^2 \delta_{k,k'}$$

which can also be written as $\int_0^b dx\, x\, \varphi_{\nu,k}(x)\, \varphi_{\nu,k'}(x) = \delta_{k,k'}$. So here then is our final summary:

$$f(x) = \Sigma_{k=1}^{\infty} A_{k,\nu}\; J_\nu(x_{\nu,k}x/b) \qquad \text{// expansion}$$

$$A_{k,\nu} = 2/[b^2 J_{\nu+1}(x_{\nu,k})^2]\; \int_0^b dx\, x\, f(x)\, J_\nu(x_{\nu,k}x/b) \qquad \text{// projection}$$

$$\left[b\, J_{\nu+1}(x_{\nu,k})/\sqrt{2}\,\right]^{-2} \int_0^b dx\, x\, J_\nu(x_{\nu,k}x/b)\, J_\nu(x_{\nu,k'}x/b) \;=\; \delta_{k,k'} \qquad \text{// orthogonality}$$

$$\Sigma_{k=1}^{\infty} \left[b\, J_{\nu+1}(x_{\nu,k})/\sqrt{2}\,\right]^{-2} J_\nu(x_{\nu,k}x/b)\, J_\nu(x_{\nu,k}x'/b) \;=\; \delta(x-x')/x \qquad \text{// completeness}$$

Since $J_\nu(x_{\nu,k}x/b) = 0$ at $x = b$, this expansion is useful for f(x) such that $f(b) = 0$. A typical application is the potential inside a metal cylinder of radius b. Usually x is written as ρ, and perhaps $k_n = x_{\nu,n}/b$. In this language, we could rewrite the above this way (change sum index from k to n, since have another k)

$$f(\rho) = \Sigma_{n=1}^{\infty} A_{n,\nu}\; J_\nu(k_n\rho) \qquad \text{// expansion}$$

$$A_{n,\nu} = 2/[b^2 J_{\nu+1}(bk_n)^2]\; \int_0^b d\rho\, \rho\, f(\rho)\, J_\nu(k_n\rho) \qquad \text{// projection}$$

$$[b\, J_{\nu+1}(bk_n)/\sqrt{2}\,]^{-2} \int_0^b d\rho\, \rho\, J_\nu(k_n\rho)\, J_\nu(k_{n'}\rho) = \delta_{n,n'} \qquad \text{// orthogonality}$$

$$\Sigma_{n=1}^{\infty} [b\, J_{\nu+1}(bk_n)/\sqrt{2}\,]^{-2}\, J_\nu(k_n\rho)\, J_\nu(k_n\rho') = \delta(\rho-\rho')/\rho \qquad \text{// completeness}$$

Fourier Bessel Integral Transform on (0,∞) = Hankel Transform (aka J transform)

This is what you get in the limit that b→∞ in the Fourier Bessel Series transform above, but I will just quote results from elsewhere (again, see my Jackson cylindricals doc)

Here is the Stakgold version of the Hankel Transform from pages Stak 315-316

$$F_\nu(\mu) = \int_0^\infty dx\, xJ_\nu(\mu x)f(x) \qquad f(x) = \int_0^\infty d\mu\, \mu J_\nu(\mu x)\, F_\nu(\mu)$$

where ν is a fixed value, but Re(ν) > -1/2 is required (see ET II page 12). The above transform is equivalent to this Stakgold completeness relation which you see on Stak page 316.

$$\delta(x-x')/x = \int_0^\infty d\mu\, \mu J_\nu(\mu x')\, J_\nu(\mu x)$$

But as I show in my Chap 4 end notes, if you do things the other way, you get this continuous orthogonality relation:

$$\delta(\mu-\mu')/\mu = \int_0^\infty dx\, xJ_\nu(\mu x)\, J_\nu(\mu' x) \qquad \text{// derived this myself}$$

Notice that we still have s(x) = x. The projection is now on (0,∞) and the normalization factor has gone away and the eigenfunctions $J_\nu([x_{\nu,k}/b]x)$ have become $J_\nu(\mu x)$ where μ is a continuous eigenvalue. In terms of our "general facts" noted above, we have

$$\delta(x-\xi)/x = \int d\mu\, \varphi_\mu(x)\, \bar{\varphi}_\mu(\xi) \qquad \varphi_\mu(x) = \sqrt{\mu}\; J_\nu(\mu x) \qquad \delta(\mu-\mu') = \int_0^\infty dx\, x\, \varphi_\mu(x)\, \bar{\varphi}_\mu(x)$$

The $J_\nu(x) \sim 1/\sqrt{x}$ so vanishes for large argument, so one could imagine that the Hankel expansion is good for functions that vanish at ∞. Here then is our summary, written in terms of the usual variables which appear in cylindrical coordinates

$$f(\rho) = \int_0^\infty dk\, k\, J_\nu(k\rho)\, F_\nu(k) \qquad \text{// expansion}$$

$$F_\nu(k) = \int_0^\infty d\rho\, \rho\, J_\nu(k\rho)\, f(\rho) \qquad \text{// projection}$$

$$\int_0^\infty d\rho\, \rho\, J_\nu(k\rho)\, J_\nu(k'\rho) = \delta(k-k')/k \qquad \text{// orthogonality}$$

$$\int_0^\infty dk\, k\, J_\nu(k\rho)\, J_\nu(k\rho') = \delta(\rho-\rho')/\rho \qquad \text{// completeness}$$

Note that the two directions of this transform are symmetric, a nice choice to make for allocating factors. This is probably why Stak does his F-B Series transform in that way, which I did not do above.

Comment: I don't know the group theory interpretation of this transform. I don't know the group for which the Bessel functions are the representation functions, but could probably find it somewhere.

The Exponential Fourier Series Transform

I discuss this in "diagonalization of convolution equations" in math/group theory. Here are the results, and note that f(θ) in general can be a complex function, [expansion should maybe have $e^{+in\theta}$]

$f(\theta) = \Sigma_n f^n e^{-in\theta}$ // expansion all on $(-\pi,\pi)$

$f^n = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, f(\theta)\, e^{+in\theta}$ // projection

$(1/2\pi) \int_{-\pi}^{\pi} d\theta\, e^{in\theta} e^{-in'\theta} = \delta_{nn'}$ // orthogonality

$(1/2\pi) \Sigma_n e^{-in\theta'} e^{in\theta} = \delta(\theta-\theta')$ // completeness

The expansion shows that $f(\theta+2\pi) = \Sigma_n f^n e^{-in(\theta+2\pi)} = \Sigma_n f^n e^{-in\theta} = f(\theta)$, so the expansion can only be applied to functions f(θ) which are periodic with period 2π. The normalized eigenfunctions are obviously $\varphi_n = e^{-in\theta}/\sqrt{2\pi}$. The sum Σ_n includes all positive and negative integers and 0. Notice that, even when f(θ) is real, f^n is complex unless f(θ) happens to be even, and for that reason this thing is called the "complex" Fourier Series. The functions $e^{-in\theta}$ for all integer n are the UIR representation functions of the group SO(2), where n labels the representation. It is for that reason I have n as a superscript on f^n .

This transform is also true if you negate the signs of all the exponents.

In my same diagonalization document, I point out that the complex Fourier Series will diagonalize a convolution integral equation of this form

$A(\theta) = \int_{-\pi}^{\pi} d\theta_1/2\pi\, B(\theta_1)\, C(\theta-\theta_1)$ // convolution equation

$A^n = B^n C^n$ // after diagonalization

where projections are as shown above for f^n . Thus it is easy to solve the above integral equation for B(θ) if you are given A(θ) and C(θ). The integral equation is still true if you swap B↔C.

The Fourier Series Transform for real functions.

