

LECTURE 1: DIFFERENTIAL FORMS

1. 1-FORMS ON $\mathbb{R}^n$

In calculus, you may have seen the *differential* or *exterior derivative* df of a function $f(x, y, z)$ defined to be

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

The expression df is called a *1-form*. But what does this really mean?

Definition: A *smooth 1-form* ϕ on $\mathbb{R}^n$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^n$, i.e.,

$$\phi : T\mathbb{R}^n \rightarrow \mathbb{R}$$

with the properties that

1. ϕ is linear on the tangent space $T_x\mathbb{R}^n$ for each $x \in \mathbb{R}^n$.
2. For any smooth vector field $v = v(x)$, the function $\phi(v) : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth.

Given a 1-form ϕ , for each $x \in \mathbb{R}^n$ the map

$$\phi_x : T_x\mathbb{R}^n \rightarrow \mathbb{R}$$

is an element of the dual space $(T_x\mathbb{R}^n)^*$. When we extend this notion to all of $\mathbb{R}^n$, we see that the space of 1-forms on $\mathbb{R}^n$ is dual to the space of vector fields on $\mathbb{R}^n$.

In particular, the 1-forms $dx^1, \dots, dx^n$ are defined by the property that for any vector $v = (v^1, \dots, v^n) \in T_x\mathbb{R}^n$,

$$dx^i(v) = v^i.$$

The dx^i 's form a basis for the 1-forms on $\mathbb{R}^n$, so any other 1-form ϕ may be expressed in the form

$$\phi = \sum_{i=1}^n f_i(x) dx^i.$$

If a vector field v on $\mathbb{R}^n$ has the form

$$v(x) = (v^1(x), \dots, v^n(x)),$$

then at any point $x \in \mathbb{R}^n$,

$$\phi_x(v) = \sum_{i=1}^n f_i(x) v^i(x).$$

2. p -FORMS ON $\mathbb{R}^n$

The 1-forms on $\mathbb{R}^n$ are part of an algebra, called the *algebra of differential forms* on $\mathbb{R}^n$. The multiplication in this algebra is called *wedge product*, and it is skew-symmetric:

$$dx^i \wedge dx^j = -dx^j \wedge dx^i.$$

One consequence of this is that $dx^i \wedge dx^i = 0$.

If each summand of a differential form ϕ contains p dx^i 's, the form is called a p -form. Functions are considered to be 0-forms, and any form on $\mathbb{R}^n$ of degree $p > n$ must be zero due to the skew-symmetry.

A basis for the p -forms on $\mathbb{R}^n$ is given by the set

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}.$$

Any p -form ϕ may be expressed in the form

$$\phi = \sum_{|I|=p} f_I dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

where I ranges over all multi-indices $I = (i_1, \dots, i_p)$ of length p .

Just as 1-forms act on vector fields to give real-valued functions, so p -forms act on p -tuples of vector fields to give real-valued functions. For instance, if ϕ, ψ are 1-forms and v, w are vector fields, then

$$(\phi \wedge \psi)(v, w) = \phi(v)\psi(w) - \phi(w)\psi(v).$$

In general, if $\phi_1, \dots, \phi_p$ are 1-forms and $v_1, \dots, v_p$ are vector fields, then

$$(\phi_1 \wedge \cdots \wedge \phi_p)(v_1, \dots, v_p) = \sum_{\sigma \in S_p} \text{sgn}(\sigma) \phi_1(v_{\sigma(1)}) \phi_2(v_{\sigma(2)}) \cdots \phi_p(v_{\sigma(p)}).$$

3. THE EXTERIOR DERIVATIVE

The *exterior derivative* is an operation that takes p -forms to $(p+1)$ -forms. We will first define it for functions and then extend this definition to higher degree forms.

Definition: If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable, then the exterior derivative of f is the 1-form df with the property that for any $x \in \mathbb{R}^n$, $v \in T_x \mathbb{R}^n$,

$$df_x(v) = v(f),$$

i.e., $df_x(v)$ is the directional derivative of f at x in the direction of v .

It is not difficult to show that

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

The exterior derivative also obeys the Leibniz rule

$$d(fg) = g df + f dg$$

and the chain rule

$$d(h(f)) = h'(f) df.$$

We extend this definition to p -forms as follows:

Definition: Given a p -form $\phi = \sum_{|I|=p} f_I dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, the exterior derivative $d\phi$ is the $(p+1)$ -form

$$d\phi = \sum_{|I|=p} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

If ϕ is a p -form and ψ is a q -form, then the Leibniz rule takes the form

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi.$$

Very Important Theorem: $d^2 = 0$. i.e., for any differential form ϕ ,

$$d(d\phi) = 0.$$

Proof: First suppose that f is a function, i.e., a 0-form. Then

$$\begin{aligned} d(df) &= d\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i\right) \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i \\ &= \sum_{i < j} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j}\right) dx^i \wedge dx^j \\ &= 0 \end{aligned}$$

because mixed partials commute.

