Essay on the wedge product PhL 7.16.15

I don't really have time to dig into this topic right now. I looked at about 10 sources, and none of them is willing to lay this subject out in a way that makes sense to me. One good source is the book of Spivak who defines the wedge product in this manner

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!}\,\mathrm{Alt}(\omega \otimes \eta).$$

Here the leading factor is some counting thing (k+ℓ,k), Alt means to antisymmetrize things (and average). The ω and η here are already TA symmetrized in their variable indices. See Spivak notes elsewhere. I looked more at this book, and here the wedge product seems to apply to **functions** not vectors.

The wiki exterior algebra presentation talks about **u** ^ **v** as an operation between two **vectors**, and **u** ^ **v** is called a 2-vector or a 2-blade. The magnitude is the same as that of **u** x **v** and is the area of the 2-blade. The area is regarded as oriented so there is a sign issue.

In the world of differential forms, both **Buck** and **Sjamaar** avoided using the ^ symbol or even talking about wedge products, since then they would have to explain them which I see is a tough task.

So for now, here is my reading on the wedge product as used in differential forms.

1. When we write dx ^ dy for E^2, this is a signed area ±dxdy. The x and y axes are orthogonal so this is then vaguely consistent with the **u** x **v** area idea, although of course dx and dy are not vectors.

2. Similarly dx ^ dy ^ dz is a signed volume, and similarly in any number of dimensions.

3. At some point you have to talk about actual multivariable integration. You have to decide which ordering of the variables gets a positive sign for the integration.

4. Here is a suggested rule. First, get your integration integrand into a *standard order*:

$$\int f(x_1,x_2,....x_n)\, dx_1 \wedge dx_2 \wedge .... \wedge dx_n$$

Once this is done, THEN you erase the ^ products and just do the integral like this

$$= \int f(x_1,x_2,....x_n)\, dx_1 dx_2 \ldots dx_n = \int\int\int \ldots \int f(x_1,x_2,....x_n)\, dx_1 dx_2 \ldots dx_n$$

$$= \int dx_1 \int dx_2 \ .... \int dx_n \ f(x_1,x_2,....x_n)$$

4. You can think if a point **x** = $(x_1,x_2,....x_n)$ being in R^n = R ⊗ R ⊗....⊗ R = direct product space

5. For non-Cartesian coordinates, like polars, you might still have $R^2 = R \otimes R$ and you have to decide what the standard ordering should be. A rule of thumb in that case might be to select the ordering that makes the determinant be positive. Thus would just be a convention. For example, in tensor doc I show that for polars if you take the order θ,r then det(S) = -r (page 60), so the right order is then r,θ.

6. This suggested rule is supported a bit by other documents. Here from Arapura for spherical coordinates:

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(\phi,\theta)} d\phi \wedge d\theta = \cos(\phi)\sin(\phi) d\phi \wedge d\theta$$

So

$$\iint_S dx \wedge dy = \int_0^{2\pi} \int_0^{\pi/2} \cos(\phi)\sin(\phi) d\phi d\theta = \pi$$

where he has cos(φ) where it should say cos(θ) in both places. Integral is over a spherical surface. The point here is that you maintain the ^ symbol even after you convert to non-Cartesian variables, you get them into the right order, so you have a positive determinant, and THEN you erase the ^ symbols. In the notation here with dφ ^ dθ you can think of my Cartesian parameter space where θ and φ are orthogonal, just as x and y are orthogonal, and it is R ⊗ R.

Again, you are just selecting what sign you want for your integral, what orientation.

In the above we have φ and θ as some parameters. In the Skimaar general stuff, those parameters are called t_1, t_2.....t_k. You get them into standard order the THEN erase the ^ symbols and do the integral.

Here is another Arapura example (he has moved to manifolds, but no matter)

THEOREM 19.1 (Generalized Stokes' theorem) *If M is an oriented k manifold with boundary ∂M and if α is a $(k-1)$-form defined on (an open set containing) M, then*

$$\int_M d\alpha = \int_{\partial M} \alpha$$

In order to get a feeling for how this works, let's calculate the "volume" V of the 4-dimensional ball $B = \{(x,y,z,t) \mid x^2 + y^2 + z^2 + t^2 \leq R\}$ of radius R in $\mathbb{R}^4$ in two ways. This is a 4-manifold with boundary $S = \{(x,y,z,t) \mid x^2 + y^2 + z^2 + t^2 = R\}$.

V can be expressed as the integral

$$V = \iiiint_B dxdydzdt = \int_B dx \wedge dy \wedge dz \wedge dt$$

Here the standard order of his variables is x,y,z,t and you see in the last line how we deal with getting rid of the wedge symbols when it comes time to do the integration. It is just a question of orientation.