Attempt to analyze section "Pullbacks re-examined" on page 88 PhL 8.17.15

Summary: If I am willing to simply *define* $g_J \equiv \varphi^*\alpha(\mathbf{e}_J)$ as a page 88 definition of g_J, I am able to show that this g_J agrees exactly with the Chapter 3 result for g_J which was $g_J(\mathbf{x}) \equiv \Sigma_I\ f_I[\varphi(x)]\ \det(D\varphi)_{I,J}$. But the whole business is very cloudy since in Chapter 3 things are functions of **x**, whereas in Chapter 7 page 88 things are functions of those k multilinear arguments. I am really unable to make a clean connection between the two Worlds. Maybe this will get clarified later. I gave it my best shot today. I could go off and read some other detailed source like Spivak, but that would take many days I suspect. Maybe Sjamaar will clarify things in upcoming chapters.

Start with page 87 A and page 88 B which combined say

$$\alpha_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \Sigma_I\ \alpha_{\mathbf{x}}(\mathbf{e}_I)\ dx_I(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) \qquad // \ \alpha_{\mathbf{x}}(\mathbf{e}_I) = f_I(\mathbf{x}) \tag{1a}$$

2.17.16: In case this idea is missed, here is a proof of the above expansion. Assume it is valid, then evaluate (1a) at this point in function space,

$$\alpha_{\mathbf{x}}(\mathbf{e}_1,\mathbf{e}_2.....\mathbf{e}_k) = \Sigma_I\ \alpha_{\mathbf{x}}(\mathbf{e}_I)\ dx_I(\mathbf{e}_1,\mathbf{e}_2.....\mathbf{e}_k) = \Sigma_I\ \alpha_{\mathbf{x}}(\mathbf{e}_I)\ \lambda^I(\mathbf{e}_Z) = \Sigma_I\ \alpha_{\mathbf{x}}(\mathbf{e}_I)\ \delta^I{}_Z$$

$$= \alpha_{\mathbf{x}}(\mathbf{e}_Z)$$

We then end up with a consistent result that $\alpha_{\mathbf{x}}(\mathbf{e}_Z) = \alpha_{\mathbf{x}}(\mathbf{e}_Z)$.

But in the spirit of the **x** and **y** use in the pullback discussion of Chapter 2, replace the above with

$$\alpha_{\mathbf{y}}(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \Sigma_I\ \alpha_{\mathbf{y}}(\mathbf{e}_I)\ dy_I(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) \qquad \mathbf{y} = \boldsymbol{\varphi}(\mathbf{x}) \tag{1b}$$

or

$$\alpha_{\mathbf{y}}(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \Sigma_I\ f_I(\mathbf{y})\ dy_I(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) \tag{1c}$$

which in short has the Chapter 3 pre-pullback form of

$$\alpha = \Sigma_I\ f_I(\mathbf{y})dy_I\ . \qquad // \text{ each side is a functional, no tensor function arguments} \tag{1d}$$

Evaluate (1b) at $\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k = \mathbf{e}_J$ to get

$$\alpha_{\mathbf{y}}(\mathbf{e}_J) = \Sigma_I\ \alpha_{\mathbf{y}}(\mathbf{e}_I)\ dy_I(\mathbf{e}_J)\ . \tag{2}$$

Now do the pullback mimicking Chapter 3

$$\varphi^*\alpha_{\mathbf{y}}(\mathbf{e}_J) = \Sigma_I\, \varphi^*[\alpha_{\mathbf{y}}(\mathbf{e}_I)]\, [\varphi^* dy_I(\mathbf{e}_J)] \ . \qquad (3)$$

Recall that we used to say (in Ch 3) that $\varphi^* f(\mathbf{y}) = f(\boldsymbol{\varphi}(\mathbf{x}))$ for a 0-form or function. That is to say, we just replaced the argument $\mathbf{y}$ with $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ which was the forward transformation. And for a differential we did the replacement $\varphi^* dy_I = d\varphi_I = (D\varphi)dx_I$. This is discussed best in my Ch 3 meta notes I think.

