Role of the Multilinear functions PhL 8.24.15

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I have finished reading Sja and am now looking back at various things. I think back-and-fill and cross correlate and digest and crystallize and congeal and consolidate is going to take a series of essays on my part to get this stuff understood. So this doc is today's essay.

The "Grassmann section" as I might call it (perhaps incorrectly) is a theory section called "7.2 Second Definition" on page 82 of Ch 7. Sja talks first about dual spaces and their covectors. Recall that a dual space V* is the set of all linear maps over vector space V. In our work on forms, we have some underlying space of interest, R^n which has coordinates x_i or perhaps a vector $\mathbf{v}$ with components v_i. This might by the usual (x,y) space in R^2. In Sja page 84, if V has basis $\{\mathbf{v}_i\}$, then V* has basis functionals λ_i. He shows that if you just think about the meaning of space V* which contains things like $f(\mathbf{v}_1,\mathbf{v}_2...) = \Sigma_i c_i(\mathbf{v}_i)\mathbf{v}_i$ then $(c_1,c_2...)$ is the covector for this particular functional in V* and he shows that

$$c_i(\mathbf{v}_i) = \lambda_i(\mathbf{v}_j) = \delta_{i,j} \qquad \text{or} \qquad \lambda_i(\mathbf{v}_j) = (\mathbf{e}^T{}_i)_j = (\mathbf{e}^T{}_j)_i$$

and this is a "coordinate function". These λ_i functionals form the dual basis to V. Notice that we can say that $\lambda_i \in$ V* where λ_i is a functional and V* is a space of functionals. We don't need to show the arguments. Notice also that

$$\lambda_i(\mathbf{v}) = \lambda_i(\Sigma_j\, c_j \mathbf{v}_j) = \Sigma_j\, c_j\, \lambda_i(\mathbf{v}_j) = \Sigma_j\, c_j\, \delta_{i,j} = c_i \qquad \Rightarrow \qquad \boldsymbol{\lambda}(\mathbf{v}) = \mathbf{c}^T$$

so you think of λ_i as a functional on the space V* for any vector $\mathbf{v}$ in V.

Question. As you see, one can write either $\lambda_i(\mathbf{v}) = c_i$ or $\boldsymbol{\lambda}(\mathbf{v}) = \mathbf{c}$. We claim that λ_i is one of a set of basis functionals in the functional space V*. But we need ALL of these functionals in order to generate the vector $\mathbf{c}^T$ which we thought was a vector in the dual space to V. So which is it? Is λ_i a covector in V*, or is $\mathbf{c}^T$ a covector in V* ? I have no idea!

Resolution: I went off and wrote another doc "the meaning of lamba-i.doc" [where is this doc? wedge stuff] to clear up this confusion. Here are some of the things I learned.

• Sjamaar's presentation is not really clear, symbols are under-labeled, no mention of $\mathbf{q}_i$ objects.

• The correct solution is that your take a generic basis $\mathbf{v}_i$ of V and you construct the set of dual basis vectors $\mathbf{q}_i$ as instructed in tensor doc. This set is the reciprocal basis, and is really a basis. Then you know that $\mathbf{q}_i \bullet \mathbf{v}_i = \delta_{i,j}$. You then define $\lambda_i(\mathbf{v}) \equiv \mathbf{q}_i{}^T\mathbf{v}$. This λ_i is obviously a linear functional on V, since $\mathbf{v}$ is an arbitrary element of V. And we then have $\lambda_i(\mathbf{v}_j) \equiv \delta_{i,j}$.

