Decoding one equation in Spivak PhL 2.19.16

It is unfortunate that Spivak is one of my main sources, yet sometimes I need an entire 30-page document like this one to "decode" just one of his equations. That is not the kind of material *I* want to be writing.

The equation at issue here is the following on page 89 of Spivak:

If $f: \mathbf{R}^n \to \mathbf{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbf{R}^n)$. By a minor modification we therefore obtain a 1-form df, defined by

$$df(p)(v_p) = Df(p)(v). \tag{1}$$

In some sense I know he is creating a 1-form called df from a 0-form called f.

But what is p?

What variable(s) is f a function of?

How does this equation relate to Sjamaar's idea of a flat space on the left and a manifold on the right, one instance of which is his figure on page 37?

In trying to correlate with Sjamaar, it is unfortunate as well that Sjamaar uses at least three different conventions for his variables. Let's review this issue right here. I will dig a little into each chapter in the following few pages.

1. In **Chapter 2**, Sjamaar writes things like this:

$\alpha = \Sigma_i f_I(x) dx_I$	page 17, a k-form on R^n
$f(\mathbf{x}) = f(\mathbf{x})$	a 0-form
$df = \Sigma_i \, (\partial f/\partial x_i) dx^i$	p 20, a 1-form
$d(df) = 0$	$d^2 = 0$, page 22

Everything here is a function of x, so there is some underlying x-space", no pictures are drawn, it is just some R^n . Now consider

$\alpha = \Sigma_I f_I(x) dx^I$	f_I = a scalar function with label I
$d\alpha = \Sigma_I \, df_I(x) \wedge dx^I$	// page 20 A

I never realized it before, but there must be a had in there as I have just shown. We are adding another vector to the wedge product inside dx^I. This is hidden in Sjamaar! You can now write

$df_I(x) = \Sigma_j (\partial f^I/\partial x^j) \, dx^j$ // dual vector = sum of dual vectors

Then we get

$d\alpha = \Sigma_I\, df_I(x) \wedge dx^I = \Sigma_I \Sigma_j (\partial f^I/\partial x^j)\, dx^j \wedge dx^I$ // another new result

2. In **Chapter 3**, Sjamaar changes things and writes now

$\alpha = \Sigma_i f_I(y) dy_I$ page 36

so suddenly we are operating in some y-space instead of x-space. We then have a transformation

$\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ goes with Figure on page 37

With this transformation is associated a certain pullback operator φ* which he endows with these properties

$\varphi^* f = f \circ \varphi$ f is a function **property 1**

So consider

$f(y)$ starting function in y-space

$(\varphi^* f)(x) = f(\varphi(x)) = (f \circ \varphi)(x)$ pulling it back to x-space

So we end up with (φ*f) being a function in x-space, totally clear .

The second property Sjamaar provides is this

dy_I starting basic form in y-space

$(\varphi^*(dy_I)) = d\varphi_I$ page 37 D **property 2**

An example of this would be

$(\varphi^*(dy_1 dy_2)) = d\varphi_1 d\varphi_2$

His page 38 example is important to me. There he shows

$\varphi_1(x),\ \varphi_2(x)$ each is an explicit function of x

$d\varphi^1(x) = (\partial\varphi^1/\partial x^1)(x)dx^1 + (\partial\varphi^1/\partial x^2)(x)dx^2$ = explicit
$d\varphi^2(x) = (\partial\varphi^2/\partial x^1)(x)dx^1 + (\partial\varphi^2/\partial x^2)(x)dx^2$ = explicit

So what he really is saying is this:

$(\varphi^*(dy_1 \wedge dy_2)) = d\varphi_1(\mathbf{x}) \wedge d\varphi_2(\mathbf{x}) = q(\mathbf{x})\, dx^1 \wedge dx^2$ q(x) = explicit

I know that one could write this last equations as

$d\varphi^1(x) = (D\varphi)_{11}(x)dx^1 + (D\varphi)_{12}(x)dx^2$ = explicit
$d\varphi^2(x) = (D\varphi)_{21}(x)dx^1 + (D\varphi)_{22}(x)dx^2$ = explicit