We can rewrite the above Exponential transform this way, changing nothing

$f(\theta) = \Sigma_n f^n [\cos(n\theta) - i\sin(n\theta)]$ // expansion all on $(-\pi,\pi)$

$f^n = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, f(\theta) [\cos(n\theta) + i\sin(n\theta)]$ // projection

If we write $f^n = [\ Re(f^n) + i\ Im(f^n)]$ we get

$f(\theta) = \Sigma_n\ [\ Re(f^n) + i\ Im(f^n)]\ \ [\ \cos(n\theta) - i\sin(n\theta)]$ // expansion all on $(-\pi,\pi)$

$[\ Re(f^n) + i\ Im(f^n)] = (1/2\pi) \int_{-\pi}^{\pi} d\theta\ f(\theta)\ [\ \cos(n\theta) + i\sin(n\theta)]$ // projection

If we now assume that $f(\theta)$ is **real** (the f^n will still be complex), we learn all these things from the above two lines, just by equating real and imaginary parts in both equations,

$f(\theta) = \Sigma_n\ [\ Re(f^n)\cos(n\theta) + Im(f^n)\sin(n\theta)]$

$0 = \Sigma_n\ [\ Im(f^n)\cos(n\theta) - Re(f^n)\sin(n\theta)]$

$Re(f^n) = (1/2\pi) \int_{-\pi}^{\pi} d\theta\ f(\theta)\cos(n\theta)$ // note that $Re(f^{-n}) = Re(f^n)$

$Im(f^n) = (1/2\pi) \int_{-\pi}^{\pi} d\theta\ f(\theta)\sin(n\theta)$ // note that $Im(f^{-n}) = -\ Im(f^n)$

where Σ_n is still over all integers n, including +, - and 0. It is at this point that we shall reflect the negative integer sum and segregate the n = 0 term using this idea

$\Sigma_n\ h_n = \Sigma_{n=-\infty}^{-1} h_n + h_0 + \Sigma_{n=1}^{\infty}\ h_n = h_0 + \Sigma_{n=1}^{\infty}\ (h_n + h_{-n})$

Consider then this application:

$h_n = [\ Re(f^n)\cos(n\theta) + Im(f^n)\sin(n\theta)]$

$h_{-n} = [\ Re(f^{-n})\cos(-n\theta) + Im(f^{-n})\sin(-n\theta)] = [\ Re(f^n)\cos(n\theta) - Im(f^n)\,(-)\sin(n\theta)] = h_n$

So for this particular h_n, we find that $\Sigma_n\ h_n = h_0 + 2\ \Sigma_{n=1}^{\infty}\ h_n = \Sigma_{n=0}^{\infty}\ \varepsilon_n\ h_n$ where we have introduced the **Neumann factor (number**)

$\varepsilon_n = 2 - \delta_{n,0}$

Thus our transform above can be written

$f(\theta) = \Sigma_{n=0}^{\infty}\ \varepsilon_n\ [\ Re(f^n)\cos(n\theta) + Im(f^n)\sin(n\theta)]$

$Re(f^n) = (1/2\pi) \int_{-\pi}^{\pi} d\theta\ f(\theta)\cos(n\theta)$ $\quad$ $Im(f^n) = (1/2\pi) \int_{-\pi}^{\pi} d\theta\ f(\theta)\sin(n\theta)$

It is a *historical convention* to define a_n and b_n in this manner

$a_n = 2Re(f^n)$ $\quad$ $b_n = 2Im(f^n)$ $\quad$ => $\quad$ $2f^n = a_n + i\ b_n$, $a_{-n} = a_n$, $\quad$ $b_{-n} = b_n$

and then the transform becomes

$$f(\theta) = (1/2)\,\Sigma_{n=0}^{\infty}\,\varepsilon_n\,[a_n\cos(n\theta) + b_n\sin(n\theta)] = a_0/2 + \Sigma_{n=1}^{\infty}\,[a_n\cos(n\theta) + b_n\sin(n\theta)]$$

$$a_n = (1/\pi)\int_{-\pi}^{\pi} d\theta\, f(\theta)\cos(n\theta) \qquad b_n = (1/\pi)\int_{-\pi}^{\pi} d\theta\, f(\theta)\sin(n\theta)$$

The convention causes the expansion of f(θ) to have a stray numerical factor only on the a_0 term.

Now, what can we say about the orthogonality and completeness equations? Well, the fact that f(θ) is assumed to be real instead of a general complex function does not affect these conditions, which we copy down here from above in the exponential transform statement

$$(1/2\pi)\int_{-\pi}^{\pi} d\theta\, e^{in\theta}\, e^{-in'\theta} = \delta_{nn'} \qquad \text{// orthogonality}$$

$$(1/2\pi)\ \Sigma_n\, e^{-in\theta'}\, e^{in\theta} = \delta(\theta-\theta') \qquad \text{// completeness}$$

We can of course write these out in terms of sines and cosines if we want.

Orthogonality: The orthogonality can then be written in a few different ways

$$\int_{-\pi}^{\pi} d\theta\,\cos[(n-n')\theta] = 2\pi\,\delta_{nn'}$$

$$\int_{-\pi}^{\pi} d\theta\,[\cos(n\theta)\cos(n'\theta) + \sin(n\theta)\sin(n'\theta)] = 2\pi\,\delta_{nn'}$$

$$\int_{-\pi}^{\pi} d\theta\,\cos(n\theta)\cos(n'\theta) + \int_{-\pi}^{\pi} d\theta\,\sin(n\theta)\sin(n'\theta) = 2\pi\,\delta_{nn'} \qquad (*)$$

Using Schaum page 96 top, we know that (beware the n=0 cosine case) for n and n' being any integers,

$$\int_{-\pi}^{\pi} d\theta\,\cos(n\theta)\cos(n'\theta) = (2\pi/\varepsilon_n)\,\delta_{nn'} \qquad \text{// orthogonalities}$$

$$\int_{-\pi}^{\pi} d\theta\,\sin(n\theta)\sin(n'\theta) = (\pi)\,\delta_{nn'}(1-\delta_{n,0})$$

$$\int_{-\pi}^{\pi} d\theta\,\sin(n\theta)\cos(n'\theta) = 0$$

One might wonder whether the first two of these are compatible with (*) :

$$(2\pi/\varepsilon_n)\,\delta_{nn'} + (\pi)\,\delta_{nn'}(1-\delta_{n,0}) = ?= 2\pi\,\delta_{nn'}$$

$$(2/\varepsilon_n) + (1-\delta_{n,0}) = ?= 2$$

Well, for n = 0 we get 2 + 0 = 2, and for n > 1 we get 2/2 + 1 = 2, so the answer is yes! Now the Schaum results show overlap integrals of the "basis functions" so these are the real orthogonality equations I would say, and I will list them in the summary below.

Completeness: In the completeness equation we have $h_n = e^{-in(\theta'-\theta)}$ with $h_{-n} = h_n^*$, so our rule above says

$$\Sigma_n h_n = h_0 + \Sigma_{n=1}^{\infty} (h_n+h_{-n}) = h_0 + \Sigma_{n=1}^{\infty} 2\,\mathrm{Re}(h_n)$$

or

$$\Sigma_n e^{-in\theta'} e^{in\theta} = 1 + 2\,\Sigma_{n=1}^{\infty} \cos[n(\theta-\theta')] = \Sigma_{n=0}^{\infty} \varepsilon_n \cos[n(\theta-\theta')]$$

Thus our completeness may be written many different ways

$$\Sigma_{n=0}^{\infty} \varepsilon_n \cos[n(\theta-\theta')] = 2\pi\,\delta(\theta-\theta')$$
$$\Sigma_{n=0}^{\infty} \varepsilon_n \{ \cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\} = 2\pi\,\delta(\theta-\theta')$$
$$\Sigma_{n=0}^{\infty} \varepsilon_n \cos(n\theta)\cos(n\theta') + \Sigma_{n=0}^{\infty} \varepsilon_n \sin(n\theta)\sin(n\theta') = 2\pi\,\delta(\theta-\theta')$$
$$\Sigma_{n=0}^{\infty} \varepsilon_n \cos(n\theta)\cos(n\theta') + \Sigma_{n=1}^{\infty} 2\,\sin(n\theta)\sin(n\theta') = 2\pi\,\delta(\theta-\theta')$$
$$1 + \Sigma_{n=1}^{\infty} 2\cos(n\theta)\cos(n\theta') + \Sigma_{n=1}^{\infty} 2\,\sin(n\theta)\sin(n\theta') = 2\pi\,\delta(\theta-\theta')$$

So here then is our summary for the Fourier Series Transform on $(-\pi,\pi)$:

$$f(\theta) = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\theta) + \Sigma_{n=1}^{\infty} b_n \sin(n\theta) \qquad \text{// expansion}$$
$$= (1/2)\,\Sigma_{n=0}^{\infty} \varepsilon_n [a_n \cos(n\theta) + b_n \sin(n\theta)] \quad \text{where } \varepsilon_n = 2-\delta_{n,0}$$

$$a_n = (1/\pi) \int_{-\pi}^{\pi} d\theta\, f(\theta) \cos(n\theta) \qquad b_n = (1/\pi) \int_{-\pi}^{\pi} d\theta\, f(\theta) \sin(n\theta) \qquad \text{// projections}$$

$$\int_{-\pi}^{\pi} d\theta \cos(n\theta) \cos(n'\theta) = (2\pi/\varepsilon_n)\,\delta_{nn'} \qquad \text{// orthogonalities}$$

$$\int_{-\pi}^{\pi} d\theta \sin(n\theta) \sin(n'\theta) = \pi\,\delta_{nn'}(1-\delta_{n,0})$$

$$\int_{-\pi}^{\pi} d\theta \sin(n\theta) \cos(n'\theta) = 0$$

$$1 + 2\,\Sigma_{n=0}^{+\infty} [\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')] = 2\pi\,\delta(\theta-\theta') \qquad \text{// completeness}$$

(see other completeness forms shown above)

If we take $L = \pi$, the above agrees with Schaum page 131. Schaum scales things to $x = \theta(L/\pi)$ and so replaces θ by $\pi x/L$ and then the interval is $(-L,L)$. The above expansion and projections are then

$$f(x) = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\pi x/L) + \Sigma_{n=1}^{\infty} b_n \sin(n\pi x/L) \qquad \text{// expansion}$$

$$a_n = (1/L) \int_{-L}^{L} dx\, f(x) \cos(n\pi x/L) \qquad b_n = (1/L) \int_{-L}^{L} dx\, f(x) \sin(n\pi x/L) \qquad \text{// projections}$$

and then f(x) must be a function with period 2L. Normally one considers the interval (-L,L), and given any function defined there, it can be trivially be made periodic by replication to all other intervals.