• There are really two different "dual spaces". You can think of either $\mathbf{v}_i$ or $\mathbf{q}_i$ as a basis for V, and then you can think of $\mathbf{q}_i{}^T$ as a basis for what I call V^T which is the corresponding "row vector space". Sometimes people regard this space V^T as the "dual space of V". But the dual space of Sjamaar is a slightly different dual space. It is the space V* of linear functionals which act on V. Our λ_i of $\lambda_i(\mathbf{v}) \equiv$

$q_i^T v$ is a particular linear functional. I show in the doc (and Sjamaar shows) that the set of λ_i form a basis for functionals in V*. Since $\lambda_i(\mathbf{v}) \equiv \mathbf{q}_i^T\mathbf{v}$, we can associate λ_i as a basis vector in V* with $\mathbf{q}_i^T$ being a basis vector in V^T. A general functional could be written as $\lambda(\mathbf{v}) = \mathbf{q}^T\mathbf{v}$ so for each $\mathbf{q}^T$ in V^T you end up with a specific functional λ in V*. So the size of the two spaces V* and V^T is the same. Of course V^T and V also have the same size. Thus dim(V*) = dim(V) as Sjamaar shows. To summarize, there are two dual spaces. One dual space has vectors which are row vectors like $\mathbf{q}^T$ and I call that V^T. The other dual space has elements which are scalar functionals like λ (not vectors).

• You could if you wanted combined all the scalar functionals λ_i into a vector $\boldsymbol{\lambda}$, but this is never done and is just not a useful way to think of things, but I do it in a few places below regarding determinants.

So far, nothing mysterious, but the reader wonders what is the relevance to the above λ_i discussion to differential forms? There is no answer at this point.

Alternating functions and A^kV

Sja then goes on the define A^kV as the space of alternating linear functionals on V having k vector arguments. Within this space, he then draws attention to a certain type of element which has this form

$$"\lambda_1\lambda_2....\lambda_k"(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) \equiv \det[\lambda_i(\mathbf{v}_j)] = \det[\,\boldsymbol{\lambda}(\mathbf{v}_1), \boldsymbol{\lambda}(\mathbf{v}_2).....\boldsymbol{\lambda}(\mathbf{v}_k)\,] \quad \boldsymbol{\lambda} = (\lambda_1,\lambda_2....\lambda_k)^T$$

where on the right I show the column vectors of the associated matrix. This is just a particular example. Here is the matrix whose determinant is taken,

$$\begin{array}{llllll} \lambda_1(\mathbf{v}_1) & \lambda_1(\mathbf{v}_2) & \lambda_1(\mathbf{v}_3) & ..... & \lambda_1(\mathbf{v}_k) \\ \lambda_2(\mathbf{v}_1) & \lambda_2(\mathbf{v}_2) & \lambda_2(\mathbf{v}_3) & ..... & \lambda_2(\mathbf{v}_k) \\ ... & & & & \\ \lambda_k(\mathbf{v}_1) & \lambda_k(\mathbf{v}_2) & \lambda_k(\mathbf{v}_3) & ..... & \lambda_k(\mathbf{v}_k) \end{array}$$

Here is another A^kV element:

$$"\lambda_2\lambda_1....\lambda_k"(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) \equiv \det[\lambda_i(\mathbf{v}_j)] = \det[\,\boldsymbol{\lambda}(\mathbf{v}_1), \boldsymbol{\lambda}(\mathbf{v}_2).....\boldsymbol{\lambda}(\mathbf{v}_k)\,] \quad \text{now } \boldsymbol{\lambda} = (\lambda_2,\lambda_1....\lambda_k)^T$$

$$= - "\lambda_1\lambda_2....\lambda_k"(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k)$$

This is a different A^kV element. In terms of A^kV, the determinant guarantees that $"\lambda_1\lambda_2....\lambda_k"(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k)$ is an "alternating function" of the vector arguments $\mathbf{v}_i$. Swapping two arguments swaps the two corresponding columns of the matrix and thus negates the function.

But that same determinant ALSO causes the function $"\lambda_{i_1}\lambda_{i_2} .....\lambda_{i_k}"$ to have this same alternating property if you switch any two indices! Such a swap causes a swap of two rows of the matrix! This fact then shows that if you consider a function like $"\lambda_1\lambda_1....\lambda_k"$ where two indices are the same, that function is 0 ! I claim then that you could write

$$"\lambda_{i_1}\lambda_{i_2}.....\lambda_{i_k}"(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) = \varepsilon_{i_1 i_2 .....i_k}\,"\lambda_1\lambda_2....\lambda_k"(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k)$$

In this family of functions there are k*k..*k = k^k members, but only k*(k-1)....*1 = k! are nonzero. So all the nonvanishing functions are trivially related to the first function "$\lambda_1\lambda_2....\lambda_k$".

