Sjamaar Exercise 2.18 PhL 7.7.15

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Part 1: Show dα = 0 by brute force.

1. Write

$$\alpha = (E_1dx_1 + E_2dx_2 + E_3dx_3)dx_4 + (B_1dx_2dx_3 + B_2dx_3dx_1 + B_3dx_1dx_2)$$

2. Apply operator d to the two fields shown,

$$dE_i = \partial_j E_i dx_j + \partial_4 E_i\, dx_4$$

$$dB_i = \partial_j B_i dx_j + \partial_4 B_i\, dx_4$$

Then

$$dE_1 = \partial_j E_1 dx_j + \partial_4 E_1\, dx_4 = \partial_1 E_1 dx_1 + \partial_2 E_1 dx_2 + \partial_3 E_1 dx_3 + \partial_4 E_1\, dx_4$$

$$dE_2 = \partial_j E_2 dx_j + \partial_4 E_2\, dx_4 = \partial_1 E_2 dx_1 + \partial_2 E_2 dx_2 + \partial_3 E_2 dx_3 + \partial_4 E_2\, dx_4$$

$$dE_3 = \partial_j E_3 dx_j + \partial_4 E_3\, dx_4 = \partial_1 E_3 dx_1 + \partial_2 E_3 dx_2 + \partial_3 E_3 dx_3 + \partial_4 E_3\, dx_4$$

$$dB_1 = \partial_j B_1 dx_j + \partial_4 B_1\, dx_4 = \partial_1 B_1 dx_1 + \partial_2 B_1 dx_2 + \partial_3 B_1 dx_3 + \partial_4 B_1\, dx_4$$

$$dB_2 = \partial_j B_2 dx_j + \partial_4 B_2\, dx_4 = \partial_1 B_2 dx_1 + \partial_2 B_2 dx_2 + \partial_3 B_2 dx_3 + \partial_4 B_2\, dx_4$$

$$dB_3 = \partial_j B_3 dx_j + \partial_4 B_3\, dx_4 = \partial_1 B_3 dx_1 + \partial_2 B_3 dx_2 + \partial_3 B_3 dx_3 + \partial_4 B_3\, dx_4$$

3. Insert these into dα

$$d\alpha = (dE_1dx_1 + dE_2dx_2 + dE_3dx_3)dx_4 + (dB_1dx_2dx_3 + dB_2dx_3dx_1 + dB_3dx_1dx_2)$$

$$\begin{aligned}
&= (\partial_1 E_1 dx_1 + \partial_2 E_1 dx_2 + \partial_3 E_1 dx_3 + \partial_4 E_1\, dx_4)\, dx_1dx_4 \\
&\quad + (\partial_1 B_1 dx_1 + \partial_2 B_1 dx_2 + \partial_3 B_1 dx_3 + \partial_4 B_1\, dx_4)\, dx_2dx_3 \\
&+ (\partial_1 E_2 dx_1 + \partial_2 E_2 dx_2 + \partial_3 E_2 dx_3 + \partial_4 E_2\, dx_4)\, dx_2dx_4 \\
&\quad + (\partial_1 B_2 dx_1 + \partial_2 B_2 dx_2 + \partial_3 B_2 dx_3 + \partial_4 B_2\, dx_4)\, dx_3dx_1 \\
&+ (\partial_1 E_3 dx_1 + \partial_2 E_3 dx_2 + \partial_3 E_3 dx_3 + \partial_4 E_3\, dx_4)\, dx_3dx_4 \\
&\quad + (\partial_1 B_3 dx_1 + \partial_2 B_3 dx_2 + \partial_3 B_3 dx_3 + \partial_4 B_3\, dx_4)\, dx_1dx_2
\end{aligned}$$

I have marked in red terms that vanish. So rewrite with all these terms removed

$$
\begin{aligned}
d\alpha = (\ & \partial_2 E_1 dx_2 + \partial_3 E_1 dx_3)\, dx_1 dx_4 \\
& + (\partial_1 B_1 dx_1 + \partial_4 B_1\, dx_4)\, dx_2 dx_3 \\
+ (\ & \partial_1 E_2 dx_1 + \partial_3 E_2 dx_3)\, dx_2 dx_4 \\
& + (\partial_2 B_2 dx_2 + \partial_4 B_2\, dx_4)\, dx_3 dx_1 \\
+ (\ & \partial_1 E_3 dx_1 + \partial_2 E_3 dx_2\ )\, dx_3 dx_4 \\
& + (\ \partial_3 B_3 dx_3 + \partial_4 B_3\, dx_4)\, dx_1 dx_2
\end{aligned}
$$