The Fourier Cosine Series Transform

It often happens that you have some $f(\theta)$ which you know is an **even** real function of θ. In this case, we specialize the Fourier series above since we have $b_n = 0$. All we need do is copy down the above results

and edit them. Everything is still on the range (-π,π), but since f(θ) is even, we reflect the projection integral getting a factor of 2 :

$f(\theta) = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\theta) = (1/2) \Sigma_{n=0}^{\infty} \varepsilon_n a_n \cos(n\theta)$ // expansion

$a_n = (2/\pi) \int_0^{\pi} d\theta\, f(\theta) \cos(n\theta)$ where $\varepsilon_n = 2-\delta_{n,0}$ // projection

$\int_0^{\pi} d\theta \cos(n\theta)\cos(n'\theta) = (\pi/\varepsilon_n)\, \delta_{nn'}$ // orthogonality

$\Sigma_{n=0}^{+\infty} \varepsilon_n \cos[n(\theta-\theta')] = 2\pi\, \delta(\theta-\theta')$ (wrong) // completeness

$\Sigma_{n=0}^{+\infty} \varepsilon_n \cos(n\theta)\cos(n\theta) = \pi\, \delta(\theta-\theta')$ (right) // completeness

See Repair section below in this document for this correction!

Since f(θ) for θ in (0,π) is all that we need to compute the projection, we *can* correspondingly restrict the expansion to (0,π) and then we can think of *the entire four lines* above as applying just to θ in (0,π).

Notice that the orthogonality is just a subset of our three orthogonality conditions above, since we don't need to know anything about sine stuff. And notice that completeness is unchanged, since it is a "property" of the cosine function shown. Completeness does not "know" that f(θ) might be even or odd. See previous section for other ways to write this completeness relation.

Now, suppose f(θ) is NOT an even function. We can *create* an even function f(|θ|) . For this function our transform above is then (where we assume the second case above, that we have θ in (0,π))

$f(|\theta|) = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\theta) = (1/2) \Sigma_{n=0}^{\infty} \varepsilon_n a_n \cos(n\theta)$ // expansion

$a_n = = (2/\pi) \int_0^{\pi} d\theta\, f(|\theta|) \cos(n\theta)$ where $\varepsilon_n = 2-\delta_{n,0}$ // projection

But since we are restricting now to θ in (0,π), we can remove the abs values and just write
{ $\theta = x\,(\pi/L)$ is alternative so get $(2/L) \int_0^{L} dx$ }

Fourier Cosine Series Transform on (0,π)

$f(\theta) = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\theta) = (1/2) \Sigma_{n=0}^{\infty} \varepsilon_n a_n \cos(n\theta)$ // expansion

$a_n = (2/\pi) \int_0^{\pi} d\theta\, f(\theta) \cos(n\theta)$ where $\varepsilon_n = 2-\delta_{n,0}$ // projection

$\int_0^{\pi} d\theta \cos(n\theta)\cos(n'\theta) = (\pi/\varepsilon_n)\, \delta_{nn'}$ // orthogonality

$\Sigma_{n=0}^{+\infty} \varepsilon_n \cos[n(\theta-\theta')] = 2\pi\, \delta(\theta-\theta')$ // completeness

where now f(θ) is *any* function, not necessarily an even function. The point is only that we can apply the Fourier Cosine Series Transform to any function f(θ), but if we want to relate this to the full Fourier Series Transform, we have to think in terms of the "evenized" function f(|θ|).

To clarify this statement, consider these alternate definitions for our projections:

$F_e[f(\theta),n] = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, f(\theta)\, e^{+in\theta}$ // exponential f^n

$F_s[f(\theta),n] = (2/\pi) \int_{0}^{\pi} d\theta\, f(\theta) \sin(n\theta)$ // sine b_n

$F_c[f(\theta),n] = (2/\pi) \int_{0}^{\pi} d\theta\, f(\theta) \cos(n\theta)$ // cosine a_n

This notation is a little better than just having things like b_n since you can "see" the function you have projected. Then here is the claim we have just made: [the factor of 1/2 arises from our projection convention such as $a_n = 2\mathrm{Re}(f^n)$]

$$F_e[f(|\theta|),n] = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, f(|\theta|)\, e^{+in\theta} = (1/\pi) \int_{0}^{\pi} d\theta\, f(\theta) \cos(n\theta) = (1/2)\, F_c[f(\theta),n]$$

which is to say

$$F_c[f(\theta),n] = 2\, F_e[f(|\theta|),n]$$

Example: Consider $f(\theta) = \sin(\theta)$ which is a quintessential ODD function on $(-\pi,\pi)$ and Maple tells us that

$$a_n = (2/\pi) \int_{0}^{\pi} d\theta \sin(\theta) \cos(n\theta) = (2/\pi)(-1)[1 + \cos(\pi n)] / (n^2-1) = -(2/\pi)[1 + (-1)^n] / (n^2-1)$$

For n even, this is $(-4/\pi)/(n^2-1)$ so we get projections onto all even cosine terms in our expansion despite the fact that $\sin(\theta)$ is an "odd function". Of course we have really processed $\sin(|\theta|)$ if you want to think of the full $(-\pi,\pi)$ interval, and this is indeed a function even in θ. But when thought of on $(0,\pi)$, we can apply this Fourier Cosine Series transform to ANY function $f(\theta)$ without regard to evenness or oddness, the only requirement is that the projections must be convergent integrals. Now it is true that if you expand $\sin(\theta)$ in a Fourier Series *Sine* Transform below, you will get exactly one term in your expansion, whereas using the Fourier Series Cosine Transform above, you get an infinite number of terms in the expansion, so yes, the sine transform is more efficient in this case, but both transforms are valid!

The Fourier Sine Series Transform

This is used if $f(\theta)$ is real and **odd**. Again, we have just write the regular thing and edit as needed:

$f(\theta) = \Sigma_{n=1}^{\infty} b_n \sin(n\theta)$ // expansion

$b_n = (1/\pi) \int_{-\pi}^{\pi} d\theta\, f(\theta) \sin(n\theta)$ // projection

$\int_{-\pi}^{\pi} d\theta \sin(n\theta) \sin(n'\theta) = \pi\, \delta_{nn'}(1 - \delta_{n,0})$ // orthogonality

$\Sigma_{n=0}^{+\infty} \varepsilon_n \cos[n(\theta-\theta')] = 2\pi\, \delta(\theta-\theta')$ // completeness

Similarly to what we did above, we can say { $\theta = x\,(\pi/L)$ is alternative so get $(2/L)\int_0^L dx$ }

Fourier Sine Series Transform on θ in (0,π)

$f(\theta) = \Sigma_{n=1}^{\infty} b_n \sin(n\theta)$ // expansion

$a_n = (2/\pi) \int_0^{\pi} d\theta\, f(\theta) \sin(n\theta)$ // projection

$\int_0^{\pi} d\theta \sin(n\theta) \sin(n'\theta) = (\pi/2)\,\delta_{nn'}(1-\delta_{n,0})$ // orthogonality

$\Sigma_{n=0}^{+\infty}\varepsilon_n \cos[n(\theta-\theta')] = 2\pi\,\delta(\theta-\theta')$ // completeness

The above applies to any f(θ) which has a convergent projection, need not be even or odd. But given any f(θ) we can construct an "oddized" function sign(θ)f(|θ|) and apply the full Fourier Series Transform to this odd function and thereby obtain our sine transform results above with $a_n = 0$. We will find that

$$F_e[\text{sign}(\theta)f(|\theta|),n] = (1/2\pi) \int_{-\pi}^{\pi} d\theta\, f(\text{sign}(\theta)|\theta|)\, e^{+in\theta}$$

$$= i\,(1/\pi) \int_0^{\pi} d\theta\, f(\theta) \sin(n\theta) = i\,(1/2)\, F_s[f(\theta),n]$$

which is to say (and second line compares to the cosine transform case)

$F_s[f(\theta),n] = -i\,2\,F_e[\text{sign}(\theta)f(|\theta|),n]$
$F_c[f(\theta),n] = 2\,F_e[f(|\theta|),n]$

