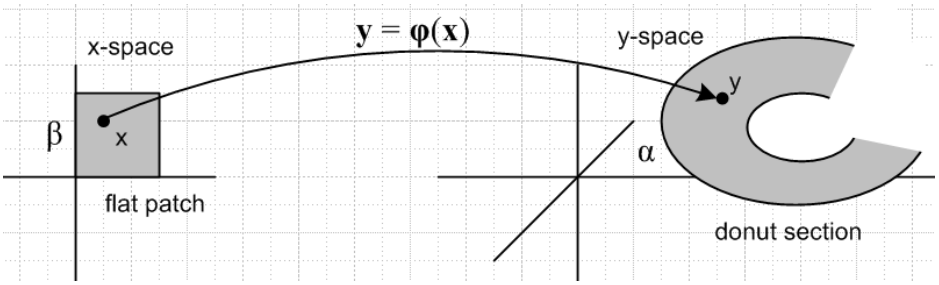
**Pullbacks** PhL 2.11.16

My source for these notes is section 3.2 of Sjamaar translated into my meta notes on Chapter 3. I will use the Sjamaar notation at first, then ponder other notations. The key picture is that of page 37 which I here simplify:




This is just an example, where we have a 2D patch mapping onto a portion of the surface of a donut. Note that x-space is $R^2$ whereas y-space is $R^3$. As x ranges over the patch, y ranges over the donut section.

The whole issue here could be called "the transformation of a form" from y-space to x-space. We start off with a differential form α in y-space which is this

$\alpha = \Sigma_I f_I(y) dy_I$ // hats present on the right

In our specific example this would be a 2-form, but just imagine it to be a k-form and both surfaces above are k-dimensional.

**1. Transformation of a k-form from y-space to x-space**

Suppose we ask: how can you express the form α in terms of x-space objects? Omitting many details, we can do this as follows

$\alpha = \Sigma_I f_I(y(x)) d\varphi_I = \Sigma_I f_I(y(x)) J_I(x) dx_I = \Sigma_I g_I(x) dx_I$

where $g_I(x) = f_I(y(x)) J_I(x)$

For a 0-form you would say

$\alpha = f(y) = f(y(x)) = g(x)$

In this case, we would be include to say

$f(\mathbf{y})$ is a 0-form in y-space
$g(\mathbf{x})$ is a 0-form in x-space

We have "transformed" the 0-form from y-space to x-space. Note that $\mathbf{y} \in R^3$ while $\mathbf{x} \in R^2$. We still have the idea that x wanders over the flat patch, while y wanders over the partial donut surface.

Since f(**y**) is not the same as g(**x**) in a function sense, it is hard to call the 0-form α in both spaces. We want to say

| | | |
|---|---|---|
| $\alpha = f(y)$ | = a 0-form in y-space | element of $\Omega^k$(partial donut surface V) |
| $\beta = g(x)$ | = a 0-form in x-space | element of $\Omega^k$(flat patch U) |

So when we include the spaces in our view of the form, these are just plain different 0-forms, so we should give them different names like α and β. It is true that when $y = \varphi(x)$, then $\alpha = \beta$, but as "forms" they are different because the functions are different. The forms also have different domains within their respective spaces, as indicated in the right column above. Forms defined on different regions are different.

The same idea holds for the more general forms first considered, so we say

| | | |
|---|---|---|
| $\alpha = \Sigma_I\, f_I(y)dy_I$ | = a k-form in y-space | |
| $\beta = \Sigma_I\, g_I(x)dx_I$ | = a k-form in x-space | // $g_I(x) = f_I(y(x))\, J_I(x)$ |

The functions are different, the spaces are different, so we give these forms different names α and β. Another way to name these two forms might be

| | | |
|---|---|---|
| $\alpha = \Sigma_I\, f_I(y)dy_I$ | = a k-form in y-space | |
| $\alpha_\varphi = \Sigma_I\, g_I(x)dx_I$ | = a k-form in x-space | // $g_I(x) = f_I(y(x))\, J_I(x)$ |

because $\alpha_\varphi$ depends on both α and on the transformation φ.

**2. Invention of the pullback operator $\varphi^*$**

We can imagine that $\alpha_\varphi$ arises from the action of an operator we shall call $\varphi^*$ such that

$$\beta = \varphi^*\alpha$$

or

$$\alpha_\varphi = \varphi^*\alpha$$

This newly minted operator is basically defined by its action on a differential form:

$$\alpha = \Sigma_I\, f_I(y)dy_I$$

But what is this action? We pretend that we can write it this way:

$$\varphi^*\alpha = \varphi^*\ (\Sigma_I\, f_I(y)dy_I) = \Sigma_I \varphi^*[f_I(y)dy_I] = \Sigma_I\, \varphi^*[f_I(y)]\varphi^*[dy_I]$$