In matrix notation these two equations would then be

$d\boldsymbol{\varphi}(x) = (D\varphi(x))\, d\mathbf{x}$ but beware, *in general* $(D\varphi)$ is not a square matrix: n cols, m rows

Now if the space on the right on page 37 was $R^m = R^1$ then we have $y = \varphi(\mathbf{x})$ and there is only a single y variable and a single scalar φ function. Then we have just

$d\varphi(x) = (D\varphi)_{11}(x)dx^1 + (D\varphi)_{12}(x)dx^2$

and in matrix notation

$d\varphi(x) = (D\varphi(x))\, d\mathbf{x}$ $(D\varphi(x))$ = a matrix consisting of a single row. (example of non-square)

Sja proves the three little pullback rules shown middle of page 38.

Comment: In the last equation above, notice that φ is a transformation function, it is not a function that is part of some form like f_I, so this seems unrelated to our Spi clip I am studying in this doc. The φ function instead is related to the notion of doing a pullback.

However: almost the very next equation in Spi says this:

4-7 Theorem. *If* $f: \mathbf{R}^n \to \mathbf{R}$ *is differentiable, then*

$$df = D_1 f \cdot dx^1 + \cdots + D_n f \cdot dx^n. \tag{2}$$

and I would write this as

$df(x) = (Df(x))\, d\mathbf{x}$

and then this has exactly the form I show above. Spi seems to be proving his theorem 4.7 from the previous clip (1), given that $dx^i = \lambda^i$.

In Ch 3 Sja goes on to use property 1 and property 2 to pull back a general case k-form. This action is sort of buried within Sja page 40. I will outline the steps right here for a k-form

$\alpha = \Sigma'_I\ f_I(y)\, dy^I$ in this sum, each dy^I has k-factors

$\varphi^*\alpha = \Sigma'_I\ \varphi^*(f_I(y))\, \varphi^*(dy^I) = \Sigma'_I f_I(\varphi(x))\, d\varphi_I$

where I use Σ' for my ordered sums. Now I do have this equation from above,

$d\boldsymbol{\varphi}(x) = (D\varphi(x))\, d\mathbf{x}$ a non-square matrix in general, it has n cols and m rows

but this $d\boldsymbol{\varphi}(x)$ is NOT the $d\boldsymbol{\varphi}_I$ product shown above. I am able to show elsewhere that in fact

$d\varphi^I = \Sigma'_J \det (D\varphi(x)^I{}_J)\, dx^J$ det is of a kxk minor of the mxn matrix Dφ

where both $d\varphi_I$ and dx_I are in ordered form, and that is what makes the det appear. I do not have a problem with that. So we end up then with

$\varphi^*\alpha = \Sigma'_I\ f_I(\varphi(x))\ [\Sigma'_J \det (D\varphi(x)^I{}_J)\, dx^J\ ]$

$= \Sigma'_J\ [\ \Sigma'_I\ f_I(\varphi(x)) \det (D\varphi(x)^I{}_J)\ ]\ dx^J$

$= \Sigma'_J\ g_J(x) dx^J$ $g_J(x) = [\ \Sigma'_I\ f_I(\varphi(x)) \det (D\varphi(x)^I{}_J)\ ]$

So I am very happy with Sja Chapter 3 really. This is the pulled back form in x-space.

3. In **Chapter 4,** Sjamaar changes notation yet again. He now has a new version of figure page 37

left side t-space domain U = a < t < b

right side x-space R^n but coordinate there is concealed, but is (x,y) in an example

I am sure Sja intends to use x on the right. So his transformation is now

$\mathbf{x} = \mathbf{c}(t)$ // replacing $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ of Chapter 3.

So in this new notation context, and considering he is doing only 1-forms so k = 1, what do my equations above look like?