Now we could use the wedge symbol by definition to write

$$(\lambda_1 \wedge \lambda_2 \wedge.... \wedge\lambda_k)(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) \equiv "\lambda_1\lambda_2....\lambda_k"(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) = \det[\lambda_i(\mathbf{v}_j)]$$

This is another way to write the NAME of this A^kV element. It is still a function of all the arguments. I would say that this was NOT true:

$$(\lambda_1 \wedge \lambda_2 \wedge.... \wedge\lambda_k)(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) = (\lambda_1(\mathbf{v}_1) \wedge \lambda_2(\mathbf{v}_2) \wedge.... \wedge\lambda_k(\mathbf{v}_k)) \qquad \text{wrong!}$$

because that gives the impression that the result is a function only of $\lambda_1(\mathbf{v}_1)$ when in fact it is a function of all the objects $\lambda_1(\mathbf{v}_i)$.

Now: If you realize that the object "$\lambda_{i_1}\lambda_{i_2}.....\lambda_{i_k}$" is a functional in A^kV, you don't have to show its arguments because you know, since it is a k-multilinear functional, you can always display the arguments, but the object "$\lambda_{i_1}\lambda_{i_2}.....\lambda_{i_k}$" is the name of a functional in A^kV . In analogy, you can think of f(x) as a function of x, but the function is f and you don't have to always say f(x) to talk about function f. Nevertheless, obviously if you *evaluate* the functional "$\lambda_{i_1}\lambda_{i_2}.....\lambda_{i_k}$" at some point in V^k, you get some particular real number, and different evaluation points will have different real numbers.

With this in mind, we can simply say

$$(\lambda_1 \wedge \lambda_2 \wedge.... \wedge\lambda_k) \in A^kV$$

The reader now pauses again to ask: all the above is quite clear and fine, but what does it have to do with differential forms? Well, suppose we simply attach a new name to λ_i as follows

$$"dx_i" \equiv \lambda_i$$

and we don't at this point associate dx_i with a "differential". Then we have

$$(dx_1 \wedge dx_2 \wedge.... \wedge dx_k) \in A^kV$$

Consider then the following object

$$\alpha(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) \equiv f_{12..k}\,(dx_1 \wedge dx_2 \wedge.... \wedge dx_k)\,(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k)$$

where $f_{12..k}$ is a real number which is a coefficient. I think if would be incorrect to require $f_{12..k}$ to also be an alternating function $f_{12..k}(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k)$. Right now I think $f_{12..k}$ is just a constant real number which does not depend on $(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k)$.

Without showing arguments

$$\alpha \equiv f_{12..k}\ (dx_1 \wedge dx_2 \wedge .... \wedge dx_k)$$

This is of course an element of A^kV because any linear combination of $(dx_1 \wedge dx_2 \wedge .... \wedge dx_k)$ is in A^kV. This expression "looks like" a particular differential form of the type we studied in Buck, though there were no wedge symbols.

If we now incorporate the technology of the "increasing multi index I" with all that has been said above, we can write this as the most general possible "differential form α" of order k:

$$\alpha = \Sigma_I\ f_I\ dx_I \qquad\qquad dx_I = dx_{i_1} \wedge dx_{i_2} ..... \wedge dx_{i_k} \text{ with } i_1 < i_2 ....< i_k$$

Again we can show arguments if we want

$$\alpha(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) = \Sigma_I\ f_I\ dx_I(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k)$$

$$\alpha(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k) = \Sigma_I\ f_I\ "dx_{i_1} \wedge dx_{i_2} ..... \wedge dx_{i_k}"(\mathbf{v}_1,\mathbf{v}_2...\mathbf{v}_k)$$