The Exponential Fourier Integral Transform

I will use my "diagonalization and convolution.." doc here for reference. The group here is the non-compact group sometimes called T(1), translations in one dimension. The results are these, where I maintain the same variable name θ as in the SO(2) compact group case,

$f(\theta) = \int_{-\infty}^{\infty} d\sigma\, f^{\sigma} e^{-i\sigma\theta}$ // expansion

$f^{\sigma} = (1/2\pi) \int_{-\infty}^{\infty} d\theta\, f(\theta)\, e^{+i\sigma\theta}$ // projection

$\int_{-\infty}^{\infty} d\theta e^{i(\sigma-\sigma')\theta} = 2\pi\,\delta(\sigma-\sigma')$ // orthogonality

$\int_{-\infty}^{\infty} d\sigma\; e^{-i\sigma(\theta'-\theta)} = 2\pi\,\delta(\theta'-\theta)$ // completeness

Normally one uses different variable names, but I was imitating the SO(2) case above. So:

$f(x) = \int_{-\infty}^{\infty} dk\, f^{k} e^{-ikx}$ // expansion

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(x)\, e^{+ikx} \qquad \text{// projection}$$

$$\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = \int_{-\infty}^{\infty} dx\, \cos[(k-k')x] = 2\pi\, \delta(k-k') \qquad \text{// orthogonality}$$

$$\int_{-\infty}^{\infty} dk\, e^{-ik(x'-x)} = \int_{-\infty}^{\infty} dk\, \cos[k(x-x')] = 2\pi\, \delta(x-x') \qquad \text{// completeness}$$

In both orthogonality and completeness, the sine term is odd in the integration variable, so we can restate them as shown. The transform is valid if the signs of all exponents are changed. Some authors put $1/\sqrt{2\pi}$ in both expansion and projection, and some put the entire $1/2\pi$ into the expansion instead of the projection. Only the product of the factors is fixed.

Diagonalization works just as in the SO(2) case, and we have

$$A(x) = \int_{-\infty}^{\infty} dy/2\pi\, B(y)\, C(x-y) \qquad \text{// convolution equation}$$

$$A^{n} = B^{n}\, C^{n} \qquad \text{// after diagonalization}$$

with projections as given above. One must be careful with the 2π factors when diagonalizing an equation, because the form of the diagonalized equation changes depending on the projection convention.

Warning: In my spectral theory doc, I use this Fourier Transform convention,

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inversion} \qquad (1.2)$$

which I don't now want to make be my standard. You see that the exponent the opposite of what I now use, and the 2π is in the "wrong place".

The Fourier Cosine Integral Transform

See "confusion about completeness relations.doc" for details on this transform! In the first form presented below, both k and z are defined only for $(0,\infty)$ and so there are no "second terms" in the orthogonality and completeness equations. See also section 4(a) "Digression on the Fourier Cosine Transform" of the following doc: [which I just read today 1.10.11, and it treats only this z>0,k>0 case].

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Summary Case 1: For z in $(0,\infty)$ and k in $(0,\infty)$:

$$f(z) = \sqrt{2/\pi} \int_{0}^{\infty} dk\, \cos(kz)\, f^{k} \qquad \text{// expansion}$$

$f^{k} = \sqrt{2/\pi} \int_0^\infty dz \cos(kz) f(z)$ // projection

$\int_0^\infty dz \cos(kz) \cos(k'z) = (\pi/2)\delta(k-k')$ // orthogonality

$\int_0^\infty dk \cos(kz) \cos(kz') = (\pi/2)\delta(z-z')$ // completeness

The transform appears in Stak p 293 (4.71) and completeness as in (4.70). Notice that everything here is symmetric. Variable z is used since this thing often finds application in the z dimension of cylindrical coordinates. You can of course partition the constants differently, for example putting 1 in one direction and $(2/\pi)$ in the other direction of the transform. Only the product of the two constants is fixed.

• The expansion shown above cannot represent a discontinuity that might be present in the original f(z) from which the projection f^{k} was obtained. If we want to allow that f(z) might have a discontinuity, we can replace the expansion with this more precise statement (which is I think a Plancherel theorem)

$[f(z+) + f(z-)]/2 = \sqrt{2/\pi} \int_0^\infty dk \cos(kz) f^{k}$ // expansion

It is not unusual to have a discontinuity at z = 0 in some applications, in which case $f(0) \neq f(0+)$.

• Suppose we can differentiate through the integral in our original expansion above, to get

$f'(z) = -\sqrt{2/\pi} \int_0^\infty dk \sin(kz)\, k\, f^{k}$

One must be warned that if the integral then diverges due to the extra factor of k, this is not allowed. But assuming it is allowed, as with our original expansion, we cannot create a discontinuity in this way, and we really have

$[f'(z+) + f'(z+)]/2 = -\sqrt{2/\pi} \int_0^\infty dk \sin(kz)\, k\, f^{k}$

Now if f'(z) is continuous at z = 0, we conclude that f'(0) = 0 due to the sin(kz) factor. Thus, we cannot apply the Fourier Cosine expansion to a function whose derivative is continuous at z = 0 but $f'(0) \neq 0$. We shall see a similar restriction for the Fourier Sine transform below. In applications, one often deals with functions which do NOT have continuous f'(z) at z = 0, and then we don't have this restriction. We will end up with f'(0) = 0, but we will have $f'(0+) \neq 0$.

• Since f(z) is defined only on $(0,\infty)$, we don't talk about f(z) being an even function of z. We can apply this transform to any f(z) as long as the projection converges.

• It is interesting to see how this may be derived as a special case of the full Fourier transform. Start with (here I use a particular allocation of the constants)

$$f(z) = \int_{-\infty}^{\infty} dk\, f^{k}\, e^{-ikz} \qquad \text{// expansion}$$

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dx\, f(z)\, e^{+ikz} \qquad \text{// projection}$$

If f(z) is **even**, we go through a few steps and conclude that f^{k} is also even and we end up with

$$f(z) = \int_{-\infty}^{\infty} dk\, f^{k} \cos(kz) \qquad \text{// expansion} \quad \text{pair 1}$$

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dz\, f(z) \cos(kz) \qquad \text{// projection}$$

We can then fold both integrals since both integrands are even, and we get

$$f(z) = 2 \int_{0}^{\infty} dk\, f^{k} \cos(kz) \qquad \text{// expansion}$$

$$f^{k} = (1/\pi) \int_{0}^{\infty} dz\, f(z) \cos(kz) \qquad \text{// projection}$$

If we then distribute the constants equally, we get $\sqrt{2/\pi}$ in each direction and we reproduce the transform quoted above, QED.

• A subtlety that is lurking but does not come up in the above derivation is this: if you want for some reason to express one or both integrals as (-∞,∞) range integrals, as in the pair 1 above, for example, you must be aware of the "second terms" in the various delta equations. If you omit these, you will find that inserting one direction into the other does not give the right answer! See the doc referenced at the start of this section. Here are some of those relations from that doc:

$$(1/\pi) \int_{-\infty}^{\infty} dz \cos(k'z) \cos(kz) = \delta(k' - k) + \delta(k' + k) \qquad (1)$$

$$(1/\pi) \int_{-\infty}^{\infty} dk \cos(kz') \cos(kz) = \delta(z' - z) + \delta(z' + z) \qquad (5)$$

• Almost always one restricts k to the (0,∞) range, but sometimes we let z have the full range (-∞.∞). Only when we do this can we talk about f(z) being "an even function". If it is only defined on (0,∞), the notion of an even function is meaningless. So here is another form of the transform for this situation:

$$f(z) = 2 \int_{0}^{\infty} dk\, f^{k} \cos(kz) \qquad \text{// expansion} \quad \text{pair 2}$$

$$f^{k} = (1/2\pi) \int_{-\infty}^{\infty} dz\, f(z) \cos(kz) \qquad \text{// projection}$$

Obviously we could cancel the two factors of 2 and we could then equalize the constants to get:

Summary Case 2: For z in (-∞,∞) and k in (0,∞) where f(z) is even in z:

$$f(z) = \sqrt{1/\pi} \int_0^\infty dk\, f^k \cos(kz) \qquad \text{// expansion pair 2}$$

$$f^k = \sqrt{1/\pi} \int_{-\infty}^\infty dz\, f(z) \cos(kz) \qquad \text{// projection}$$

$$\int_{-\infty}^\infty dz \cos(kz) \cos(k'z) = \pi\, \delta(k-k') \qquad \text{// orthogonality}$$

$$\int_0^\infty dk \cos(kz) \cos(kz') = (\pi/2)[\, \delta(z-z') + \delta(z+z')] \qquad \text{// "completeness"}$$

and here we have to include the second term in the completeness relation.