$\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ → $\mathbf{x} = \mathbf{c}(t)$ y→x x→ t

$\alpha = \Sigma'_I\ f_I(y)\, dy^I$ → $\alpha = f_I(x)\, dx^I = f_i(x)\, dx^i$ dx^I = just one factor

$\varphi^*\alpha = \Sigma'_J\ [\ \Sigma'_I\ f_I(\varphi(x)) \det (D_{\mathbf{x}}\varphi(x)^I{}_J)\ ]\ dx^J$

→ $c^*\alpha = \Sigma'_J\ [\ \Sigma'_I\ f_I(c(t)) \det (D_t c(t)^I{}_J)\ ]\ dt^J$

$= [\ \Sigma'_I\ f_I(\mathbf{c}(t)) \det [(D_t\mathbf{c}(t))^I{}_1)]\ ]\ dt$

$= [\ \Sigma_i\ f_i(\mathbf{c}(t)) \det [(D_t\mathbf{c}(t))^i{}_1)]\ ]\ dt$

$= [\ \Sigma_i\ f_i(\mathbf{c}(t))\ (D_t\mathbf{c}(t))^i{}_1)\ ]\ dt$ // det of 1x1 matrix is matrix

$$= [\ \Sigma_{i=1}^{n}\ f_i(\mathbf{c}(t))\ \partial_t c^i(t)\ ]\ dt \qquad \text{// agrees Sja p 47 *(4.1)}$$

He then writes

$$\int_c \alpha = \int_c f_i(\mathbf{x})\ dx^i = \int_C \mathbf{f}(\mathbf{x}) \bullet d\mathbf{x} = \int_a^b c^*\alpha = \int_a^b dt\ [\ \Sigma_{i=1}^{n}\ f_i(\mathbf{c}(t))\ \partial_t c^i(t)\ ]$$

This is the "line integral" of a function defined on a curve over that curve c. The dt integral is then the pullback of this integral where it is pulled back from x-space to t-space. This is the first time Sjamaar address any kind of integral in his pdf. This is the integral of a 1-form.

4. In **Chapter 5,** Sjamaar generalizes the above Chapter 4 work and the context is now

$$\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x}) \qquad \rightarrow \qquad \mathbf{x} = \mathbf{c}(\mathbf{t}) \qquad \mathbf{y} \rightarrow \mathbf{x} \quad \mathbf{x} \rightarrow t$$

where **x** has n components and **t** has k components. The integer n is not mentioned for some reason but I am sure it is still n. Our pull back stuff is then

$$\alpha = \Sigma'_I\ f_I(\mathbf{y})\ dy^I \qquad \rightarrow \qquad \alpha = \Sigma'_I\ f_I(\mathbf{x})\ dx^I$$

$$\varphi^*\alpha = \Sigma'_J\ [\ \Sigma'_I\ f_I(\varphi(\mathbf{x}))\ \det\ (D\varphi(\mathbf{x})^I{}_J)\ ]\ dx^J$$

$$\rightarrow\ c^*\alpha = \Sigma'_J\ [\ \Sigma'_I\ f_I(\mathbf{c}(\mathbf{t}))\ \det\ (Dc(\mathbf{t})^I{}_J)\ ]\ dt^J$$

Sja then considers this thing integrated over a "unit cube" in t-space,.

$$\int_c \alpha = \int_c \Sigma'_I\ f_I(\mathbf{y})\ dy^I = \int_{[0,1]^k} c^*\alpha$$

$$= \int_{[0,1]^k}\ \Sigma'_J\ [\ \Sigma'_I\ f_I(\mathbf{c}(\mathbf{t}))\ \det\ (Dc(\mathbf{t})^I{}_J)\ ]\ dt^J$$

$$= \Sigma'_J\ \Sigma'_I \int_{[0,1]^k}\ dt^J\ \ [f_I(\mathbf{c}(\mathbf{t}))\ \det\ ((Dc(\mathbf{t}))^I{}_J)\ ]$$

$$= \int_{[0,1]^k}\ \Sigma'_J\ g_J(\mathbf{t})\ dt^J$$

Here in $\int_c \alpha$ the **c** represents a surface or manifold of dimension k. The letter **c** is bolded both here and in the 1-form Chapter 4 because it is a set of points in R^n , be it a curve or a surface.