But the arguments are not needed if you are just talking about functionals in A^kV . So we go back to

$$\alpha = \Sigma_I\ f_I\ dx_I \quad = \text{most general element of } A^kV$$

At this point, the reader *must admit* that in terms of these functional labels $dx_i = \lambda_i$ the resulting α certainly "looks like" a differential form like f(x,y)dxdy which you could integrate over a piece of R^2. But "looks like" is not very precise. These dx_I have nothing whatsoever to do with calculus differentials! Let's compare these two animals,

$$\beta = f(x_1,x_2)dx_1dx_2 \qquad\qquad \text{// a calculus integrand}$$

$$\alpha(\mathbf{v}_1,\mathbf{v}_2) = f\,[dx_1 \wedge dx_2](\mathbf{v}_1,\mathbf{v}_2) \qquad\qquad \text{// a differential form}$$

where recall that $(\mathbf{v}_1,\mathbf{v}_2)$ is some set of basis vectors in R^2. Let's rewrite as

$$\alpha(\mathbf{r}_1,\mathbf{r}_2) = f\,[dx_1 \wedge dx_2](\mathbf{r}_1,\mathbf{r}_2) \qquad\qquad \mathbf{r}_1, \mathbf{r}_2 \text{ in } R^2$$

Well, not very clear is it! But now let's change the scenario. Go back to

$$\alpha(\mathbf{v}_1,\mathbf{v}_2) = f\,[dx_1 \wedge dx_2](\mathbf{v}_1,\mathbf{v}_2)$$

where $(\mathbf{v}_1,\mathbf{v}_2)$ is some basis in R^2.

Comment: I had the passing idea that you could regard arguments $(\mathbf{v}_1,\mathbf{v}_2)$ as being two arbitrary vectors in the tangent space $T_{\mathbf{x}}M$, but I don't think that is right. They are just arbitrary arguments which any $AV^{\mathbf{k}}$ functional must have.

Now let's just assume that for a manifold M, the above "form" varies depending on the point **x** on M you choose. We now just "tag" things with x in this arbitrary manner

$$\alpha_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2) = f_{\mathbf{x}}\,[dx_1 \wedge dx_2]_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2)$$

where all three items are then functions of **x**, as indicated by this label. This step seems quite arbitrary. We can write it this way

$$\alpha_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2) = f(\mathbf{x})\,[dx_1 \wedge dx_2]_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2)$$

so at least we have f(x) in there, looking more like calculus. If we omit these generic arguments we have

$$\alpha_{\mathbf{x}} = f(\mathbf{x})\,[dx_1 \wedge dx_2]_{\mathbf{x}}$$

which compare to the above calculus thing. I think here we have to assume that our manifold M is just the plane, so then $[dx_1 \wedge dx_2]_{\mathbf{x}}$ if differentials might be the same at all points. Then $\mathbf{x} = (x_1,x_2)$ and we get

$$\alpha_{\mathbf{x}} = f(x_1,x_2)\,[dx_1 \wedge dx_2] \qquad \text{or} \qquad \alpha_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2) = f(x_1,x_2)\,[dx_1 \wedge dx_2](\mathbf{v}_1,\mathbf{v}_2)$$

This now seems quite close now to our calculus result

$$\beta = f(x_1,x_2)dx_1dx_2.$$

Remark: The object α is a theoretical object in $A^{\mathbf{k}}V$, whereas β is some real-world calculus thing which I can integrate to get a number. Sjamaar has provided the reader with many technical tools, but has failed to make this connection. The question is this: what really does the function $"\lambda_1\lambda_2"(\mathbf{v}_1,\mathbf{v}_2)$, which we can write as $[\lambda_1 \wedge \lambda_2](\mathbf{v}_1,\mathbf{v}_2)$, have to do with the calculus area patch dx_1dx_2 ?

In his differential forms Chapter 2, we get lots of good tools. We learn what "d" and "*" do to a differential form. We learn that $d^2\alpha = 0$. We are told about closed and exact forms. But the English language explanation of "what is really going on here" is completely missing, His discussion of forms is divorced from the physical world.

Pullbacks at least relate integrals over curves to flat-space integrals

But as noted above, I have lots of trouble just with a simple 2D flat space integral. Still, I will comment on the pullback stuff here.