The Fourier Sine Integral Transform

The development here is almost identical to that for the cosine case, so I will skip the details. The main differences are these: (1) if you need "second terms" in delta equations, those second terms have minus signs. (2) continuous-at-0 functions f(z) must have the property f(0) = 0 since all the sine terms in the expansion have this property. (3) If you use the full (-∞,∞) for z, you are then talking about f(z) that are *odd*. Here are the results, and the reader can fill out all the comments of the previous section:

Summary Case 1: For z in (0,∞) and k in (0,∞):

$$f(z) = \sqrt{2/\pi} \int_0^\infty dk \sin(kz)\, f^k \qquad \text{// expansion}$$

$$f^k = \sqrt{2/\pi} \int_0^\infty dz \sin(kz)\, f(z) \qquad \text{// projection}$$

$$\int_0^\infty dz \sin(kz) \sin(k'z) = (\pi/2)\delta(k-k') \qquad \text{// orthogonality}$$

$$\int_0^\infty dk \sin(kz) \sin(kz') = (\pi/2)\delta(z-z') \qquad \text{// completeness}$$

Summary Case 2: For z in (-∞,∞) and k in (0,∞) where f(z) is odd in z:

$$f(z) = \sqrt{1/\pi} \int_0^\infty dk\, f^k \sin(kz) \qquad \text{// expansion pair 2}$$

$$f^k = \sqrt{1/\pi} \int_{-\infty}^\infty dz\, f(z) \sin(kz) \qquad \text{// projection}$$

$$\int_{-\infty}^\infty dz \sin(kz) \sin(k'z) = \pi\, \delta(k-k') \qquad \text{// orthogonality}$$

$$\int_0^\infty dk \sin(kz) \sin(kz') = (\pi/2)[\, \delta(z-z') - \delta(z+z')] \qquad \text{// "completeness"}$$

As before, we should really be saying this for the expansion

$$[f(z+) + f(z-)]/2 = \sqrt{1/\pi} \int_0^\infty dk\, f^k \sin(kz)$$

And as before, we find that we cannot do a Fourier Sine transform on a function f(z) which is continuous at z = 0 but which does not vanish there! It is often the case that f(z) is not continuous there, and so we are allowed to have f(0) = 0 but f(0+) ≠ 0. This happens in the half-rod heat problem Ex 7.16 Stak.

Fourier Series Transforms involving an arbitrary compact Lie Group G

Also from doc "diagonalization of convolution..." . For compact groups, the projections are elements of finite matrices. The symbol σ is the label of a UIR which is a set of Casimirs if you will, often just one. For each σ, there is a matrix of some size N = d^σ of projection elements, so in general there are three labels on the projection instead of just 1 as in our cases above

We can write (I don't know how to write orthogonality in matrix notation)

matrix form:

$f^\sigma = \int dg\, f(g)\, D^\sigma(g^{-1})$ // projection

$f(g) = \Sigma_\sigma\, d^\sigma\, tr\,[\, f^\sigma\, D^\sigma(g)]$ // expansion

$\Sigma_\sigma\, d^\sigma\, tr[D^\sigma(g'^{-1})D^\sigma(g)] = \delta(g-g')$ // completeness

showing matrix indices:

$f^\sigma{}_{kk'} = \int dg\, f(g)\, D^\sigma{}_{kk'}(g^{-1}) = \int dg\, f(g)\, D^\sigma{}_{k'k}(g)^*$ // projection

$f(g) = \Sigma_\sigma\; d^\sigma\, \Sigma_{k,k'}\, f^\sigma{}_{kk'}\, D^\sigma{}_{k'k}(g)$ // expansion

$\Sigma_\sigma\, d^\sigma\;\; D^\sigma(g'^{-1})_{nm} D^\sigma(g)_{mn} = \delta(g-g')$ // completeness

$\int dg\;\; D^{\sigma'}(g)_{nm}\, D^\sigma(g^{-1})_{kk'} = (1/d^\sigma)\, \delta_{\sigma\sigma'}\;\; \delta_{mk}\;\; \delta_{nk'}$ // orthogonality

The "group property" leads to an "addition theorem":

$\Sigma_b\, D^\sigma(g_1)_{ab}\, D^\sigma(g_2)_{bc} = D^\sigma(g_1 g_2)_{ac}$ or $D^\sigma(g_1)\, D^\sigma(g_2) = D^\sigma(g_1 g_2)$

There is a definite Haar measure dg that makes this all work for a given group.

For non-compact groups, like T(1), the expansion is an integral instead of a series.

My thesis talks about some of this stuff, but is mostly focused on diagonalization which works like this:

$A(g) = \int dg_1 B(g_1) C(g_2)$ where $g_2 = g_1^{-1} g$. // convolution equation $g = g_1 g_2$ (*)

$A^\sigma{}_{kk'} = \Sigma_{k''}\, C^\sigma{}_{kk''}\, B^\sigma{}_{k''k'}$ //diagonalized equation

$A^\sigma = C^\sigma\; B^\sigma$ // same, but in matrix form

Spherical Harmonics Series Transform

The most famous 3D compact Lie group is SO(3). I copy here a few "facts" from my TK binder

g = Euler angles in a certain convention = (α,β,γ)

$dg = d\alpha\ d(\cos\beta)\ d\gamma$ // should there be some 2π's here??

$R(\alpha,\beta,\gamma) = R_z(\alpha)\ R_y(\beta)\ R_z(\gamma)$ $\qquad R_i(\theta) = e^{-i\theta J_i}$

$\langle j'm' | R_z(\alpha)\ R_y(\beta)\ R_z(\gamma) |jm\rangle = D^j_{m'm}(\alpha,\beta,\gamma) = e^{-i\alpha m'}\ d^j_{m'm}(\beta)\ e^{-i\gamma m}$

I give the orthog and completeness of the d functions in my notes. The general projection and expansion formulas follow the general form stated in the previous section, where

$f^n_{kk'} = \int dg\ f(g)\ D^n_{kk'}(g^{-1}) = \int dg\ f(g)\ D^n_{k'k}(g)^*$ // projection

$f(g) = \Sigma_{n=0}^{\infty}\ (2n+1)\ \Sigma_{k,k'}\ f^n_{kk'}\ D^n_{k'k}(g)$ // expansion

When f(g) is a function only of angles α and β (usually called φ and θ), the projection will vanish unless the second index is 0, so only $D^j_{m0}(\alpha,\beta)$ will appear in the corresponding expansion,

$D^j_{m0}(\alpha,\beta,\gamma) = D^j_{m0}(\alpha,\beta) = [\ f(\ell,-m)]^{1/2}\ P_\ell^m(\cos\beta)\ e^{-im\alpha}$

and this is the connection to the spherical harmonics (see TK notes).

Jackson gives his convention for these things on page 65,

$Y_{nm}(\theta,\varphi) = [\ (2n+1)/\sqrt{4\pi}\ ]\ f(n,-m)/\ P_n^m(\cos\beta)\ e^{im\varphi}$

and writes both orthogonality and completeness this way:

$\int d\Omega\ Y_{n'm'}(\theta,\varphi)^*\ Y_{nm}(\theta,\varphi) = \delta_{nn'}\delta_{mm'}$ // orthogonality

$\Sigma_{nm}\ Y_{nm}(\theta',\varphi')^*\ Y_{nm}(\theta,\varphi) = \delta(\varphi-\varphi')\ \delta(\cos\theta-\cos\theta')$ // completeness

From these we can derive a projection expansion pair. Here is one way to do it:

$f(\theta,\varphi) = \Sigma_{nm}\ f_{nm}\ Y_{nm}(\theta,\varphi)$ // expansion

$f_{nm} = \int d\Omega\ Y_{nm}(\theta,\varphi)^*\ f(\theta,\varphi)$ // projection

So here is a summary of this situation:

$f(\theta,\varphi) = \Sigma_{nm}\ f_{nm}\ Y_{nm}(\theta,\varphi)$ // expansion

$$f_{nm} = \int d\Omega\, Y_{nm}(\theta,\varphi)^*\, f(\theta,\varphi) \qquad \text{// projection}$$

$$\int d\Omega\, Y_{n'm'}(\theta,\varphi)^*\, Y_{nm}(\theta,\varphi) = \delta_{nn'}\delta_{mm'} \qquad \text{// orthogonality}$$

$$\Sigma_{nm}\, Y_{nm}(\theta',\varphi')^*\, Y_{nm}(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta') \qquad \text{// completeness}$$

Stakgold and others have different conventions, I have notes on spherical harmonics elsewhere.

