Pullbacks in terms of 88A PhL 2.18.16

1. Getting Spivak and Sjamaar to agree.

Equation 88a of Sjamaar says this:

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}...\mathbf{v_k}) = \alpha_{\boldsymbol{\varphi(x)}}([D\varphi(x)]\mathbf{v_1}, [D\varphi(x)]\mathbf{v_2},.... [D\varphi(x)]\mathbf{v_k})$$

I think this is in the context of the figure on page 37 where U and x-space is on the left, and where V and y-space is on the right, and where y = φ(x) is the transformation.

This looks a lot like Spivak's two equations,

$$f_*(v_p) \doteq (Df(p)(v))_{f(p)}$$
$$f^*\omega(p)(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1), \ldots, f_*(v_k)).$$

Let's try to translate Spivak into Sjammar notation. Assume that

Sjamaar	Spivak
α	ω
φ	f
x	p
φ(x) = y	f(p) = y

With just these three connections, Spivak's first equation becomes

$$\varphi_*(\mathbf{v_p}) = ((D\varphi(\mathbf{x})(\mathbf{v}))_{\boldsymbol{\varphi(x)}} = ((D\varphi(\mathbf{x})(\mathbf{v}))_{\mathbf{y}}$$

Now y is a "point on the manifold on the right side of page 37", so I understand perhaps why Spivak shows it as a marking label. I think since $(D\varphi(\mathbf{x})(\mathbf{v})$ is linear, I can translate further

$$((D\varphi(\mathbf{x})(\mathbf{v}))_{\mathbf{y}} = (D\varphi(\mathbf{x}))_{\mathbf{y}}\,\mathbf{v}$$ = a matrix times a vector **v**

With this assumption, Spivak's second equation becomes

$$\varphi^*\alpha^*(\mathbf{x})(\mathbf{v_1},\mathbf{v_2}...\mathbf{v_k}) = \alpha(\boldsymbol{\varphi}(\mathbf{x}))\,(\,(D\varphi(\mathbf{x}))_{\mathbf{y}}\,\mathbf{v_1}, (D\varphi(\mathbf{x}))_{\mathbf{y}}\,\mathbf{v_2} ..... (D\varphi(\mathbf{x}))_{\mathbf{y}}\,\mathbf{v_k})$$

Now let's further change Spivak's notation so that the argument of a form is made a subscript and not a paren argument. Then Spivak becomes

$$\varphi^*\alpha^*_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}...\mathbf{v_k}) = \alpha_{\boldsymbol{\varphi(x)}}\,(\,(D\varphi(\mathbf{x}))_{\mathbf{y}}\,\mathbf{v_1}, (D\varphi(\mathbf{x}))_{\mathbf{y}}\,\mathbf{v_2} ..... (D\varphi(\mathbf{x}))_{\mathbf{y}}\,\mathbf{v_k})$$

which we can compare with Sjamaar

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}...\mathbf{v_k}) = \alpha_{\boldsymbol{\varphi(x)}}([D\varphi(x)]\mathbf{v_1}, [D\varphi(x)]\mathbf{v_2},.... [D\varphi(x)]\mathbf{v_k})$$

So maybe I should be interpreting the original Spivak LHS as $(f^*\omega)(p)$.

I can take Spivak's $(D\varphi(\mathbf{x}))_{\mathbf{y}}$ in light of page 37 as saying you compute the matrix $D\varphi(x)$ where derivatives are with respect to $x^{\mathbf{i}}$ coordinates, but then you evaluate the result at the x which corresponds to y on the manifold, $x = \varphi^{-1}(x)$.

So I think I have a concordance between Spivak and Sjamaar on this pullback equation.

2. What does this equation mean?

Here it is I think in its simplest form re page 37,

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}...\mathbf{v_k}) = \alpha_{\mathbf{y}}([D\varphi(x)]\mathbf{v_1}, [D\varphi(x)]\mathbf{v_2},.... [D\varphi(x)]\mathbf{v_k})$$

The tensor function on the left, $(\varphi^*\alpha)_{\mathbf{x}}$, is obtained from the tensor function on the right by shuffling the arguments as shown. There are no determinants here.

$D\varphi$ is the matrix of derivatives, it is the R matrix for the transformation, the differential matrix. So then I could write in shorter notation

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}...\mathbf{v_k}) = \alpha_{\mathbf{y}}(R\mathbf{v_1}, R\mathbf{v_2},.... R\mathbf{v_k})$$

or

$$<(\varphi^*\alpha)_{\mathbf{x}} | \mathbf{v_1},\mathbf{v_2}...\mathbf{v_k} > = <\alpha_{\mathbf{y}} | R\mathbf{v_1}, R\mathbf{v_2},.... R\mathbf{v_k} > = <\alpha_{\mathbf{y}} | R | \mathbf{v_1},\mathbf{v_2}...\mathbf{v_k} >$$

$$= <R^{\mathbf{T}}\alpha_{\mathbf{y}} | \mathbf{v_1},\mathbf{v_2}...\mathbf{v_k} >$$

in my Dirac notation. Then in the dual space where these k-forms live,

$$<(\varphi^*\alpha)_{\mathbf{x}}| = <R^{\mathbf{T}}\alpha_{\mathbf{y}}|$$

We start with a form $\alpha_{\mathbf{y}}$ defined at point y on page 37, we apply operator $R^{\mathbf{T}}$ to it, and we end up with the form $<\beta| = <(\varphi^*\alpha)_{\mathbf{x}}|$ which is defined at point x on the left on page 37. This is the pulled back form. The form $\alpha_{\mathbf{y}}$ has been pulled back from y-space to x-space where it is the form $(\varphi^*\alpha)_{\mathbf{x}}$.