The pullback chapters 3 and 4 are perhaps closest to real world integrals. $\int_c \alpha = \int_a^b g(t)\,dt$. It seems that only after you do a "pullback" do you get something that can be integrated in the real world. Here he pulls back the abstract integral of the abstract form α over a general curve c to get a physical integral. So you never integrate actually "on the curve", you always integrate in the "flat" pullback space.

He jumped widely from 1-forms to k-forms at the end of Ch 2 and maybe we should have had a visit to the world of 2-forms as done in Buck. Let's try to specialize Ch 2 ending to 2-forms. First, here is the "general theory"

$$\alpha = \Sigma_I f_I(\mathbf{x})\, dx_I$$

$$\varphi^*\alpha = \Sigma_I \varphi^*(f_I(\mathbf{x}))\, \varphi^*(dx_I) = \Sigma_I f_I(\boldsymbol{\varphi}(\mathbf{x}))\ \Sigma_J \det(D\varphi)_{I,J}\, dx_J$$

$$= \Sigma_J [\ \Sigma_I f_I(\boldsymbol{\varphi}(\mathbf{x}))\det(D\varphi)_{I,J}]\, dx_J = \Sigma_J g_J(\mathbf{x})\, dx_J$$

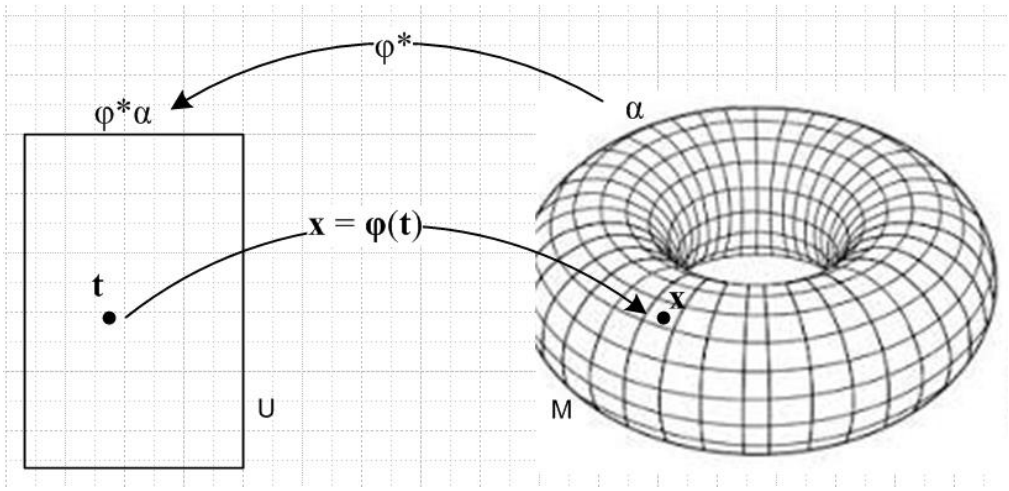
If you wanted to do an integral, you would write

$$\int_M \alpha = \int_U \varphi^*\alpha = \int_U \Sigma_J g_J(\mathbf{x})\, dx_J$$

Here M is the curved "surface" while U is the "flat space" over which we know how to integrate [but I don't yet really have the flat-space integral form-to-integral connection] . In the above notation, this flat space has coordinates x_J. The mapping $\varphi(\mathbf{x})$ is what takes you from the flat integrating space to the general surface M.

Now Sjamaar is IMHO guilty of doing a bad choice of variable names going from one chapter to the next, which added lots of confusion to me. A calculus book that used y = f(x) in one chapter and x = f(y) in another chapter would also be confusing. So let's now rewrite the above "general theory" where the flat space U has coordinate **t** and then $\mathbf{t} = \boldsymbol{\varphi}(\mathbf{x})$ is the mapping from M to U. Note how in Ch 3 he instead on page 37 uses $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$. So I will make a better picture:

α φ*α **x** **t** $\mathbf{t} = \boldsymbol{\varphi}(\mathbf{x})$



In this picture the two arrows are oppositely directed. The domain U is "flat" and we know how to integrate something over that domain. Now rewrite the above "general theory"