Note that dt^J is still a wedge product!! The pullback above has many terms in the Σ'_J sum, and each of those corresponding dt^J is an ordered wedge product of k of the dt^i selected from your set of n.

Sjamaar now wants to consider a special case where m = k, so then the sum above has only one term and dt^J is the full volume differential in the left "flat" space. Then the above becomes

$$\int_c \alpha = \int_{[0,1]^k} g(\mathbf{t})\, dt^1 \wedge dt^2 \wedge \ldots \wedge dt^k$$

He then comes to the issue of Stokes' Theorem. For that theorem, he wants to replace α by dα above as the k form, so he has

$$\int_c d\alpha = \int_{[0,1]^k} g(\mathbf{t})\, dt^1 \wedge dt^2 \wedge \ldots \wedge dt^k = \int_{[0,1]^k} c^*(d\alpha) = \int_{[0,1]^k} d(c^*\alpha) \qquad \text{p 63 A}$$

Since dα is a k-form, d(c*α) is also a k-form in the pulled back space, and that means that c*α must be a k-1 form, so it must then be true that

$$c^*\alpha = \Sigma_{i=1}^k \, g_i(t) \; dt^1 \wedge dt^2 \wedge \ldots \widehat{dt^i} \ldots \wedge dt^k \qquad \text{p 63 B}$$

where red italic dt^i means dt^i is missing. The sum is just over all the possibly allowed differentials for a k-1 form. I will now try to regurgitate the page 63 proof of Stokes theorem for k-cubes and chains. We have

$$d(c^*\alpha) = d\left[\Sigma_{i=1}^k \, g_i(t) \; dt^1 \wedge dt^2 \wedge \ldots \widehat{dt^i} \ldots \wedge dt^k \right]$$

$$= \Sigma_{i=1}^k \, dg_i(t) \wedge \; dt^1 \wedge dt^2 \wedge \ldots \widehat{dt^i} \ldots \wedge dt^k$$

$$= \Sigma_{i=1}^k \, \left[\Sigma_{j=1}^k (\partial g_i(t)/\partial t^j)\, dt^j\right] \wedge \; dt^1 \wedge dt^2 \wedge \ldots \widehat{dt^i} \ldots \wedge dt^k$$

I have updated the Ch 2 notes above to show the extra hat here! Now if $j \neq i$, the total wedge product will vanish, so we have only the j = i terms surviving and we get

$$d(c^*\alpha) = \Sigma_{i=1}^k \, (\partial g_i(t)/\partial t^i)\, dt^i \wedge \; dt^1 \wedge dt^2 \wedge \ldots \widehat{dt^i} \ldots \wedge dt^k$$

We now slide the dt_i into the red hole and pick up a sign $(-1)^{i-1} = (-1)^{i+1}$ because you have to slide i-1 places. Then we have

$$d(c^*\alpha) = \Sigma_{i=1}^k \, (-1)^{i+1} \; (\partial g_i(t)/\partial t^i) \; (dt^1 \wedge dt^2 \wedge \ldots \wedge dt^k)$$

where we are back to the full volume element. Putting this into the above we get

$$\int_c d\alpha = \int_{[0,1]^k} d(c^*\alpha)$$

$$= \Sigma_{i=1}^k \, (-1)^{i+1} \int_{[0,1]^k} (\partial g_i(t)/\partial t^i) \; (dt^1 \wedge dt^2 \wedge \ldots \wedge dt^k) \qquad \text{// page 63 C}$$

This then is the pullback of $\int_c$ dα, where dα is a k-form, to t-space. I will pause here because this is where I got into trouble and sent Sjamaar an email and I am not sure I ever cleared that matter up. If that were cleared up, I could continue to obtain Stokes' Theorem on a k-cube. But right now I am trying to stay focused on a different matter which is the dual space stuff, so let this lie for now.

I will skip Chapter 6 hoping that the second manifold definition $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{c}$ won't be important for my purpose. We then come to

5. In **Chapter 7,** Sjamaar puts his differential forms onto manifolds, somehow. In the "first definition" this is done by treating the manifold as the union of a bunch of patches labeled by i. On each patch the form takes some value α_i and we have certain consistency conditions.