Hypothesize now in our new situation this parallel idea,

$$\varphi^*[\alpha_{\mathbf{y}}(\mathbf{e}_I)] = \alpha_{\boldsymbol{\varphi}(\mathbf{x})}(\mathbf{e}_I) \qquad \text{// that is, } y \to \varphi(x) \text{ in the label of } \alpha \qquad (4)$$

or $\varphi^*[\, f_I(y)] = f_I(\varphi(x))$

and

$$[\varphi^* dy_I(\mathbf{e}_J)] = dy_I(\, (D\varphi)\mathbf{e}_{j1},\ (D\varphi)\mathbf{e}_{j2}, .,,,,\ (D\varphi)\mathbf{e}_{jk}) \qquad (5)$$

Comment 2.15.16: Compare this to our Ch 3 meta notes result

$$\varphi^* dy_I = d\varphi_I(\mathbf{x}) = \Sigma_J \det(R_{I,J})\, dx_J = \Sigma_J\, [\det[(D\varphi)_{I,J}]\, dx_J \qquad \text{// ordered sum on J}$$

In Chapter 7 the above would be interpreted as a statement about functionals,

$$\varphi^* dy_I = d\varphi_I(\mathbf{x}) = \Sigma_J \det(R_{I,J})\, \lambda^J = \Sigma_J\, [\det[(D\varphi)_{I,J}]\, \lambda^J$$

and you could close this onto e_K to get the following tensor function statement,

$$[\varphi^* dy_I](\mathbf{e}_K) = d\varphi_I(\mathbf{x})(\mathbf{e}_K) = \Sigma_J \det(R_{I,J})\, \lambda^J(\mathbf{e}_K) = \Sigma_J\, [\det[(D\varphi)_{I,J}]\, \lambda^J(\mathbf{e}_K)$$

$$= \Sigma_J\, [\det[(D\varphi)_{I,J}]\, \delta^J{}_K = \det[(D\varphi)_{I,K}]$$

which would say

$$[\varphi^* dy_I](\mathbf{e}_J) = d\varphi_I(\mathbf{x})(\mathbf{e}_J) = \det[(D\varphi)_{I,J}]$$

Now maybe we can show that

$$\det[(D\varphi)_{I,J}] \ =?= \ dy_I(\, (D\varphi)\mathbf{e}_{j1},\ (D\varphi)\mathbf{e}_{j2}, .,,,,\ (D\varphi)\mathbf{e}_{jk}) = dy_I(\ (D\varphi)\mathbf{e}_J\ )$$

$$= \lambda^I\, [(D\varphi)\mathbf{e}_J\,] = \lambda^{i1}\, [\, (D\varphi)\mathbf{e}_{j1}]\, \lambda^{i2}\, [(D\varphi)\mathbf{e}_{j2}] \ .....$$

$$= [\, (D\varphi)\mathbf{e}_{j1}]^{i1}\ [\, (D\varphi)\mathbf{e}_{j2}]^{i2} \ ....$$

but I don't see how det is going to appear. Pause on this. End comment. [but I do this detail below!]

Based on these assumptions, we then find that

$$\varphi^*\alpha_{\mathbf{y}}(\mathbf{e}_J) = \Sigma_I\, \varphi^*[\alpha_{\mathbf{y}}(\mathbf{e}_I)]\, [\varphi^* dy_I(\mathbf{e}_J)]$$

$$= \Sigma_I \ \alpha_{\varphi(x)}(\mathbf{e}_I) \ dy_I((D\varphi)\mathbf{e}_{j1}, \ (D\varphi)\mathbf{e}_{j2}, .,,,, \ (D\varphi)\mathbf{e}_{jk}) \quad . \qquad (6)$$

Now looking back at (1b), make substitutions in (1b) to write

$$\alpha_{\varphi(x)}((D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}) = \Sigma_I \ \alpha_{\varphi(x)}(\mathbf{e}_I) \ dx_I((D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}) \ . \qquad (7)$$