$\alpha = \Sigma_I\, f_I(\mathbf{x})\, dx_I$ = defined for the right side picture

$\varphi^*\alpha = \Sigma_I\, \varphi^*(f_I(\mathbf{x}))\, \varphi^*(dx_I) = \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{t}))\, \Sigma_J\, \det(D\varphi)_{I,J}\, dt_J$

$= \Sigma_J [\, \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{t}))\det(D\varphi)_{I,J}]\, dt_J = \Sigma_J\, g_J(\mathbf{t})\, dt_J$

Then here is how we do an integral

$\int_M \alpha = \int_U \varphi^*\alpha = \int_U \Sigma_J\, g_J(\mathbf{t})\, dt_J \qquad g_J(\mathbf{t}) = \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{t}))\det(D\varphi)_{I,J}$

Here the dt_J integral is a flat space one we know how to do! And $g_J(\mathbf{t})$ is just some scalar function.

Now let's tone this down from the general case to k = 2

$\alpha = f_{12}(\mathbf{x})\, dx_1 \wedge dx_2$ = defined for the right side picture

$\varphi^*\alpha = \varphi^*(f_{12}(\mathbf{x}))\, \varphi^*(dx_1 \wedge dx_2) = [f_{12}(\boldsymbol{\varphi}(\mathbf{t}))\, \det(D\varphi)_{12,12}]\, (dt_1 \wedge dt_2) = g_J(\mathbf{t})\, (dt_1 \wedge dt_2)$

I *assume* that the wedge product survives here. There is always the orientation sign question which Sja worries about a lot, but let's say we identify the front side of U with the outside of M. Then I claim

$$\int_U g_J(\mathbf{t})\, (dt_1 \wedge dt_2) = \int_{a2}^{b2} \int_{a1}^{b1} g_J(t_1,t_2)\, dt_1 dt_2 = \int_{a2}^{b2} [\, \int_{a1}^{b1} g_J(t_1,t_2)\, dt_1]\, dt_2$$

as in the Sja Remark 5.3. I know that the above is the "rule", but I don't at all see how we can replace the theoretical wedge product $dt_1 \wedge dt_2$ with some real-world differentials $dt_1 dt_2$ multiplied together.

At this point, I don't really associate $(dt_1 \wedge dt_2)$ with a rectangular patch of area! So there is work still to be done, the picture is incomplete. Consider then for *general* vector arguments that

$$(\lambda_1 \wedge \lambda_2)(\mathbf{v}_1, \mathbf{v}_2) = \det \begin{pmatrix} \lambda_1(\mathbf{v}_1) & \lambda_1(\mathbf{v}_2) \\ \lambda_2(\mathbf{v}_1) & \lambda_2(\mathbf{v}_2) \end{pmatrix}$$

Now specialize this to the case that $\mathbf{v}_1$ and $\mathbf{v}_2$ are a basis for R^2 (possibly non-orthogonal). Then we can say

$$(\lambda_1 \wedge \lambda_2)(\mathbf{v}_1, \mathbf{v}_2) = \det \begin{pmatrix} \lambda_1(\mathbf{v}_1) & \lambda_1(\mathbf{v}_2) \\ \lambda_2(\mathbf{v}_1) & \lambda_2(\mathbf{v}_2) \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

How exactly should I interpret this result? Note in passing that

$$(\lambda_1 \wedge \lambda_2)(\mathbf{v}_1, \mathbf{v}_1) = \det \begin{pmatrix} \lambda_1(\mathbf{v}_1) & \lambda_1(\mathbf{v}_1) \\ \lambda_2(\mathbf{v}_1) & \lambda_2(\mathbf{v}_1) \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

So I continue to be puzzled by the claim that

$(\lambda_1 \wedge \lambda_2)(\mathbf{v}_1, \mathbf{v}_2) = 1$ for any pair of basis vectors $(\mathbf{v}_1, \mathbf{v}_2)$

which would translate into, for example,

$(dt_1 \wedge dt_2)(\mathbf{v}_1, \mathbf{v}_2) = 1$ for any pair of basis vectors $(\mathbf{v}_1, \mathbf{v}_2)$

Comment: By not using $\wedge$ symbols, both Sjamaar and Buck are able to masquerade various activities which would I think be clearer with the symbols.

