Relation between Sjamaar Section 6.2, Bucks 6.4 p 349 and Lagrange doc PhL 2.14.16

These three pieces of text are all dancing around the same subject, but I have not yet made the connection between the three objects. Now is a good time to do that.

Bucks Section 6.4. They talk mainly about functions z = f(x,y) and how you might find extrema. The subject is addressed on pages 349-359. At that point constraints are first mentioned and p 359-363 then talks about the Lagrange multipliers idea.

In the first part we learn that extrema must occur at "critical points" defined to be places where $\partial_i f$ vanish in all dimensions. Various examples are given. There is no mention of the rank of any matrix. Bucks do talk about rank on page 231 with regard to transformations. However, on page 362 top they do note that at a critical point, the rank of the differential dT does drop down.

In Lagrange Doc I state up front that extremal points of a function f must occur at places where the rank of the R matrix drops below full rank. I define the R matrix in terms of a transformation which $\mathbf{u} = \mathbf{F}(\mathbf{x})$ where the functions u_i are f and the various constraint functions a,b,c..q :

$f = u_1(x_1, x_2, ...x_N)$ $R_{11} = \partial u_1/\partial x_1 = \partial f/\partial x_1 = f_1$ $R_{12} = f_2$ etc.
$a = u_2(x_1, x_2, ...x_N)$ $R_{21} = \partial u_2/\partial x_1 = \partial a/\partial x_1 = a_1$ $R_{22} = a_2$ etc
$b = u_3(x_1, x_2, ...x_N)$ $R = (Du)$
...
$q = u_S(x_1, x_2, ...x_N)$ // S equations in N variables (1.10)

In Lagrange doc below (1.4) I regard $f = u_1(x_1, x_2, ...x_N)$ as a surface of N dimensions in E^{N+1} where that last axis is $x_{N+1} = f$. If I extrude the surface a = 0 into E^{N+1}, these two surfaces in E^{N+1} intersect in a surface of dimension N-1. So with one constraint, surface is N-1 and with S-1 constraints a=0,b=0...q=0 surface of intersection has dimension N-(S-1) = N-S+1. So in the space E^{N+1} where the last (vertical) axis is f, the above set of S equations specifies a certain surface of dimension N-S+1. This is the only way I can think of to say that the entire set of equations above specifies a surface. Within E^N I can say that the last S-1 equations (the constraint equations) specify a surface in E^N which is "the constraint surface".

Now in Sjamaar we have this set of equations

$c_1 = \varphi_1(x_1, x_2, ...x_N)$
$c_2 = \varphi_2(x_1, x_2, ...x_N)$
...
$c_m = \varphi_m(x_1, x_2, ...x_N)$ // m equations in N variables

This set of m equations specifies a surface of degree N-m in E^N.

Plan A: In this plan, I will try to identify the $\mathbf{c} = \boldsymbol{\varphi}(\mathbf{x})$ specified Sjamaar surface with the constraint surface of Lagrange doc. To do this, I write

$a = u_1(x_1, x_2, ...x_N) = \varphi_1(x_1, x_2, ...x_N) - c_1$
$b = u_2(x_1, x_2, ...x_N) = \varphi_2(x_1, x_2, ...x_N) - c_2$ $R = (Du) = (D\varphi)$

...

$q = u_m(x_1, x_2, ...x_N) = \varphi_m(x_1, x_2, ...x_N) - c_m$

This connection makes sense in the sense that a = 0, b=0....q=0 *matches* $\varphi_1(x) = c_1$ ....$\varphi_m(x) = c_m$, if we identify the Sjamaar m with the Lagrange doc S-1. So now the Sjamaar specified surface is identified exactly with the Lagrange doc constraint surface. In Sjamaar we are supposed to consider (Du) = (Dφ) at points on the surface and see if this matrix has full rank at all points on the surface. However, in Lagrange doc we are supposed to study the rank of matrix R whose first row involves f and which I have omitted in this Plan A. So this seems *not very useful* since the two matrices R and (Dφ) are completely different in these two scenarios, though in both scenarios we are interested in where the respective matrix falls below full rank.

Plan B. Since the previous plan seems to make no connection at all between Sjamaar and Lagrange doc, I will take a quick look at the E^{N+1} surface situation and try to make some connection there. I take the full set of S equations of Lagrange doc and I regard them as specifying a surface in E^{N+1}. An example of such a surface would be my red curve in Lagrange doc. But the Sjamaar surface is always in E^N so how would I ever get these on a same footing? OK, here is a dim idea. Lower the Lagrange dimension so that the Lagrange doc equations are then

$x_N = u_1(x_1, x_2, ...x_{N-1}) = f$

$a = u_2(x_1, x_2, ...x_{N-1})$

$b = u_3(x_1, x_2, ...x_{N-1})$

...

$q = u_S(x_1, x_2, ...x_{N-1})$

So now this set of S equations specifies a surface in E^N. But these equations don't have the same form as the Sjamaar equations since all but one of them only go up to x_{N-1}.

My Conclusion: I don't think there *is* any simple connection between the Lagrange doc application and the Sjamaar Chapter 6.2 application! Both applications refer to the rank of a matrix of derivatives, but that is about the only connection I see. I thought there would be some very simple and direct connection, so I guess I was wrong.