It is the second definition where the dual space business arises and that is my main interest. I want to nail down his notation on the various spaces here, as I have done in each of the above chapter sections.

On page 88 Sja talks about φ: U→V and about φ(x). His example says $\alpha = \Sigma'_I\ f_I(\mathbf{y})\ dy^I$ so I think he is exactly in the Chapter 3 (page 37 figure) framework where $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ is the transformation. Recall above that in this notation environment we had

$$\alpha = \Sigma'_I\ f_I(\mathbf{y})\ dy^I$$

$$\varphi^*\alpha\ = \Sigma'_J\ g_J(\mathbf{x})dx^J \qquad \text{where} \qquad g_J(\mathbf{x}) = [\ \Sigma'_I\ f_I(\varphi(\mathbf{x}))\ \det\ (D\varphi(x)^I{}_J)\ ]$$

OK, so much for Sjamaar's various notations. I will now process page 87-88 from scratch.

Whenever dx^i does NOT mean $d\lambda^i$, I will put it in red. So

$$dx^i\ = \lambda^i$$

dx^i = a calculus differential

What is Example 7.15 trying to show? Write

$$\lambda = \Sigma_I a_I\ dx^I \qquad dx^I = \lambda^{i_1} \wedge \lambda^{i_2}\ \ldots.\ \wedge\ \lambda^{i_k} \quad \text{in dual space } \Lambda^k$$

$$\lambda = \Sigma'_I\ a_{i_1 i_2 \ldots i_k}\ \lambda^{i_1} \wedge \lambda^{i_2}\ \ldots.\ \wedge\ \lambda^{i_k}$$

My language for this in wedge doc is

$$\mathcal{T}_\wedge = \Sigma_{i_1 i_2 \ldots . i_k}\ T_{i_1 i_2 \ldots . i_k}\ (\lambda^{i_1} \wedge \lambda^{i_2} \ldots..\wedge \lambda^{i_k})\ . \qquad \mathcal{T}_\wedge = \Sigma_I\ T_I\ \lambda_\wedge{}^I \qquad (8.4.4)$$

$$\mathcal{T}_\wedge = \Sigma_{1 \le i_1 < i_2 < \ldots . < i_k \le n}\ A_{i_1 i_2 \ldots i_k}\ (\lambda^{i_1} \wedge \lambda^{i_2} \ldots..\wedge \lambda^{i_k})\ . \qquad \mathcal{T}_\wedge = \Sigma'_I\ A_I\ \lambda_\wedge{}^I \qquad (8.4.7)$$

Now the objects T could be fields and could therefore depend on x , or they could be constants.

I am not sure why Sja wants the a_I to be constants in his $\lambda = \Sigma_I a_I\, dx^I$. But let's now add the generic arguments from V^k to get our tensor functions, so then

$$\lambda(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) = \Sigma'_I\, a_{i_1 i_2 \ldots i_k}\, (\lambda^{i_1} \wedge \lambda^{i_2} \wedge .... \wedge \lambda^{i_k})(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)$$

OK, if a_I is constant not depending on the v_I, then λ is linear and alternating. I agree. There is no mention yet of a_I possibly depending on **x** in some x-space. Now write the above more generally as

$$\lambda(\mathbf{v}_J) = \Sigma_I a_I\, dx^I(\mathbf{v}_J) = \Sigma_I a_I\, \lambda\wedge^I(\mathbf{v}_J)$$