So I end this essay for now being still confused about the relation between $(\lambda_1 \wedge \lambda_2)(\mathbf{v}_1, \mathbf{v}_2)$ of the Grassmann world and $dt_1 dt_2$ being the area of a differential area patch on a flat surface. Those $(\mathbf{v}_1, \mathbf{v}_2)$ arguments seem to play no role in anything. Maybe there is a homomorphism between the Grassmann forms world and the calculus world, and if so, someone should state that fact.

So I have now to look at other sources once again. I have squeezed all I can out of Buck and Sjamaar.

Arapura. He starts off doing what seems to be normal calculus stuff, but then suddenly on page 8 bottom he shows his first $\wedge$ symbol. He digresses to discuss $\mathbf{u} \wedge \mathbf{v}$. He refers to this as a 2-vector which we are supposed to think in terms of a real physical n-piped which is spanned by $\mathbf{u}$ and $\mathbf{v}$. This object has an area, it has a plane in which it lies (normal to that plane would be $\mathbf{u} \times \mathbf{v}$ **),** and the boundary has two circulation senses. This discussion makes no connection to any AV^2 2-multilinear functionals.

This is fine, but then suddenly he writes,

$$F(x,y,z)dx \wedge dy + G(x,y,z)dy \wedge dz + H(x,y,z)dz \wedge dx$$

where dx seems to be a scalar and not a vector. So now all that **u** $\wedge$ **v** vector stuff seems not to relate. So right at this point, Arapura has dropped the ball at least for what I want to know.

Here is an interesting piece of his work:

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$$\begin{aligned} dx \wedge dy &= \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) \wedge \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right) \\ &= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du \wedge dv \\ &= \frac{\partial(x,y)}{\partial(u,v)} du \wedge dv \end{aligned}$$

This seems to suggest that dx is our ordinary differential dx and we then maintain the wedge between differentials. There is no mention of "dxdy"($\mathbf{v_1}$,$\mathbf{v_2}$) as an alternating function in AV^2.

Next comes this also interesting result:

DEFINITION 11.1 *The integral of a 2-form on S is given by*

$$\iint_S F dx\wedge dy + G dy\wedge dz + H dz\wedge dx = \iint_D \left[F\frac{\partial(x,y)}{\partial(u,v)} + G\frac{\partial(y,z)}{\partial(u,v)} + H\frac{\partial(z,x)}{\partial(u,v)}\right] dudv$$

Probably this can be written as a pullback thing, but the point I make is that on the right we have dudv of ordinary calculus, there is no wedge there. Like Sjamaar, he *defines* the thing on the left to be what is on the right. Here comes the Catch 22:

In practice, the integral of a 2-form can be calculated by first converting it to the form $f(u,v)du \wedge dv$, and then evaluating $\int\int_D f(u,v)\, dudv$.

which seems to say you can drop the wedge for a flat space integral. More on this theme

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(r,\theta)} dr \wedge d\theta = r dr \wedge d\theta$$

$$\iint_S dx \wedge dy = \int_0^{2\pi} \int_0^1 r dr d\theta = \pi$$

So again you can just drop the wedge when you feel like it!

Here is more

16 Triple integrals and the divergence theorem

Recall that a 3-form is an expression $f(x,y,z)dx \wedge dy \wedge dz$. Given a solid region $V \subset \mathbb{R}^3$, we define

$$\iiint_V f(x,y,z)dx \wedge dy \wedge dz = \iiint_V f(x,y,z)dxdydz$$

Again this suggests the for a flat space integral like in R^3 you can drop the wedges.

Arapura only references Spivak and Flanders, like Sjamaar. Donu Arapura is at Purdue, and this pdf is listed on his list of "expository materials". He is a smilin' guy,



So OK, he is not going to provide the connection. However, his stuff lends credence to the idea that "in flat space you can drop the wedge symbols".