Next, note that dx^i really does mean $d(x^i)$, where x^i is the i th coordinate function. So by the argument above, $d(dx^i) = 0$. Now suppose that

$$\phi = \sum_{|I|=p} f_I dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Then by the Leibniz rule,

$$\begin{aligned}
 d(d\phi) &= d\left(\sum_{|I|=p} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}\right) \\
 &= \sum_{|I|=p} [d(df_I) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} - df_I \wedge d(dx^{i_1}) \wedge \cdots \wedge dx^{i_p} + \cdots] \\
 &= 0. \quad \square
 \end{aligned}$$

Definition: A p -form ϕ is *closed* if $d\phi = 0$. ϕ is *exact* if there exists a $(p-1)$ -form η such that $\phi = d\eta$.

By the Very Important Theorem, every exact form is closed. The converse is only partially true: every closed form is *locally* exact. This means that given a closed p -form ϕ on an open set $U \subset \mathbb{R}^n$, any point $x \in U$ has a neighborhood on which there exists a $(p-1)$ -form η with $d\eta = \phi$.

4. DIFFERENTIAL FORMS ON MANIFOLDS

Given a smooth manifold M , a *smooth 1-form* ϕ on M is a real-valued function on the set of all tangent vectors to M such that

1. ϕ is linear on the tangent space $T_x M$ for each $x \in M$.
2. For any smooth vector field v on M , the function $\phi(v) : M \rightarrow \mathbb{R}$ is smooth.

So for each $x \in M$, the map

$$\phi_x : T_x M \rightarrow \mathbb{R}$$

is an element of the dual space $(T_x M)^*$.

Wedge products and exterior derivatives are defined similarly as for $\mathbb{R}^n$. If $f : M \rightarrow \mathbb{R}$ is a differentiable function, then we define the exterior derivative of f to be the 1-form df with the property that for any $x \in M$, $v \in T_x M$,

$$df_x(v) = v(f).$$

A local basis for the space of 1-forms on M can be described as before in terms of any local coordinate chart $(x^1, \dots, x^n)$ on M , and it is possible to show that the coordinate-based notions of wedge product and exterior derivative are in fact independent of the choice of local coordinates and so are well-defined.

More generally, suppose that M_1, M_2 are smooth manifolds and that $F : M_1 \rightarrow M_2$ is a differentiable map. For any $x \in M_1$, the differential dF (also denoted F_*) : $T_x M_1 \rightarrow T_{F(x)} M_2$ may be thought of as a *vector-valued* 1-form, because it is a linear map from $T_x M_1$ to the vector space $T_{F(x)} M_2$. There is an analogous map in the opposite direction for differential forms, called the *pullback* and denoted F^* . It is defined as follows.

Definition: If $F : M_1 \rightarrow M_2$ is a differentiable map, then

1. If $f : M_2 \rightarrow \mathbb{R}$ is a differentiable function, then $F^*f : M_1 \rightarrow \mathbb{R}$ is the function

$$(F^*f)(x) = (f \circ F)(x).$$

2. If ϕ is a p -form on M_2 , then $F^*\phi$ is the p -form on M_1 defined as follows: if $v_1, \dots, v_p \in T_x M_1$, then

$$(F^*\phi)(v_1, \dots, v_p) = \phi(F_*(v_1), \dots, F_*(v_p)).$$

In terms of local coordinates $(x^1, \dots, x^n)$ on M_1 and $(y^1, \dots, y^m)$ on M_2 , suppose that the map F is described by

$$y^i = y^i(x^1, \dots, x^n), \quad 1 \leq i \leq m.$$

Then the differential dF at each point $x \in M_1$ may be represented in this coordinate system by the matrix

$$\left[\frac{\partial y^i}{\partial x^j} \right].$$

The dx^j 's are forms on M_1 , the dy^i 's are forms on M_2 , and the pullback map F^* acts on the dy^i 's by

$$F^*(dy^i) = \sum_{j=1}^n \frac{\partial y^i}{\partial x^j} dx^j.$$

The pullback map behaves as nicely as one could hope with respect to the various operations on differential forms, as described in the following theorem.

Theorem: Let $F : M_1 \rightarrow M_2$ be a differentiable map, and let ϕ, η be differential forms on M_2 . Then

1. $F^*(\phi + \eta) = F^*\phi + F^*\eta$.
2. $F^*(\phi \wedge \eta) = F^*\phi \wedge F^*\eta$.
3. $F^*(d\phi) = d(F^*\phi)$.

5. THE LIE DERIVATIVE

The final operation that we will define on differential forms is the Lie derivative. This is a generalization of the notion of directional derivative of a function.

Suppose that $v(x)$ is a vector field on a manifold M , and let $\varphi : M \times (-\varepsilon, \varepsilon) \rightarrow M$ be the *flow* of v . This is the unique map that satisfies the conditions

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(x, t) &= v(\varphi(x, t)) \\ \varphi(x, 0) &= x.\end{aligned}$$

In other words, $\varphi_t(x) = \varphi(x, t)$ is the point reached at time t by flowing along the vector field $v(x)$ starting from the point x at time 0.