Now (6) and (7) have the same RHS so we may equate the LHS's to get

$$\varphi^*\alpha_y(\mathbf{e}_J) = \alpha_{\varphi(x)}[(D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}] \qquad (8a)$$

I guess you could *define*

$(\varphi^*\alpha)_x \equiv \varphi^*[\alpha_y] = \varphi^*[\alpha_{\varphi(x)}]$ functions of k vector arguments. Then you get

$$(\varphi^*\alpha)_x(\mathbf{e}_J) = \alpha_{\varphi(x)}[(D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}] \qquad (8a)$$

It seems to me a weird way to label the left side, but it is certainly allowable. I guess the idea is that the object $(\varphi^*\alpha)$ is some new form, and it should be a tack-on **x** label like any other form, hence $(\varphi^*\alpha)_x$

My (8a) above is **waypoint 1** which I have checked in red on page 88 as ✓1. Now recall,

$$\alpha_y(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \Sigma_I \ f_I(\mathbf{y}) \ dy_I(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) \qquad (1c)$$

Substitute into this $\mathbf{v}_1 = (D\varphi)\mathbf{e}_{j1}$ and $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ to get

$$\alpha_{\varphi(x)}((D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}) = \Sigma_I \ f_I(\boldsymbol{\varphi}(\mathbf{x})) \ dy_I[(D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}] \qquad (9)$$

Now use this in equation (8a) to obtain

$$(\varphi^*\alpha)_x(\mathbf{e}_J) = \Sigma_I \ f_I(\boldsymbol{\varphi}(\mathbf{x})) \ dy_I[(D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}] \qquad (10)$$

This matches **waypoint 2** so another red check #2 goes on page 88 near 88 A.

Now go back to page 85C where we would say

$$"dy_1dy_2 \ ...dy_k"(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \det[\ dy_i(v_j)]$$

$$dy_I(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \det[\ dy_I(\mathbf{v}_j)] \qquad (11a)$$

The matrix inside the determinant here is this

$dy_{i1}(\mathbf{v}_1)$	$dy_{i1}(\mathbf{v}_2)$	...	$dy_{i1}(\mathbf{v}_k)$
$dy_{i2}(\mathbf{v}_1)$	$dy_{i2}(\mathbf{v}_2)$	...	$dy_{i2}(\mathbf{v}_k)$
...			

$$dy_{ik}(\mathbf{v}_1) \qquad dy_{ik}(\mathbf{v}_2) \qquad \ldots \qquad dy_{ik}(\mathbf{v}_k) \tag{11b}$$

Now in (11a) and (11b) replace $\mathbf{v}_1 \rightarrow (D\varphi)\mathbf{e}_{j1}$ and so on to get this new equation and matrix

$$dy_I((D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}) = \det[\, dy_I((D\varphi)\mathbf{e}_{j1})]$$

or in shorthand,

$$dy_I(\, (D\varphi)\mathbf{e}_J) = \det\, [dy_I((D\varphi)e_J)] \tag{12a}$$

where the matrix is

$$\begin{matrix} dy_{i1}((D\varphi)\mathbf{e}_{j1}) & dy_{i1}((D\varphi)\mathbf{e}_{j2}) & \ldots & dy_{i1}((D\varphi)\mathbf{e}_{jk}) \\ dy_{i2}((D\varphi)\mathbf{e}_{j1}) & dy_{i2}((D\varphi)\mathbf{e}_{j2}) & \ldots & dy_{i2}((D\varphi)\mathbf{e}_{jk}) \\ \ldots & & & \\ dy_{ik}((D\varphi)\mathbf{e}_{j1}) & dy_{ik}((D\varphi)\mathbf{e}_{j2}) & \ldots & dy_{ik}((D\varphi)\mathbf{e}_{jk}) \end{matrix} \tag{12b}$$