Recently I have been thinking about this expansion as being in terms of spherical atoms in the spherical coordinates system, and I use the $P_n^m(z)$ functions directly. These have their own orthogonality and completeness which derives from the SL problem in the θ dimension (z = cosθ),

$$\int_{-1}^{1} dz\, P_n^m(z) P_k^m(z) = \delta_{n,k}\, K_n^m \qquad n,k = m,\ m+1,\ m+2\ \ldots\ldots\ \infty \qquad \text{// orthogonality}$$

$$K_n^m = (n+1/2)^{-1}\,(n+m)!\,/\,(n-m)!$$

$$\Sigma_{n=m}^{\infty}(1/K_n^m)\, P_n^m(z')\, P_n^m(z) = \delta(z'-z) \qquad \text{// completeness}$$

One thinks of θ and φ as the two oscillatory dimensions of spherical coordinates and each is a SL problem with its eigenfunctions and these get combined in the spherical harmonics.

The Legendre Series Transform and Raw Spherical Harmonic Expansion/Projection

In my doc called "Legendre SL Problem and Spherical Harmonics.doc" I derive the following facts which are valid for m = any integer (of either sign or 0)

spectrum: m = integer and $\nu = |m|, |m|+1, \ldots$

$$\int_{-1}^{1} dz\, P_n^m(z) P_{n'}^m(z) = \delta_{n,n'}\, K_n^m \qquad n,n' = |m|, |m|+1, \ldots$$

$$\Sigma_{n=|m|}^{\infty}(1/K_n^m)\, P_n^m(z')\, P_n^m(z) = \delta(z'-z) \qquad K_n^m = (n+1/2)^{-1}\, f(n,m)$$

We start now with the expansion which is pretty well known

$$f(\theta,\varphi) = \Sigma_{n=0}^{\infty}\, \Sigma_{m=-n}^{n}\, f_{nm}\ P_n^m(\cos\theta)\, e^{im\varphi}$$

where we assume f is a real function. In this case, either $e^{im\varphi}$ or $e^{-im\varphi}$ is acceptable since the imaginary part of the sum must be 0. Suppose we start by applying $\int d\varphi\, e^{-im'\varphi}$ to both sides where m' is an arbitrary integer. We then get

$$\int d\varphi\, e^{-im'\varphi} f(\theta,\varphi) = \Sigma_{n=0}^{\infty}\, \Sigma_{m=-n}^{n}\, f_{nm}\ P_n^m(\cos\theta) \int d\varphi\, e^{-im'\varphi}\, e^{im\varphi}$$

and we can now use our Expo Fourier Series transform orthogonality above to write this as

$$= (2\pi) \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} f_{nm}\ P_n^m(\cos\theta)\ \delta_{m,m'}$$

$$= (2\pi) \Sigma_{n=0}^{\infty}\ f_{nm'}\ P_n^{m'}(\cos\theta)\ \theta(\ -n \leq m' \leq n)$$

$$= (2\pi) \Sigma_{n=0}^{\infty}\ f_{nm'}\ P_n^{m'}(\cos\theta)\ \theta(\ |m'| \leq n)$$

$$= (2\pi) \Sigma_{n=|m'|}^{\infty}\ f_{nm'}\ P_n^{m'}(\cos\theta)$$

where the Heaviside-like notation means we only get a hit in the Σ_m sum when m' is in the range shown, for each value of n. We then have (changing m' back to m again)

$$\int d\varphi\ e^{-im\varphi}\ f(\theta,\varphi) = (2\pi) \Sigma_{n=|m|}^{\infty} f_{nm}\ P_n^m(\cos\theta)$$

Next, we apply $\int dz\ P_{n'}^m(z)$ to both side to get (n' is a non-negative integer)

$$\int d\varphi \int dz\ e^{-im\varphi}\ P_{n'}^m(z)\ f(\theta,\varphi) = (2\pi) \Sigma_{n=|m|}^{\infty} f_{nm} \int dz\ P_{n'}^m(z)\ P_n^m(z)$$

$$= (2\pi) \Sigma_{n=|m|}^{\infty} f_{nm}\ \delta_{n,n'}\ K_{n'}^m$$

$$= (2\pi)\ f_{n'm}\ K_{n'}^m\ \theta(|m| \leq n')$$

where we use Heaviside type notation to show that we don't get a hit if n' is too small for m. So let's then assume that we choose n' to be "large enough" , that is, $n' \geq |m|$. Then we have shown, were we replace n' with n as label

$$\int d\varphi \int dz\ f(\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z) = (2\pi)\ f_{nm}\ K_n^m \qquad n \geq |m| \qquad K_n^m = (n+1/2)^{-1}\ f(n,m)$$

So we have found the following projection: $K_n^m = (n+1/2)^{-1}\ f(n,m) = 2(2n+1)^{-1}\ f(n,m)$

$$f_{nm} = (1/K_n^m)\ (1/2\pi) \int d\varphi \int dz\ f(\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

$$= (2n+1)\ (1/2)\ f(n,-m)\ (1/2\pi) \int d\varphi \int dz\ f(\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

$$= (2n+1)\ \ f(n,-m)\ (1/4\pi) \int d\varphi \int dz\ f(\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

$$= (2n+1)\ \ f(n,-m)\ (1/4\pi) \int d\Omega\ f(\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

If $f(\theta,\varphi)$ depends only on θ, this becomes

$$f_{nm} = \delta_{m0}\ (2n+1)\ (1/4\pi)\ \int d\Omega\ f(\theta,-)\ P_n^m(z)$$

Let's verify our results both ways. First, we have

$$f(\theta,\varphi) = \Sigma_{n=0}^{\infty}\ \Sigma_{m=-n}^{n}\ f_{nm}\ P_n^m(\cos\theta)\ e^{im\varphi}$$

$$= \Sigma_{n=0}^{\infty}\ \Sigma_{m=-n}^{n}\ P_n^m(\cos\theta)\ e^{im\varphi}\ \{(2n+1)\ \ f(n,-m)\ (1/4\pi)\ \int d\Omega'\ f(\theta',\varphi')\ e^{-im\varphi'}\ P_n^m(z')\}$$

$$= \Sigma_{m=-\infty}^{\infty}\ \Sigma_{n=|m|}^{\infty}\ P_n^m(\cos\theta)\ e^{im\varphi}\ \{(2n+1)\ \ f(n,-m)\ (1/4\pi)\ \int d\Omega'\ f(\theta',\varphi')\ e^{-im\varphi'}\ P_n^m(z')\}$$

where we have taken the key step of reordering the double summation. Then

$$= \int d\Omega'\ \ \{\Sigma_{m=-\infty}^{\infty}\ e^{im\varphi}\ e^{-im\varphi'}\ \}\ \Sigma_{n=|m|}^{\infty}\ P_n^m(\cos\theta)\ \{(2n+1)\ \ f(n,-m)\ (1/4\pi)\ f(\theta',\varphi')\ P_n^m(z')\}$$

Now we are allowed to use our expo Fourier series completeness $(1/2\pi)\ \ \Sigma_n\ e^{-in\theta'}\ e^{in\theta} = \delta(\theta-\theta')$ so

$$= \int d\Omega'\ 2\pi\ \delta(\varphi-\varphi')\ \Sigma_{n=|m|}^{\infty}\ P_n^m(\cos\theta)\ \{(2n+1)\ \ f(n,-m)\ (1/4\pi)\ f(\theta',\varphi')\ P_n^m(z')\}$$

$$= \int dz'\ 2\pi\ \Sigma_{n=|m|}^{\infty}\ P_n^m(\cos\theta)\ \{(2n+1)\ \ f(n,-m)\ (1/4\pi)\ f(\theta',\varphi)\ P_n^m(z')\}$$

$$= \int dz'\ \ \Sigma_{n=|m|}^{\infty}\ P_n^m(z)\ \{(2n+1)\ \ f(n,-m)\ (1/2)\ f(\theta',\varphi)\ P_n^m(z')\}$$

$$= \int dz'\ \ f(\theta',\varphi)\ \ \{\ \Sigma_{n=|m|}^{\infty}\ P_n^m(z)\ \{(2n+1)\ \ f(n,-m)\ (1/2)\ P_n^m(z')\}\ \}$$

$$= \int dz'\ \ f(\theta',\varphi)\ \ \{\ \Sigma_{n=|m|}^{\infty}\ P_n^m(z)\ \{(n+1/2)\ \ f(n,-m)\ \ P_n^m(z')\}\ \}$$

$$= \int dz'\ \ f(\theta',\varphi)\ \ \{\ \Sigma_{n=|m|}^{\infty}\ P_n^m(z)\ (1/K_n^m)\ \ P_n^m(z')\}\ \}$$

$$= \int dz'\ \ f(\theta',\varphi)\ \ \{\ \delta(z'-z)\ \}$$

$$= f(\theta,\varphi)$$

Now go the other way, where we make the same Σ_{nm} order interchange right away

$$f_{nm} = (1/K_n^m)\,(1/2\pi) \int d\varphi \int dz\, f(\theta,\varphi)\, e^{-im\varphi}\, P_n^m(z)$$