We still have 12 terms. Next, bring in the dx trailing factors

$$
\begin{aligned}
d\alpha = (\ & \partial_2 E_1 dx_2\, dx_1 dx_4 + \partial_3 E_1 dx_3\, dx_1 dx_4) \\
& + (\partial_1 B_1 dx_1 dx_2 dx_3 + \partial_4 B_1\, dx_4 dx_2 dx_3) \\
+ (\ & \partial_1 E_2 dx_1 dx_2 dx_4 + \partial_3 E_2 dx_3 dx_2 dx_4) \\
& + (\partial_2 B_2 dx_2 dx_3 dx_1 + \partial_4 B_2\, dx_4 dx_3 dx_1) \\
+ (\ & \partial_1 E_3) dx_1\, dx_3 dx_4 + \partial_2 E_3 dx_2\, dx_3 dx_4) \\
& + (\ \partial_3 B_3 dx_3\, dx_1 dx_2 + \partial_4 B_3\, dx_4 dx_1 dx_2)
\end{aligned}
$$

Now put all triple factors into standard order with apropos signs

$$
\begin{aligned}
d\alpha = (\ & -\partial_2 E_1 dx_1 dx_2\, dx_4 - \partial_3 E_1 dx_1 dx_3\, dx_4) \\
& + (\partial_1 B_1 dx_1 dx_2 dx_3 + \partial_4 B_1\, dx_2 dx_3 dx_4) \\
+ (\ & \partial_1 E_2 dx_1 dx_2 dx_4 - \partial_3 E_2 dx_2 dx_3 dx_4) \\
& + (\partial_2 B_2 dx_1 dx_2 dx_3 - \partial_4 B_2\, dx_1 dx_3 dx_4) \\
+ (\partial_1 E_3) & dx_1 dx_3 dx_4 + \partial_2 E_3 dx_2 dx_3 dx_4) \\
& + (\ \partial_3 B_3 dx_1 dx_2 dx_3 + \partial_4 B_3\, dx_1 dx_2 dx_4)
\end{aligned}
$$

Now group like terms

$$
\begin{aligned}
d\alpha = {} & dx_1 dx_2 dx_3\, [\, \partial_3 B_3 + \partial_2 B_2 + \partial_1 B_1] \\
& + dx_1 dx_2 dx_4\, [\, -\partial_2 E_1 + \partial_1 E_2 + \partial_4 B_3] \\
& + dx_1 dx_3 dx_4\, [\ -\partial_3 E_1 - \partial_4 B_2 + \partial_1 E_3] \\
& + dx_2 dx_3 dx_4\, [\, \partial_4 B_1 - \partial_3 E_2 + \partial_2 E_3\,]
\end{aligned}
$$

Now put terms into a reasonable ordering

$$
\begin{aligned}
d\alpha = {} & dx_1 dx_2 dx_3\, [\, \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3] \\
& + dx_1 dx_2 dx_4\, [\ \partial_1 E_2 - \partial_2 E_1 + \partial_4 B_3] \\
& - dx_1 dx_3 dx_4\, [\ \partial_3 E_1 - \partial_1 E_3 + \partial_4 B_2\,]
\end{aligned}
$$

$+ \; dx_2dx_3dx_4 \, [\, \partial_2 E_3 \; - \; \partial_3 E_2 \; + \partial_4 B_1]$

Do some more

$d\alpha = \; dx_1dx_2dx_3 \, [\, \text{div } \mathbf{B}]$

$+ \; dx_1dx_2dx_4 \, [\, [\text{curl } E]_3 + \; \partial_4 B_3]$

$- \; dx_1dx_3dx_4 \, [\, [\text{curl } E]_2 \; + \partial_4 B_2 \,]$

$+ \; dx_2dx_3dx_4 \, [\, [\text{curl } E]_1 \; + \partial_4 B_1]$

Here are the two homogeneous Maxwell equations,

$\text{div } \mathbf{B} = 0$

$[\text{curl } E]_k \; + \partial_4 B_k = 0$

Thus, when we apply these two Maxwell equations ONLY, we obtain the desired result

$d\alpha = 0$

Part 2: Show $d\alpha = 0$ in a reasonable manner

1. Starting definitions

$\alpha \equiv \; (E_i dx_i)dx_4 + B_i(*dx_i)$ in R^4 2-form, sums only for R^3

$\beta \equiv - \, (H_i dx_i)dx_4 \; + D_i(*dx_i)$ in R^4 2-form, sums only for R^3

$\gamma \; \equiv \; J_i(*dx_i) \, dx_4/c \; - \rho dV$ in R^4 3-form, sum only for $R^3 dV = dx_1dx_2dx_3$

NOTE: $(*dx_i)$ is always a 3D object as if we were in spatial R^3.