But for example

$$dy_{i1}((D\varphi)\mathbf{e}_{j2}) = e^{T}_{i1}((D\varphi)\mathbf{e}_{j2} = (D\varphi)_{i1,j2} \tag{13}$$

Then from (12a) and (12b) we end up with

$$dy_I(\, (D\varphi)\mathbf{e}_J) = \det(D\varphi)_{I,J}$$

Now insert this into (10) to get

$$(\varphi^*\alpha\,)_{\mathbf{x}}(\mathbf{e}_J) = \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{x}))\, dy_I[(D\varphi)\mathbf{e}_{j1}, (D\varphi)\mathbf{e}_{j2}..... (D\varphi)\mathbf{e}_{jk}] \qquad \textit{(10)}$$

$$= \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{x}))\ \det(D\varphi)_{I,J}$$

$$= \Sigma_I\, \varphi^* f_I(\mathbf{x})\ \det(D\varphi)_{I,J} \tag{14}$$

So here is what I have shown in all the above (where $y = \varphi(x)$)

$$(\varphi^*\alpha\,)_{\mathbf{x}}(\mathbf{e}_J) \equiv \varphi^*[\alpha_{\mathbf{y}}(\mathbf{e}_J)] = \Sigma_I\, \varphi^* f_I(\mathbf{x})\ \det(D\varphi)_{I,J} \tag{15a}$$

This agrees with **waypoint** red checkmark #4. Another way to write the above is

$$(\varphi^*\alpha\,)_{\mathbf{x}}(\mathbf{e}_J) \equiv \varphi^*[\alpha_{\mathbf{y}}(\mathbf{e}_J)] = \Sigma_I\ f_I(\boldsymbol{\varphi}(\mathbf{x}))\ \det(D\varphi)_{I,J} \tag{15b}$$

However, nowhere have I referred to this as g_J. I have simply derived it as you see it in (15).

Now what was our result from Chapter 3 on pullbacks?

$$\varphi^*\alpha = \varphi^*[\ \Sigma_I\ f_I\, dy_I\ ] = \Sigma_I\ \varphi^*[f_I]\,\varphi^*[dy_I] = \Sigma_I\, f_I[\varphi(x)]\, d\varphi_I = \Sigma_I\, f_I[\varphi(x)]\, \Sigma_J\ \det(D\varphi)_{I,J}\, dx_J$$

$$= \Sigma_{I,J}\ f_I[\varphi(x)]\, \det(D\varphi)_{I,J}\, dx_J$$

$$= \Sigma_J\, g_J(\mathbf{x})\, dx_J \quad \text{where} \quad g_J(\mathbf{x}) \equiv \Sigma_I\ f_I[\varphi(x)]\, \det(D\varphi)_{I,J} \tag{16}$$

This does show that

$$g_J(\mathbf{x})^{\text{of Chapter 3}} = (\varphi^*\alpha)_{\mathbf{x}}(\mathbf{e}_J)^{\text{of page 88}} \tag{17}$$

Now, in Chapter 3 we had, as shown above in (16)

$$\varphi^*\alpha = \Sigma_J\, g_J(\mathbf{x})\, dx_J \qquad g_J(\mathbf{x}) \equiv \Sigma_I\ f_I[\varphi(x)]\, \det(D\varphi)_{I,J} \qquad \text{// chapter 3} \tag{18}$$

What is the page 88 equation corresponding to the left equation in 18?

I want to find something like this:

$$\varphi^*\alpha = \Sigma_I\, f_I[\varphi(x)]\, \Sigma_J\ \det(D\varphi)_{I,J}\, dx_J \qquad \text{// Chapter 3 pullback result}$$

I do have this result with *general* arguments:

$$\varphi^*\alpha_{\mathbf{y}}(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \Sigma_I\ \alpha_{\varphi(\mathbf{x})}(\mathbf{e}_I)\ dy_I(\ (D\varphi)\mathbf{v}_1,\ (D\varphi)\mathbf{v}_2,\ .,,,,\ (D\varphi)\mathbf{v}_k) \qquad .$$

$$= \Sigma_I\, f_I(\varphi(x))\ dy_I(\ (D\varphi)\mathbf{v}_1,\ (D\varphi)\mathbf{v}_2,\ .,,,,\ (D\varphi)\mathbf{v}_k) \tag{19}$$