Then for sure

$$\lambda(\mathbf{e}_J) = \Sigma_I a_I\, \lambda\wedge^I(\mathbf{e}_J) = \Sigma_I a_I\, \delta^I{}_J = a_J$$

and we get the main result of Example 7.15 which is this

$$a_J = \lambda(\mathbf{e}_J)$$

I am very familiar with this result in wedge doc where it appears as

$$\mathcal{T}(\mathbf{e}_{j_1},\mathbf{e}_{j_2}, ... \mathbf{e}_{j_k}) = \langle T \mid \mathbf{e}_{j_1},\mathbf{e}_{j_2}, ... \mathbf{e}_{j_k} \rangle$$

$$= \Sigma_{i_1 i_2 \ldots . i_k}\, T_{i_1 i_2 \ldots . i_k}\, (\mathbf{e}_{j_1})^{i_1} (\mathbf{e}_{j_2})^{i_2} .... (\mathbf{e}_{j_k})^{i_k}$$

$$= T_{j_1 j_2 \ldots . j_k} . \qquad \mathcal{T}(\mathbf{e}_J) = T_J \quad (2.11.f.9)$$

I will now review how I arrived at this last result:

$\mathcal{T}\wedge = \Sigma_I\, T_I\, \lambda\wedge^I$ // symmetric tensor expansion

$\mathcal{T}\wedge(\mathbf{v}_J) = \Sigma_I\, T_I\, \lambda\wedge^I(\mathbf{v}_J)$

$\mathcal{T}\wedge(\mathbf{e}_J) = \Sigma_I\, T_I\, \lambda\wedge^I(\mathbf{e}_J) = T_J$

So it is important to have Σ_I symmetric expansion in this proof. So in Example 7.15, you really have to assume that $\lambda = \Sigma_I a_I dx^I$ is a full symmetric sum, and only then does it follow that $a_I = \lambda(\mathbf{e}_I)$. Since Sja uses Σ_J for both symmetric and ordered sums, things are not quite clear which he means, and now we know for thios case.

Thus we can write

$$\lambda = \Sigma_I a_I\, dx^I = \Sigma_I\, \lambda(\mathbf{e}_I)\, dx^I$$

Now we want to generalize so that this form has an x-dependent coefficient, so we are talking then about a form in x-space, which is NOT the starting position for p 37, but write it anyway as

$$\lambda_{\mathbf{x}} = \Sigma_{\mathbf{I}} a_{\mathbf{I}}(\mathbf{x})\, dx^{\mathbf{I}} = \Sigma_{\mathbf{I}} \lambda_{\mathbf{x}}(\mathbf{e}_{\mathbf{I}})\, dx^{\mathbf{I}}$$

or

$$\alpha_{\mathbf{x}} = \Sigma_{\mathbf{I}} f_{\mathbf{I}}(\mathbf{x})\, dx^{\mathbf{I}} = \Sigma_{\mathbf{I}} \alpha_{\mathbf{x}}(\mathbf{e}_{\mathbf{I}})\, dx^{\mathbf{I}}$$

Notice that $\alpha_{\mathbf{x}}$ is a dual space functional on the left, while $\alpha_{\mathbf{x}}(\mathbf{e}_{\mathbf{I}})$ on the right is this functional *evaluated* at $e_{\mathbf{I}}$. I could easily put this into Dirac notation.

So I am now at the starting blocks on page 88 top. I want x to be the variable of the pull back space on left side of p 37, so lets start now with

$$\alpha_{\mathbf{y}} = \Sigma_{\mathbf{I}} f_{\mathbf{I}}(\mathbf{y})\, dy^{\mathbf{I}} = \Sigma_{\mathbf{I}} \alpha_{\mathbf{y}}(\mathbf{e}_{\mathbf{I}})\, dy^{\mathbf{I}} \qquad \text{a k-form in y-space of page 37 fig}$$

where in our new coordinates we have $dy^{\mathbf{i}} = \lambda^{\mathbf{i}}$.

What is this form when it is pulled back to x-space? We know from work above that

$$\alpha_{\mathbf{y}} = \Sigma'_{\mathbf{I}}\ f_{\mathbf{I}}(\mathbf{y})\, dy^{\mathbf{I}} \qquad (0)$$

$$(\varphi^*\alpha)_{\mathbf{x}} = \Sigma'_{\mathbf{J}}\, g_{\mathbf{J}}(\mathbf{x}) dx^{\mathbf{J}} \qquad \text{where} \qquad g_{\mathbf{J}}(\mathbf{x}) = [\ \Sigma'_{\mathbf{I}}\ f_{\mathbf{I}}(\varphi(\mathbf{x}))\ \det\ (D\varphi(x)^{\mathbf{I}}{}_{\mathbf{J}})\ ]$$

where I have now added little subscripts on the forms indicating where they are being evaluated.

