Sjamaar Exercise 2.18 PhL 7.7.15

This has mistakes in it, and the whole method I think is not very good, just archive it.
See v2 doc which at least shows that $d\alpha = 0$.

This is a good test to see if I really know what I am doing with manipulating forms. I am flunking the test so far below. So now we are in a separate doc.

$\alpha \equiv (E_i dx_i) dx_4 + B_i (*dx_i)$ in R^4 2-form, sums only for R^3

$\beta \equiv -(H_i dx_i) dx_4 + D_i (*dx_i)$ in R^4 2-form, sums only for R^3

$\gamma \equiv J_i (*dx_i)\, dx_4/c - \rho dV$ in R^4 3-form, sum only for R^3 $dV = dx_1 dx_2 dx_3$

Now compute $d\alpha$ hopefully in a smart way. Restate the above

$\alpha \equiv (\mathbf{E} \bullet d\mathbf{x})\, dx_4 + \mathbf{B} \bullet (*d\mathbf{x})$ in R^4 2-form, sums only for R^3

$\beta \equiv -(\mathbf{H} \bullet d\mathbf{x}) dx_4 + \mathbf{D} \bullet (*d\mathbf{x})$ in R^4 2-form, sums only for R^3

$\gamma \equiv \mathbf{J} \bullet (*d\mathbf{x})\, dx_4/c - \rho dV$ in R^4 3-form, sum only for R^3 $dV = dx_1 dx_2 dx_3$

Now let's define some auxiliary forms

$\delta \equiv \mathbf{E} \bullet d\mathbf{x}$ $\varepsilon = \mathbf{B} \bullet d\mathbf{x}$ $\varphi \equiv \mathbf{J} \bullet d\mathbf{x}$ $\kappa \equiv \mathbf{D} \bullet d\mathbf{x}$ $\theta \equiv \mathbf{H} \bullet d\mathbf{x}$

$*(d\delta) = [\text{curl } \mathbf{E}] \bullet d\mathbf{x}$ $*(d\varepsilon) = [\text{curl } \mathbf{B}] \bullet d\mathbf{x}$ $*(d\theta) = [\text{curl } \mathbf{H}] \bullet d\mathbf{x}$ // PL4

Then our definitions of $\alpha.\beta$ and γ may be stated

$\alpha = \delta\, dx_4 + *(\varepsilon)$

$\beta = -\theta\, dx_4 + *(\kappa)$

$\gamma = (1/c) *(\varphi)\, dx_4 - \rho dV$

Meanwhile, try to translate the Maxwell equations into the forms language:

div $\mathbf{B} = 0$ => $*[d(*\varepsilon)] = 0$ where $\varepsilon \equiv \mathbf{B} \bullet d\mathbf{x}$ // from PL3

div $\mathbf{D} = 4\pi\rho$ => $*[d(*\kappa)] = 4\pi\rho$ where $\kappa \equiv \mathbf{D} \bullet d\mathbf{x}$ // from PL3

curl $\mathbf{E} \bullet d\mathbf{x} = -(1/c^2)(\partial_4 \mathbf{B}) \bullet d\mathbf{x}$ => $*(d\delta) = -(1/c^2)\, \partial_4 \varepsilon$

curl $\mathbf{H} \bullet d\mathbf{x} = (4\pi/c)\, \mathbf{J} \bullet d\mathbf{x} + (1/c^2)\, \partial_4 \mathbf{D} \bullet d\mathbf{x}$ => $*(d\theta) = (4\pi/c)\varphi + (1/c^2)\, \partial_4 \kappa$

OK, so here is what the four Maxwell equations say in form-space

$*[d(*\varepsilon)] = 0 \qquad *(d\delta) = -(1/c^2)\,\partial_4\varepsilon$

$*[d(*\kappa)] = 4\pi\rho \qquad *(d\theta) = (4\pi/c)\varphi + (1/c^2)\,\partial_4\kappa$

Now pursue the first of these for a while

$*[d(*\varepsilon)] = 0$

Now use result (2.12) page 28 which I proved elsewhere which says

$*(*\alpha) = (-1)^{kn+k}\,\alpha$ for α = any k-form in R^n

Question: what is the meaning of *0 ? Suppose

$\alpha = \Sigma_I\, f_I\, dx_I$ where all $f_I = 0$. Then $\alpha = 0$. Then

$*\alpha = \Sigma_I\, f_I\, (*dx_I) = \Sigma_I\, 0\, (*dx_I) = 0$

so I think this is true

$*0 = 0.$

So go back to

$*[d(*\varepsilon)] = 0$

Apply * to both sides to get

$**[d(*\varepsilon)] = *0 = 0$

Now use little theorem to get

$**[d(*\varepsilon)] = (-1)^{kn+k}\, d(*\varepsilon)$ where k is the order of the form $d(*\varepsilon)$

Thus we get

$(-1)^{kn+k}\, d(*\varepsilon) = 0$

which is a very long winded way of arriving at our result

$d(*\varepsilon) = 0$

In other words

$*[d(*\varepsilon)] = 0 \quad \Rightarrow \quad d(*\varepsilon) = 0$

Here then is a restatement of our four Maxwell equations:

$[d(*\varepsilon)] = 0 \qquad *(d\delta) = -(1/c^2)\,\partial_4 \varepsilon$

$*[d(*\kappa)] = 4\pi\rho \qquad *(d\theta) = (4\pi/c)\varphi + (1/c^2)\,\partial_4 \kappa$

and here is a restatement of our defined forms

$\alpha = \delta\, dx_4 + *(\varepsilon)$

$\beta = -\theta\, dx_4 + *(\kappa)$

$\gamma = (1/c)\, *(\varphi)\, dx_4 - \rho dV$

One claim Sjamaar makes is that $d\alpha = 0$. Let's now try to show that to be true. Consider

$d\alpha = d[\delta\, dx_4 + *(\varepsilon)] = d(\delta\, dx_4) + d(*\varepsilon) = d(\delta\, dx_4)$ // since Max $[d(*\varepsilon)] = 0$

$= (d\delta)\, dx_4$ dx_4 is constant relative to d

But I don't see why this should be 0. We don't know anything about $d\delta$. I would have to show

$d\delta = 0$

or

$d(\mathbf{E} \bullet d\mathbf{x}) = 0$

or

$(d\mathbf{E}) \bullet d\mathbf{x} = 0$

or

$\Sigma_{i=1}^{3}\, dE_i dx_i = 0$

But

$dE_i = \Sigma_{k=1}^{4} (\partial E_i/\partial x_k) dx_k$

so get

$\Sigma_{i=1}^{3}\, \Sigma_{k=1}^{4} (\partial E_i/\partial x_k) dx_k dx_i = 0$?

I have (sadly) done something wrong so more hours need to be sunk into this example.