If these are general vectors $\mathbf{v}_i$, I don't know what to do next! But if I evaluate the arguments at $\mathbf{v}_1 = \mathbf{e}_{j1}$ and so on I get

$$\varphi^*\alpha_{\mathbf{y}}(\mathbf{e}_{j1},\mathbf{e}_{j2}.....\mathbf{e}_{jk}) = \Sigma_I\ \alpha_{\varphi(\mathbf{x})}(\mathbf{e}_I)\ dy_I(\ (D\varphi)\mathbf{e}_{j1},\ (D\varphi)\mathbf{e}_{j2},\ .,,,,\ (D\varphi)\mathbf{e}_{jk})\ .$$

$$= \Sigma_I\ f_I(\varphi(x))\ \det(D\varphi)_{I,J}$$

or

$$\varphi^*\alpha_{\mathbf{y}}(\mathbf{e}_J) = \Sigma_I\ f_I(\varphi(x))\ \det(D\varphi)_{I,J}$$

Plan A. Now if we were to mult both sides by dx_J and sum over ordered index J, we would get

$$\Sigma_J\, \varphi^*\alpha_{\mathbf{y}}(\mathbf{e}_J)dx_J = \Sigma_{I,J}\ f_I(\varphi(x))\ \det(D\varphi)_{I,J}\, dx_J = \varphi^*\alpha \quad \text{of chapter 3.}$$

Plan B. I could try this

$$\det(D\varphi)_{I,J} = \Sigma_J\ \det(D\varphi)_{I,J}\, \delta_{I,J} = \Sigma_J\ \det(D\varphi)_{I,J}\, dx_J(\mathbf{e}_I)$$

Then my page 88 result becomes

Chapter 3: $\varphi^*[dy_I] = d\varphi_I = \Sigma_J \det(D\varphi)_{I,J}\, dx_J$

Page 88: $[\varphi^* dy_I(\mathbf{e}_J)] = \Sigma_J \det(D\varphi)_{I,J}\, dx_J(\mathbf{e}_I)$

Now they look close. But it is not really correct.

Plan C. Try continuing from (18). We started above with

$$\alpha_{\mathbf{y}}(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \Sigma_I\, f_I(\mathbf{y})\, dy_I(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) \qquad \textit{(1c)}$$

Had we applied the pullback to this thing directly with general arguments, we would have gotten

$$\varphi^*\alpha_{\mathbf{y}}(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = \Sigma_I\ \alpha_{\varphi(\mathbf{x})}(\mathbf{e}_I)\ dy_I(\ (D\varphi)\mathbf{v}_1,\ (D\varphi)\mathbf{v}_2, .,,,,,\ (D\varphi)\mathbf{v}_k) \quad .$$

$$= \Sigma_I\, f_I(\varphi(x))\ dy_I(\ (D\varphi)\mathbf{v}_1,\ (D\varphi)\mathbf{v}_2, .,,,,,\ (D\varphi)\mathbf{v}_k) \qquad (19)$$

Can we just apply the multilinear rule to this last function? Then consider

$$dy_I(\ (D\varphi)\mathbf{v}_1,\ (D\varphi)\mathbf{v}_2, .,,,,,\ (D\varphi)\mathbf{v}_k)$$

The rule is only helpful if you could express $(D\varphi)\mathbf{v}_1$ as a linear combination of vectors. But certainly

$(D\varphi)\mathbf{v}_1 = \Sigma_k\, q_{1k}\, \mathbf{e}_k$ // since e_k is a basis