Recall that if $f : M \rightarrow \mathbb{R}$ is a smooth function, then the directional derivative of f at x in the direction of v is

$$\begin{aligned}v(f) &= \lim_{t \rightarrow 0} \frac{f(\varphi_t(x)) - f(x)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(\varphi_t^*(f) - f)(x)}{t}.\end{aligned}$$

Similarly, given a differential form ϕ we define the *Lie derivative* of ϕ along the vector field $v(x)$ to be

$$\mathcal{L}_v \phi = \lim_{t \rightarrow 0} \frac{\varphi_t^* \phi - \phi}{t}.$$

Fortunately there is a practical way to compute the Lie derivative. First we need the notion of the left-hook of a differential form with a vector field. Given a p -form ϕ and a vector field v , the *left-hook* $v \lrcorner \phi$ of ϕ with v (also called the *interior product* of ϕ with v) is the $(p-1)$ -form defined by the property that for any $w_1, \dots, w_{p-1} \in T_x \mathbb{R}^n$,

$$(v \lrcorner \phi)(w_1, \dots, w_{p-1}) = \phi(v, w_1, \dots, w_{p-1}).$$

For instance,

$$\frac{\partial}{\partial x} \lrcorner (dx \wedge dy + dz \wedge dx) = dy - dz.$$

Now according to *Cartan's formula*, the Lie derivative of ϕ along the vector field v is

$$\mathcal{L}_v \phi = v \lrcorner d\phi + d(v \lrcorner \phi).$$

Exercises

1. Classical vector analysis avoids the use of differential forms on $\mathbb{R}^3$ by converting 1-forms and 2-forms into vector fields by means of the following one-to-one correspondences. ($\varepsilon_1, \varepsilon_2, \varepsilon_3$ will denote the standard basis $\varepsilon_1 = [1, 0, 0]$, $\varepsilon_2 = [0, 1, 0]$, $\varepsilon_3 = [0, 0, 1]$.)

$$f_1 dx^1 + f_2 dx^2 + f_3 dx^3 \longleftrightarrow f_1 \varepsilon_1 + f_2 \varepsilon_2 + f_3 \varepsilon_3$$

$$f_1 dx^2 \wedge dx^3 + f_2 dx^3 \wedge dx^1 + f_3 dx^1 \wedge dx^2 \longleftrightarrow f_1 \varepsilon_1 + f_2 \varepsilon_2 + f_3 \varepsilon_3$$

Vector analysis uses three basic operations based on partial differentiation:

1. *Gradient* of a function f :

$$\text{grad}(f) = \sum_{i=1}^3 \frac{\partial f}{\partial x^i} \varepsilon_i$$

2. *Curl* of a vector field $v = \sum_{i=1}^3 v^i(x) \varepsilon_i$:

$$\text{curl}(v) = \left(\frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3} \right) \varepsilon_1 + \left(\frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1} \right) \varepsilon_2 + \left(\frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2} \right) \varepsilon_3$$

3. *Divergence* of a vector field $v = \sum_{i=1}^3 v^i(x) \varepsilon_i$:

$$\text{div}(v) = \sum_{i=1}^3 \frac{\partial v^i}{\partial x^i}$$

Prove that all three operations may be expressed in terms of exterior derivatives as follows:

1. $df \leftrightarrow \text{grad}(f)$
2. If ϕ is a 1-form and $\phi \leftrightarrow v$, then $d\phi \leftrightarrow \text{curl}(v)$.
3. If η is a 2-form and $\eta \leftrightarrow v$, then $d\eta \leftrightarrow \text{div}(v) dx^1 \wedge dx^2 \wedge dx^3$.

Show that the identities

$$\text{curl}(\text{grad}(f)) = 0$$

$$\text{div}(\text{curl}(v)) = 0$$

follow from the fact that $d^2 = 0$.

2. Let f and g be real-valued functions on $\mathbb{R}^2$. Prove that

$$df \wedge dg = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} dx \wedge dy.$$

(You may recognize this from the change-of-variables formula for double integrals.)

3. Suppose that ϕ, ψ are 1-forms on $\mathbb{R}^n$. Prove the Leibniz rule

$$d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi.$$

4. Prove the statement above that if $F : M_1 \rightarrow M_2$ is described in terms of local coordinates by

$$y^i = y^i(x^1, \dots, x^n), \quad 1 \leq i \leq m$$

then

$$F^*(dy^i) = \sum_{j=1}^n \frac{\partial y^i}{\partial x^j} dx^j.$$

5. Let (r, θ) be coordinates on $\mathbb{R}^2$ and (x, y, z) coordinates on $\mathbb{R}^3$. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by

$$F(r, \theta) = (\cos \theta, \sin \theta, r).$$

Describe the differential dF in terms of these coordinates and compute the pullbacks $F^*(dx)$, $F^*(dy)$, $F^*(dz)$.