Let's see if we can obtain this *same* result from the very obscure starting point of page 88 A which says

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_{\mathbf{k}}) = \alpha_{\mathbf{y}}(R\mathbf{v}_1, R\mathbf{v}_2.....R\mathbf{v}_{\mathbf{k}}) \qquad (1)$$

We anticipate a result of this form

$$(\varphi^*\alpha)_{\mathbf{x}} = \Sigma_{\mathbf{J}}\, g_{\mathbf{J}}(\mathbf{x}) dx^{\mathbf{J}} \qquad (2)$$

so that

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v}_{\mathbf{I}}) = \Sigma_{\mathbf{J}}\, g_{\mathbf{J}}(\mathbf{x}) dx^{\mathbf{J}}(\mathbf{v}_{\mathbf{I}}) \qquad (3)$$

and then

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{e}_{\mathbf{I}}) = \Sigma_{\mathbf{J}}\, g_{\mathbf{J}}(\mathbf{x}) dx^{\mathbf{J}}(\mathbf{e}_{\mathbf{I}}) = g_{\mathbf{I}}(\mathbf{x}) \qquad (4)$$

Thus we can anticipate a result of this form (I will repair Σ, Σ' stuff later, trying to get first cut)

$$(\varphi^*\alpha)_{\mathbf{x}} = \Sigma_{\mathbf{J}}\, g_{\mathbf{J}}(\mathbf{x}) dx^{\mathbf{J}} = \Sigma_{\mathbf{J}}\, (\varphi^*\alpha)_{\mathbf{x}}(\mathbf{e}_{\mathbf{J}})\, dx^{\mathbf{J}} \qquad (5)$$

Now go back to (1) and evaluate at $\mathbf{e}_J$ and use (4) as well to say

$$g_J(\mathbf{x}) = (\varphi^*\alpha)_{\mathbf{x}}(\mathbf{e}_{j_1},\mathbf{e}_{j_2}....\mathbf{e}_{j_k}) = \alpha_{\mathbf{y}}(R\mathbf{e}_{j_1}, R\mathbf{e}_{j_2}.....R\mathbf{e}_{j_k}) \qquad (6) \qquad \text{waypoint 1}$$

Now install the $\alpha_{\mathbf{y}}$ expansion from (0) above to get

$$g_J(\mathbf{x}) = \alpha_{\mathbf{y}}(R\mathbf{e}_{j_1}, R\mathbf{e}_{j_2}.....R\mathbf{e}_{j_k}) = \Sigma'_I\ f_I(\mathbf{y})\ dy^I(R\mathbf{e}_{j_1}, R\mathbf{e}_{j_2}.....R\mathbf{e}_{j_k}) \qquad \text{waypoint 2} \qquad (7)$$

Now recall from wedge doc

$$(\lambda^{j_1} \wedge \lambda^{j_2} \wedge .... \wedge \lambda^{i_k})(v_{i_1},v_{i_2}....v_{i_k}) = (1/k!)\ \det[\ (v_{i_*})^{j_*}] \qquad (8.3.9a)$$

$$= (1/k!)\ \det\ (v_I{}^J) \qquad (8)$$

which implies that

$$(\lambda^{i_1} \wedge \lambda^{i_2} \wedge .... \wedge \lambda^{i_k})(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) = (1/k!)\ \det[\ v_Z{}^I\ ] = dy^I(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) \qquad (9)$$

So far then we have

$$dy^I(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) = (1/k!)\ \det[\ v_Z{}^I\ ] \qquad (10)$$