Now we know

$$(D\varphi)\mathbf{v}_1 = (D\varphi)\,\Sigma_k(\mathbf{v}_1)_k\mathbf{e}_k = \Sigma_k(\mathbf{v}_1)_k\,[\ (D\varphi)\mathbf{e}_k\ ]$$

Now dive to the next level and write

$$(D\varphi)\mathbf{e}_k = \Sigma_i\, s_{ki}\, \mathbf{e}_i$$

$$\mathbf{e}_j^{\mathbf{T}}\,(D\varphi)\mathbf{e}_k = \Sigma_i\, s_{ki}\, \mathbf{e}_j^{\mathbf{T}}\mathbf{e}_i = s_{kj} = (D\varphi)_{j,k}$$

Thus we have that

$$(D\varphi)\mathbf{e}_k = \Sigma_i(D\varphi)_{i,k}\,\mathbf{e}_i$$

Now go back a few steps and we have shown that

$$(D\varphi)\mathbf{v}_1 = \Sigma_k(\mathbf{v}_1)_k\,[\ (D\varphi)\mathbf{e}_k\ ]$$

$$= \Sigma_k(\mathbf{v}_1)_k\,\Sigma_i(D\varphi)_{i,k}\,\mathbf{e}_i$$

$$= \Sigma_i(\mathbf{v}_1)_i\,\Sigma_k(D\varphi)_{k,i}\,\mathbf{e}_k \qquad \text{// swap indices}$$

$= \Sigma_k [\Sigma_i (v_1)_i (D\varphi)_{k,i}] e_k$

$= \Sigma_k [q_{1k}] e_k \qquad q_{1k} \equiv \Sigma_i (v_1)_i (D\varphi)_{k,i}$

So I have finally succeeded in writing $(D\varphi)v_1$ as a linear combination of vectors. Then I guess I can say

$dy_I((D\varphi)v_1, (D\varphi)v_2, .,,,, (D\varphi)v_k)$

$= \Sigma_k q_{1k} \; dy_I(e_k, (D\varphi)v_2, .,,,, (D\varphi)v_k)$

Now replace summation index k by j_1 so we then have

$= \Sigma_{j1} \; q_{1,j1} \; dy_I(e_{j1}, (D\varphi)v_2, .,,,, (D\varphi)v_k)$

Then I could do this on all the arguments and end up with

$= \Sigma_{j1,j2...jk} \; q_{1,j1} \, q_{2,j2} .... \; q_{k,jk} \; dy_I(e_{j1}, e_{j2}, .,,,, e_{jk})$

Now things get fuzzy. I is an ordered index, but J here is not an ordered index. Give up.

Plan D Sja mysteriously makes this definition

$g_J^{p88} \equiv \varphi^*\alpha(e_J)$

which I interpret to mean

$g_J^{p88} \equiv (\varphi^*\alpha)_x(e_J) \equiv \varphi^*[\alpha_y(e_J)]$

Given this definition, I have shown above that

$g_J^{p88} = \Sigma_I \varphi^* f_I(x) \; \det(D\varphi)_{I,J}$ = exactly the same as g_J^{ch3}

The problem is that the definitions of the two g_J seem unrelated:

$\varphi^*\alpha = \Sigma_J g_J^{ch3}(x) \, dx_J$ // chapter 3 definition

$\varphi^*\alpha(e_J) = g_J^{p88}$ // page 88 definition

So I am not really happy with Sja's claim that "therefore our new definition agrees with the old" unless he means our new definition of g_J. He writes $\varphi^*\alpha = \Sigma_J g_J dx_J$ as page 88 C but this is not consistent with

$\varphi^*\alpha(e_J) = g_J$ page 88 D

because then you would have to get

$\varphi^*\alpha = \Sigma_J \varphi^*\alpha(\mathbf{e}_J)\, dx_J$

but dx_J does not appear in any of his page 88 equations!

Conclusion: For now I am going to have to "let this go" because I don't have enough information.

2.15.16. Reviewed the above, good to have this detail in case I someday want to do a complete rewrite of this stuff, but not crucial at the moment.