$$= (1/K_n^m)\,(1/2\pi) \int d\varphi \int dz \{ \Sigma_{m'=-\infty}^{\infty} \Sigma_{n'=|m'|}^{\infty} f_{n'm'}\ P_{n'}^{m'}(\cos\theta)\, e^{im'\varphi} \}\, e^{-im\varphi}\, P_n^m(z)$$

$$= (1/K_n^m)\,(1/2\pi)\, \Sigma_{m'=-\infty}^{\infty} \{ \int d\varphi\, e^{im'\varphi}\, e^{-im\varphi} \}\, \Sigma_{n'=|m'|}^{\infty} \int dz \{ f_{n'm'}\ P_{n'}^{m'}(\cos\theta) \}\, P_n^m(z)$$

and we replace {} by $2\pi\, \delta_{mm'}$ so get

$$= (1/K_n^m)\,(1/2\pi)\, \Sigma_{m'=-\infty}^{\infty} \{2\pi\, \delta_{mm'} \}\, \Sigma_{n'=|m'|}^{\infty} \int dz \{ f_{n'm'}\ P_{n'}^{m'}(\cos\theta) \}\, P_n^m(z)$$

$$= (1/K_n^m)\, \Sigma_{n'=|m|}^{\infty} f_{n'm} \{ \int dz\, P_{n'}^{m}(z)\, P_n^m(z) \}$$

$$= (1/K_n^m)\, \Sigma_{n'=|m|}^{\infty} f_{n'm}\ \ \delta_{n,n'}\, K_n^m$$

$$= (1/K_n^m) f_{nm}\ \ \theta(n \geq |m|)\, K_n^m$$

$$= f_{nm}\ \theta(n \geq |m|)$$

But we only care about projections which have this $n \geq |m|$, so QED.

Here then is a summary:

$$f(\theta,\varphi) = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} f_{nm}\ P_n^m(\cos\theta)\, e^{im\varphi} \qquad \text{// expansion}$$

$$f_{nm} = (2n+1)\ f(n,-m)\,(1/4\pi) \int d\Omega\, f(\theta,\varphi)\, e^{-im\varphi}\, P_n^m(z) \qquad \text{// projection}$$

$$\int_{-1}^{1} dz\, P_n^m(z) P_{n'}^m(z) = \delta_{n,n'}\, K_n^m \qquad n,n' = |m|, |m|+1, \ldots \qquad \text{// orthogonality}$$

$$\Sigma_{n=|m|}^{\infty} (1/K_n^m)\, P_n^m(z')\, P_n^m(z) = \delta(z'-z) \qquad K_n^m = (n+1/2)^{-1}\, f(n,m) \qquad \text{// completeness}$$

But I see that I have chewed off more than I intended here. So let's consider instead this expansion:

$$g(z) = \Sigma_{n=|m|}^{\infty} g_{nm}\, P_n^m(z) \qquad m = \text{bystander integer}$$

Apply $\int dz\, P_{n'}^m(z)$ to both sides to get

$$\int dz\, P_{n'}^{m}(z)\, g(z) = \Sigma_{n=|m|}^{\infty} g_{nm} \int dz\, P_{n'}^{m}(z)\, P_n^m(z) = \Sigma_{n=|m|}^{\infty} g_{nm}\ \delta_{n,n'}\ K_n^m$$

$$= g_{n'm}\ K_{n'}^{m}\ \theta(n' \geq |m|)$$

so we assume that n' is "in range, and we find that

$$g_{nm} = (1/K_n^m) \int dz\, P_n^m(z)\, g(z) \qquad \text{// projection}$$

Summary: So in this little world with any fixed bystander integer m we can say

$$g(z) = \Sigma_{n=|m|}^{\infty} g_{nm}\, P_n^m(z) \qquad \text{// expansion}$$

$$g_{nm} = (1/K_n^m) \int dz\, P_n^m(z)\, g(z) \qquad \text{// projection}$$

$$\int_{-1}^{1} dz\, P_n^m(z) P_{n'}^m(z) = \delta_{n,n'}\, K_n^m \qquad n,n' = |m|, |m|+1, \ldots \qquad \text{// orthogonality}$$

$$\Sigma_{n=|m|}^{\infty} (1/K_n^m)\, P_n^m(z')\, P_n^m(z) = \delta(z'-z) \qquad K_n^m = (n+1/2)^{-1}\, f(n,m) \qquad \text{// completeness}$$

Then to this we can add

$$f(\theta,\varphi) = \Sigma_{n=0}^{\infty}\, \Sigma_{m=-n}^{n}\, f_{nm}\ P_n^m(\cos\theta)\, e^{im\varphi} \qquad \text{// expansion}$$

$$f_{nm} = (2n+1)\ f(n,-m)\ (1/4\pi) \int d\Omega\, f(\theta,\varphi)\, e^{-im\varphi}\, P_n^m(z) \qquad \text{// projection}$$

Data from Dan Zwillinger's Book

He has two good sections which just summarize things, this is a good place to add all that info, maybe just as a check on my stuff above.

Table 77.1: Different transform pairs of the form

$$v(\xi_k) = \int_\alpha^\beta K(x, \xi_k)\, u(x)\, dx, \qquad u(x) = \sum_{\xi_k} H(x, \xi_k)\, v(\xi_k).$$

Finite cosine transform – 1, (see Miles [5, page 86]) here l and h are arbitrary, and the $\{\xi_k\}$ satisfy $\xi_k \tan \xi_k l = h$.

$$v(\xi_k) = \int_0^1 \cos(x\xi_k)\; u(x)\, dx, \quad u(x) = \sum_{\xi_k} \frac{(2 - \delta_{\xi_k 0})(\xi_k^2 + h^2)\cos(\xi_k x)}{h + l(\xi_k^2 + h^2)}\, v(\xi_k).$$

Finite cosine transform – 2, (see Butkov [1, page 161]) this is the last transform with $h = 0$, $l = 1$, so that $\xi_k = 0, \pi, 2\pi, \ldots$.

$$v(\xi_k) = \int_0^1 \cos(x\xi_k)\; u(x)\, dx, \quad u(x) = \sum_{\xi_k} (2 - \delta_{\xi_k 0}) \cos(\xi_k x)\; v(\xi_k).$$

Finite sine transform – 1, (see Miles [5, page 86]) here l and h are arbitrary, and the $\{\xi_k\}$ satisfy $\xi_k \cot(\xi_k l) = -h$.

$$v(\xi_k) = \int_0^1 \sin(x\xi_k)\; u(x)\, dx, \quad u(x) = \sum_{\xi_k} 2\frac{(\xi_k^2 + h^2)\sin(\xi_k x)}{h + l(\xi_k^2 + h^2)}\, v(\xi_k).$$

Finite sine transform – 2, (see Butkov [1, page 161]) this is the last transform with $h = 0$, $l = 1$, so that $\xi_k = 0, \pi, 2\pi, \ldots$.

$$v(\xi_k) = \int_0^1 \sin(x\xi_k)\; u(x)\, dx, \quad u(x) = \sum_{\xi_k} 2\sin(\xi_k x)\; v(\xi_k).$$

Finite Hankel transform – 1, (see Tranter [8, page 88]) here n is arbitrary and the $\{\xi_k\}$ are positive and satisfy $J_n(\xi_k) = 0$.

$$v(\xi_k) = \int_0^1 x J_n(x\xi_k)\, u(x)\, dx, \quad u(x) = \sum_{\xi_k} 2\frac{J_n(x\xi_k)}{J_{m+1}^2(\xi_k)}\, v(\xi_k).$$

Finite Hankel transform – 2, (see Miles [5, page 86]) here n and h are arbitrary and the $\{\xi_k\}$ are positive and satisfy $\xi_k J_n'(a\xi_k) + hJ_n(a\xi_k) = 0$.

$$v(\xi_k) = \int_0^a x J_n(x\xi_k)\, u(x)\, dx, \quad u(x) = \sum_{\xi_k} \frac{2\xi_k^2 J_n(x\xi_k)}{\{(h^2 + \xi_k^2)\, a^2 - m^2\}\, J_n^2(a\xi_k)}\, v(\xi_k).$$

Finite Hankel transform − 3, (see Miles [5, page 86]) here $b > a$, the $\{\xi_k\}$ are positive and satisfy $Y_n(a\xi_k)J_n(b\xi_k) = J_n(a\xi_k)Y_n(b\xi_k)$, and $Z_n(x\xi_k) := Y_n(a\xi_k)J_n(x\xi_k) - J_n(a\xi_k)Y_n(x\xi_k)$.

$$v(\xi_k) = \int_a^b xZ_n(x\xi_k)\,u(x)\,dx, \quad u(x) = \sum_{\xi_k} \frac{\pi^2}{2}\frac{\xi_k^2 J_n^2(b\xi_k)Z_n(x\xi_k)}{J_n^2(a\xi_k) - J_n^2(b\xi_k)}\,v(\xi_k).$$