2. Maxwell's Curl Equations

$\text{curl } \mathbf{E} = -(1/c)\partial_t \mathbf{B}$ $\qquad$ $\text{curl } \mathbf{H} = (4\pi/c) \, \mathbf{J} + (1/c)\partial_t \mathbf{D}$

$[\text{curl } E]_i = \; -(1/c)\partial_t B_i$ $\qquad$ $[\text{curl } H]_i \; = (4\pi/c) \, J_i + (1/c)\partial_t D_i$

Now use $x_4 = ct$ so $dx_4 = c \, dt$ so $\partial_4 = (1/c)\partial_t$. Then we have

$[\text{curl } E]_i = \; - \, \partial_4 B_i$ $\qquad$ $[\text{curl } H]_i \; = (4\pi/c) \, J_i + \partial_4 D_i$

Now write out

$$[\text{curl } E]_1 = -\partial_4 B_1 = [\partial_2 E_3 - \partial_3 E_2]$$

or

$$[\partial_2 E_3 - \partial_3 E_2] = -\partial_4 B_1 = -\varepsilon_{23k}\partial_4 B_k$$

I think this then means

$$[\partial_i E_j - \partial_j E_i] = -\varepsilon_{ijk}\partial_4 B_k$$

or

$$[\partial_i E_j - \partial_j E_i] + \varepsilon_{ijk}\partial_4 B_k = 0$$

or

$$[\partial_j E_i - \partial_i E_j] + \varepsilon_{jik}\partial_4 B_k = 0$$

or

$$[\partial_j E_i - \partial_i E_j] + \varepsilon_{jik}\partial_4 B_k = 0 \qquad (*)$$

3. Maxwell's Div Equations

$$\partial_i E_i = 4\pi\rho \qquad \partial_i B_i = 0$$

In all of the above, sums are i = 1 to 3.

4. d applied to fields

$$dE_i = \partial_j E_i dx_j + \partial_4 E_i\, dx_4 \qquad j = 1,2,3$$

$$dB_i = \partial_j B_i dx_j + \partial_4 B_i\, dx_4 \qquad j = 1,2,3$$

5. Compute dα:

$$\alpha \equiv (E_i dx_i)dx_4 + B_i(*dx_i)$$

$$d\alpha = (dE_i dx_i)dx_4 + dB_i(*dx_i)$$

$$= (\partial_j E_i dx_j + \partial_4 E_i\, dx_4)dx_i dx_4 + (\partial_j B_i dx_j + \partial_4 B_i\, dx_4)(*dx_i)$$

$$= (\partial_j E_i dx_j)dx_i dx_4 + (\partial_j B_i dx_j)(*dx_i) + \partial_4 B_i\, dx_4\, (*dx_i)$$

The factor $dx_j(*dx_i)$ vanishes unless i = j. For example, $dx_1(*dx_2) = 0$. Therefore the second term can be written $(\partial_i B_i)dx_i(*dx_i) = (\partial_i B_i)dV = 0\, dV = 0$. So we are then down to

$$d\alpha = (\partial_j E_i dx_j)dx_i dx_4 + \partial_4 B_i\, dx_4\, (*dx_i)$$

$$= [\partial_j E_i dx_j dx_i + \partial_4 B_k(*dx_k)]\, dx_4$$

Now how about this idea

$$(*dx_k) = (1/2)\, \varepsilon_{kji} dx_j dx_i$$

For example:

$$(*dx_1) = (1/2)\, \varepsilon_{1ji} dx_j dx_i = \varepsilon_{123} dx_2 dx_3 = dx_2 dx_3$$
$$(*dx_2) = (1/2)\, \varepsilon_{2ji} dx_j dx_i = \varepsilon_{231} dx_3 dx_1 = dx_3 dx_1 \qquad \text{etc}$$

Then we have

$$d\alpha = [\ \partial_j E_i dx_j dx_i + \partial_4 B_k\, (1/2)\, \varepsilon_{kji} dx_j dx_i\ ]\, dx_4$$

Then write

$$\partial_j E_i dx_j dx_i = (1/2)\, (\ \partial_j E_i - \partial_i E_j) dx_j dx_i$$

Then we get

$$d\alpha = [\ \partial_j E_i dx_j dx_i + \partial_4 B_k\, (1/2)\, \varepsilon_{kji} dx_j dx_i\ ]\, dx_4$$

$$= [(1/2)\, (\ \partial_j E_i - \partial_i E_j) dx_j dx_i + \partial_4 B_k\, (1/2)\, \varepsilon_{kji} dx_j dx_i\ ]\, dx_4$$

$$= (1/2)\, [\ \partial_j E_i - \partial_i E_j + \partial_4 B_k\, \varepsilon_{kji}\ ]\, dx_j dx_i\, dx_4$$

$$= (1/2)\, [\ \partial_j E_i - \partial_i E_j + \varepsilon_{jik} \partial_4 B_k\ ]\, dx_j dx_i\, dx_4$$

But [...] = 0 by (*) above, so we have shown $d\alpha = 0$.

3. Conclusion

I have had enough with this "exercise" . I showed $d\alpha = 0$ two ways above. The whole set of results seems just a curiosity to me right now.