Here $v_Z{}^I$ is a kxk matrix whose row indices are 1,2...k and whose column indices are i_i, i_2.. i_k. A typical element of the matrix would be $(\mathbf{v}_2)^{i_1}$. Now if we replace $\mathbf{v}_2 \rightarrow R\mathbf{e}_{j_2}$ a typical element of that matrix becomes $(\ R\mathbf{e}_{j_2})^{i_1} = R^{i_1}{}_s(\mathbf{e}_{j_2})^s = R^{i_1}{}_s\delta_{j_2}{}^s = R^{i_1}{}_{j_2}$. Therefore

$$dy^I(R\mathbf{e}_{j_1}, R\mathbf{e}_{j_2}.....R\mathbf{e}_{j_k}) = (1/k!)\ \det[\ R^I{}_J] = (1/k!)\ \det[\ (D\varphi)^I{}_J] \qquad (11)$$

We find then that

$$g_J(\mathbf{x}) = \Sigma'_I\ f_I(\mathbf{y})\ dy^I(R\mathbf{e}_{j_1}, R\mathbf{e}_{j_2}.....R\mathbf{e}_{j_k})$$

$$= \Sigma'_I\ f_I(\mathbf{y})\ (1/k!)\ \det[\ (D\varphi)^I{}_J] \qquad (12)$$

Now I think I have **finally learned the reason for the Spivak normalization** which in fact Sjamaar quietly uses on page 85 C. So I will need to work this in. In Spivak normalization we get above

$$g_J(\mathbf{x}) = \Sigma'_I\ f_I(\mathbf{y}) \det[\ (D\varphi)^I{}_J]$$

using our tensor-function starting point for a description of the pullback,

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) = \alpha_{\mathbf{y}}(R\mathbf{v}_1, R\mathbf{v}_2.....R\mathbf{v}_k)$$

Since this $g_J(\mathbf{x})$ is the same expression found in Chapter 3, it is a justified way to write the pullback in tensor function space.

I could perhaps maintain my own normalization and replace the Spivak rule with this rule

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}....\mathbf{v_k}) = k!\ \alpha_{\mathbf{y}}(R\mathbf{v_1}, R\mathbf{v_2}.....R\mathbf{v_k}) \qquad (13)$$

or something like that. The original rule appears here in Spivak

$$f^*\omega(p)(v_1, \ldots, v_k) = \omega(f(p))(f_*(v_1), \ldots, f_*(v_k)). \qquad (14)$$

where

$$f_*(v_p) \doteq (Df(p)(v))_{f(p)}.$$

which compare to

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}....\mathbf{v_k}) = \alpha_{\mathbf{y}}((D\varphi)\mathbf{v_1}, (D\varphi)\mathbf{v_2}.....(D\varphi)\mathbf{v_k})$$

The translation rules for Spivak to Sjamaar are:

f	→	φ
p	→	x
f(p)	→	φ(x)
ω(p)	→	$\alpha_{\mathbf{x}}$
ω(f(p))	→	$\alpha_{\mathbf{y}}$

so the above Spivak equation becomes

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}....\mathbf{v_k}) = \alpha_{\mathbf{y}}[\ (D\varphi)(\mathbf{x})\ \mathbf{v_1},\ \text{etc}]$$

The match is almost perfect, not sure why he has f(p) as subscript on $[\ (D\varphi)(\mathbf{x})\ \mathbf{v_1}]_{\mathbf{y}}$.

So I am finally happy with (14) above from Spivak.

Sja now transforms from $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ to $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$ on the bottom of page 88, so we have

$$(\psi^*\alpha)_{\mathbf{t}}(\mathbf{v_1},\mathbf{v_2}....\mathbf{v_k}) = \alpha_{\mathbf{x}}((D\psi(\mathbf{t}))\mathbf{v_1}, (D\psi(\mathbf{t}))\mathbf{v_2}.....(D\boldsymbol{\psi}(t))\mathbf{v_k}) \qquad \text{// page 88 C}$$

Now you think of point $\mathbf{x}$ being on a manifold and you think of $\alpha_{\mathbf{x}}$ being an element of the cotangent space which is just the dual space to $T_{\mathbf{x}}M$.

I still have many questions:

- why do I need tensor functions?

- why are we associating forms with the dual space functionals?
- clear up which sums are Σ and which are Σ'
- what is the connection to ordinary differentials?