Legendre transform, (see Miles [5, page 86]) here $\xi_k = 0, 1, 2, \ldots$.

$$v(\xi_k) = \int_{-1}^1 P_{\xi_k}(x)\,u(x)\,dx, \quad u(x) = \sum_{\xi_k} \frac{2\xi_k + 1}{2} P_{\xi_k}(x)\,v(\xi_k).$$

Table 78.1: Different integral transform pairs of the form
$v(\xi) = \int_\alpha^\beta K(x,\xi)u(x)\,dx, \qquad u(x) = \int_a^b H(x,\xi)v(\xi)\,d\xi.$

Fourier transform, (see Butkov [3, Chapter 7])

$$v(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\, u(x)\, dx, \quad u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, v(\xi)\, d\xi.$$

Fourier cosine transform, (see Butkov [3, page 274])

$$v(\xi) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(x\xi)\, u(x)\, dx, \quad u(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(x\xi)\, v(\xi)\, d\xi.$$

Fourier sine transform, (see Butkov [3, page 274])

$$v(\xi) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(x\xi)\, u(x)\, dx, \quad u(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(x\xi)\, v(\xi)\, d\xi.$$

Hankel transform, (see Sneddon [22, Chapter 5])

$$v(\xi) = \int_{0}^{\infty} xJ_\nu(x\xi)\, u(x)\, dx, \quad u(x) = \int_{0}^{\infty} \xi J_\nu(x\xi)\, v(\xi)\, d\xi.$$

Hilbert transform, (see Sneddon [22, pages 233–238])

$$v(\xi) = \int_{-\infty}^{\infty} \frac{1}{\pi(x-\xi)}\, u(x)\, dx, \quad u(x) = \int_{-\infty}^{\infty} \frac{1}{\pi(\xi-x)}\, v(\xi)\, d\xi.$$

***K*–transform**, (see Erdélyi [7])

$$v(\xi) = \int_{0}^{\infty} K_\nu(x\xi)\sqrt{\xi x}\, u(x)\, dx, \quad u(x) = \frac{1}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} I_\nu(x\xi)\sqrt{\xi x}\, v(\xi)\, d\xi.$$

Kontorovich–Lebedev transform, (see Sneddon [22, Chapter 6])

$$v(\xi) = \int_0^\infty \frac{K_{i\xi}(x)}{x}\, u(x)\, dx, \quad u(x) = \frac{2}{\pi^2} \int_0^\infty \xi \sinh(\pi\xi) K_{i\xi}(x)\, v(\xi)\, d\xi.$$

Kontorovich–Lebedev transform (alternative form), (see Jones [12])

$$v(\xi) = \int_0^\infty H_\xi^{(2)}(x)\, u(x)\, dx, \quad u(x) = -\frac{1}{2x} \int_{-i\infty}^{i\infty} \xi J_\xi(x)\, v(\xi)\, d\xi.$$

Laplace transform, (see Sneddon [22, Chapter 3])

$$v(\xi) = \int_0^\infty e^{-x\xi}\, u(x)\, dx, \quad u(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{x\xi}\, v(\xi)\, d\xi.$$

Mehler–Fock transform of order m, (see Sneddon [22, Chapter 7])

$$v(\xi) = \int_0^\infty \sinh(x) P^m_{i\xi-1/2}(\cosh x)\, u(x)\, dx,$$
$$u(x) = \int_0^\infty \xi \tanh(\pi\xi) P^m_{i\xi-1/2}(\cosh x)\, v(\xi)\, d\xi.$$

Mellin transform, (see Sneddon [22, Chapter 4])

$$v(\xi) = \int_0^\infty x^{\xi-1}\, u(x)\, dx, \quad u(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-\xi}\, v(\xi)\, d\xi.$$

Weber formula, (see Titchmarsh [24, page 75])

$$v(\xi) = \int_a^\infty \sqrt{x}\, [J_\nu(x\xi) Y_\nu(a\xi) - Y_\nu(x\xi) J_\nu(a\xi)]\; u(x)\, dx,$$
$$u(x) = \sqrt{x} \int_0^\infty \frac{J_\nu(x\xi) Y_\nu(a\xi) - Y_\nu(x\xi) J_\nu(a\xi)}{J^2_\nu(a\xi) + Y^2_\nu(a\xi)}\, v(\xi)\, d\xi.$$

Weierstrass transform, (see Hirschman and Widder [10, Chapter 8])

$$v(\xi) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^\infty e^{(\xi-x)^2/4}\, u(x)\, dx, \quad u(x) = \frac{1}{\sqrt{4\pi}} \lim_{T\to\infty} \int_{-T}^T e^{(x-i\xi)^2/4} v(i\xi)\, d\xi.$$

Unnamed transform, (see Naylor [20])

$$v(\xi) = \int_{-\infty}^\infty K_0(|\xi - x|)\, u(x)\, dx, \quad u(x) = -\frac{1}{\pi^2} \left(\frac{d^2}{dx^2} - 1\right) \int_{-\infty}^\infty K_0(|\xi - x|)\, v(\xi)\, d\xi.$$

Unnamed transform, (see Titchmarsh [24, page 83])

$$v(\xi) = \int_{-\infty}^\infty \left[J_{i\sqrt{\xi}}(e^x) + J_{-i\sqrt{\xi}}(e^x)\right]\; u(x)\, dx,$$
$$u(x) = \int_0^\infty \frac{J_{i\sqrt{\xi}}(e^x) + J_{-i\sqrt{\xi}}(e^x)}{4\sinh\left(\pi\sqrt{\xi}\right)}\, v(\xi)\, d\xi.$$

Transform Repair Done 12/8/15.

I have errors in my completeness relations in some cases. You really want to stick with the Stakgold general theory.

Although I no longer have the details in core storage, Stakgold *clearly* says this in Section 4.2

1. $\psi_n(x) = \sqrt{s(x)}\,\varphi_n(x)$ form a **complete orthonormal set**.

   That means that $\int_a^b dx\ \psi_n(x)\,\psi_m(x) = \delta_{n,m}$

2. The correct statement of **completeness** is then

   $\delta(x-\xi) = \Sigma_n\ \psi_n(x)\,\overline{\psi}_n(\xi)$

Now consider the transforms I talk about in transforms.doc.

Some of the completeness relations I have written do not have this form!

I will start repairing the sections one at a time (but won't finish right now)

The Exponential Fourier Series Transform

$(1/2\pi)\ \int_{-\pi}^{\pi} d\theta\ e^{in\theta}\,e^{-in'\theta} = \delta_{nn'}$ // orthogonality

$(1/2\pi)\ \Sigma_n\ e^{-in\theta'}\,e^{in\theta} = \delta(\theta-\theta')$ // completeness

So here we have $s(x) = 1$, and $\psi_n(\theta) = 1/\sqrt{2\pi}\ e^{in\theta}$. Then can rewrite

$$\int_{-\pi}^{\pi} d\theta\ \psi_n(\theta)\ \psi_{n'}{}^*(\theta) = \delta_{nn'}$$

$$\Sigma_n\ \psi_n(\theta')\ \psi_n{}^*(\theta) = \delta(\theta-\theta')$$

So no problems there.

The Fourier Series Transform for real functions.

This is just a translation of the previous section where I write things out and specialize to the face the the functions of interest $f(\theta)$ are real. We stay on the full interval $(-\pi,\pi)$.

The Fourier Cosine Series Transform

The interval here is now really $(0,\pi)$ and we use only cosine basis functions. So

$$\psi_n(\theta) = \sqrt{\varepsilon_n/\pi}\cos(n\theta)$$

$$\int_0^{\pi} d\theta \ \psi_n(\theta) \ \psi_{n'}^*(\theta) \ = \delta_{nn'}$$

so that

$$\sqrt{\varepsilon_n/\pi}\sqrt{\varepsilon_{n'}/\pi} \ \int_0^{\pi} d\theta \cos(n\theta)\cos(n'\theta) \ = \delta_{nn'}$$

or

$$\int_0^{\pi} d\theta \cos(n\theta)\cos(n'\theta) = (\pi/\varepsilon_n)\,\delta_{nn'}$$

which is something you can actually look up.

Therefore. the completeness relation must be:

$$\Sigma_n \ \psi_n(\theta) \ \psi_n^*(\theta') = \delta(\theta-\theta')$$

or

$$\Sigma_n \ \sqrt{\varepsilon_n/\pi}\cos(n\theta)\sqrt{\varepsilon_n/\pi}\cos(n\theta) = \delta(\theta-\theta')$$

or

$$\Sigma_n \ (\varepsilon_n/\pi)\cos(n\theta)\cos(n\theta) = \delta(\theta-\theta')$$

or

$$\Sigma_n \ \varepsilon_n\cos(n\theta)\cos(n\theta) = \pi\,\delta(\theta-\theta')$$

So I have written in this correction, and have no time right now to complete these repairs.