Later we can check self-consistency to see if this is reasonable. We then separately define the action of this new operator $\varphi^*$ on a function and on a differential :

$$\varphi^*[f_I(y)] = f_I(\varphi(x))$$

$$\varphi^*[dy_I] = d\varphi_I = J_I(x)dx_I$$

With this definition of the action of operator $\varphi^*$, we find that

$$\varphi^*\alpha = \Sigma_I f_I(\varphi(x))J(x)dx_I = \Sigma_I g_I(x)dx_I = \beta = \alpha_\varphi$$

So this operator, defined in this somewhat strange way, acts on $\alpha$ to generate $\beta$. So you can think of this operator as generating the x-space form $\beta$ from the y-space form $\alpha$. In English you say that the $\varphi^*$ pulls back the y-space k-form $\alpha$ to generate the x-space k-form $\beta$. The idea of "back" is because the function $\mathbf{y} = \varphi(\mathbf{x})$ pushes (maps) the variable $\mathbf{x}$ forward to the variable $\mathbf{y}$. The pulling back action of $\varphi^*$ goes backwards from y-space to x-space, whereas the action of $\varphi$ in $\varphi(x)$ is a push forward from x-space to y-space. Sometimes people say that $\varphi^*$ pulls the form $\alpha$ back "along $\varphi$" meaning with respect to the transformation $\varphi$.

When $\varphi^*$ is defined as above, it is not hard to show that it has these three "properties"

1 $\varphi^*(s_1\alpha + s_2\beta) = s_1\varphi^*(\alpha) + s_s\varphi^*(\beta)$ // $\varphi^*$ is a linear operator

2 $\varphi^*(\alpha_1\alpha_2) = \varphi^*(\alpha_1)\varphi^*(\alpha_2)$ // how $\varphi^*$ acts on a product of forms

3 $\varphi_1^*(\varphi_2^*\,\alpha) = (\varphi_2 \circ \varphi_1)^*(\alpha)$ // "natural rule"

I am not going to worry too much about these rules. I proved them in my raw notes. You can see how the first two rules are consistent with what I did above:

$$\varphi^*\alpha = \varphi^*\,(\Sigma_I f_I(y)dy_I) = \Sigma_I\varphi^*[f_I(y)dy_I] \quad \text{// rule 1, linearity}$$

$$= \Sigma_I \varphi^*[f_I(y)]\varphi^*[dy_I] \quad \text{// rule 2, since } f_I(y) \text{ is a 0-form and } dy_I \text{ a k-form}$$

To go the next step, we need to know the actual action of $\varphi^*$ on the two objects, as shown earlier.

**3. The Jacobian things**

In the above, I wrote

$$\varphi^*[dy_I] = d\varphi_I = J_I(x)dx_I$$

Now $dy_I$ is a k-form in y-space, and $dx_I$ is a k-form in x-space. We are doing a change from one set of variables to another set of variables. The object $J_I(x)$ is the determinant of a certain square matrix

(depends on label I) whose elements are derivatives like $\partial\varphi_i/\partial x_j = R_{ij}$ which is my R-matrix. The determinant can be shown to arise from the fact that in both $dy_I$ and $dx_I$, the wedge products are always in "increasing order". This fact converts a product of R matrix elements into a determinant of such elements. This is an important technical detail which I will write up eventually, but I don't want to get distracted on that right now, and I do have a pretty good understanding of it already.

**4. Why should anybody care about pulling back a k-form from y-space to x-space?**

In other words, what practical benefit do I get from doing such a pull-back?

I think this is the answer: when we try to define the notion of integration of "something" over a general Manifold, we don't at first have any clue as to what this even means. However, if that "something" is a k-form, and if the Manifold is a k-dimensional one, then we can use this pullback idea to define the integration of a form over a Manifold. No matter how ugly the Manifold might be, we can define a set of mappings like $\varphi$ above, and each one has its own "flat patch". We always know how to integrate something over a flat patch, so we can then define the integral of a form over a manifold to be in effect the sum of the integrals of the pulled-back forms over their respective patches. In general there won't be any "single flat patch" that works for the whole Manifold, but there will in general be some finite number of mappings $\varphi_i$ and their respective patches such that this plan works. So the answer is that the pullback is a tool that can be used to define the meaning of integration of a differential form over a Manifold.

This of course begs another question: Why would someone want to integrate an arbitrary k-form over a k-dimensional manifold? The theoretical answer is that this is just something we want to be able to do for a general case, and then we will likely find that the integration of certain particular forms over certain particular manifolds will have some real-world applications. I think that "differential geometry" includes the study of "integration over manifolds" and the whole definition of forms and manifolds fleshes this out with meaning and with practical methods. One might similarly ask why one would want to theoretically be able to integrate any real function on the real axis, when only certain functions are interesting. Or why should a 3D printer be able to make arbitrary 3D shapes when only certain shapes seem useful?