Proof of a Lemma regarding *(*dxI)

Lemma: Show that

$$*(*dx_I) = (-1)^{kn+k}\, dx_I \qquad (1)$$

Take an example

$$dx_I = dx_a\, dx_b \ldots dx_q \qquad a < b < \ldots < q \qquad \text{a k-form, k } dx_i \text{ factors} \qquad (2)$$

Note that this is in R^n, so we have k and n to worry about. Then

$$*dx_I = S\, \Pi_h\, dx_i \text{ where we have a } \textit{hole} \text{ for each of } dx_a\, dx_b \ldots dx_q \text{ and S = a certain sign.} \qquad (3)$$

Then sign S is selected so that

$$dx_I(*dx_I) = +dx_1 \ldots dx_n = dV. \qquad (4)$$

Inserting (2) and (3) into (4),

$$dx_a\, dx_b \ldots dx_q\, S\, \Pi_h\, dx_i = dV \qquad (5)$$

and therefore

$$dx_a\, dx_b \ldots dx_q\, \Pi_h\, dx_i = SdV \qquad (6)$$

1. Now how do we find S?

I did this just for interest. The resulting expression for S is not needed in Section 2 below.

Combine (2) and (3) above to get

$$dx_I\, (*dx_I) = S\, dx_a\, dx_b \ldots dx_q\, \Pi_h\, dx_i \qquad (1.1)$$

The sign is found by right-sliding each of the dx_i listed into the correct position in $\Pi\, dx_i$. Write

$$dx_a\, dx_b \ldots dx_q\, \Pi_h\, dx_i = dx_a\, dx_b \ldots dx_q\, [\, dx_1 \ldots dx_a \ldots dx_b \ldots\ldots dx_q \ldots dx_n\,] \qquad (1.2)$$

Here red shows the holes. Let's start by sliding dx_a into its red hole. We first have to slide dx_a through k-1 other dx's to get it to the right, so this gives

$$\text{LHS}(1.2) = (-1)^{k-1} dx_b \ldots dx_q\, dx_a\, [\, dx_1 \ldots dx_a \ldots dx_b \ldots\ldots dx_q \ldots dx_n\,] \qquad (1.3)$$

Now we have to slide dx_a another (a-1) positions to get it into its proper hole. So,

$$= (-1)^{k-1}\,(-1)^{a-1} dx_b \ldots dx_q\ [\, dx_1 \ldots\ldots dx_b \ldots\ldots dx_q \ldots dx_n\,] \,. \tag{1.4}$$

Now we slide dx_b first to the right, picking up $(-1)^{k-2}$ and then we pick up $(-1)^{b-1}$ so

$$= (-1)^{k-1}\,(-1)^{a-1}(-1)^{k-2}\ (-1)^{b-1} \ldots dx_q\ [\, dx_1 \ldots\ldots dx_q \ldots dx_n\,] \,. \tag{1.5}$$

We repeat this little process for each of the k elements of the dx_I group.

When we get to the last item dx_q, since it is all by itself, we don't have a "first sign" since no sliding is necessary. So I think the k-signs are these (** means exponentiation)

$$(-1)^{k-1}(-1)^{k-2} \ldots\ldots(-1)^{k-k} \quad = \Pi_{i=0}^{k-1}\,(-1)^{i} = (-1)^{**}(\Sigma_{i=0}^{k-1}\, i) = (-1)^{**}\,(k-1)k/2$$

$$= (-1)^{k(k-1)/2} \tag{1.6}$$

where I use this classic result

1. $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{1.7}$$

Can this be simplified? Write out some cases:

k	$(-1)^{k(k-1)/2}$
0	+
1	+
2	-
3	-
4	+
5	+
6	-

(1.8)

I see no simplification.

The other sign contributions are this set of k factors

$$(-1)^{a-1}(-1)^{b-1}\ldots (-1)^{q-1} \ = \ (-1)^{a+b+..+q}\ (-1)^{k} \tag{1.9}$$

The total k related factor is then

$$(-1)^{k(k-1)/2}\ (-1)^{k} = (-1)^{k+k(k-1)/2} = (-1)^{k(k+1)/2} \tag{1.10}$$

My conclusion is that

$$dx_a\, dx_b \ldots dx_q\, \Pi_h\, dx_i \ = \ (-1)^{a+b+..+q}\ \ (-1)^{k(k+1)/2}[\, dx_1 \ldots\ldots dx_n\,]$$

$$= (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2} dV \tag{1.11}$$

Now recall from (4) and (1.1) and (1.11) that

$$dV = dx_I\,(*dx_I) = S\ dx_a\, dx_b\, ...\, dx_q\, \Pi_h\, dx_i = S\ (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2} dV \tag{1.12}$$

The conclusion then is that

$$S = (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2} \tag{1.13}$$

Example 1: $n = 6,\ k = 2,\ \ dx_I = dx_2 dx_5\ ,\ \ *dx_I = S\, dx_1 dx_3 dx_4 dx_6$ (1.14)

Claim of the above is that

$$S = (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2} = (-1)^{2+5}\,(-1)^3 = (-1)^{10} = +1$$

and then we are supposed to have

$$dx_I = dx_2 dx_5 \qquad *dx_I = dx_1 dx_3 dx_4 dx_6$$

Let's manually check to verify that $dx_I\,(*dx_I) = dV$

$$dx_I\,(*dx_I) = dx_2 dx_5\ dx_1 dx_3 dx_4 dx_6 = +\, dx_5\ dx_1 dx_2 dx_3 dx_4 dx_6 = +\, dx_1 dx_2 dx_3 dx_4 dx_5\ dx_6$$

This example succeeds.

Example 2: $n = 4$ and $k = 2$, $dx_I = dx_1 dx_2$, $*dx_I = S\, dx_3 dx_4$.

Claim of the above is that

$$S = (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2} = (-1)^{1+2}(-1)^3 = (-1)^6 = +1$$

Let's manually check to verify that $dx_I\,(*dx_I) = dV$

$$dx_I\,(*dx_I) = dx_1 dx_2 dx_3 dx_4 = dV$$

This example succeeds and agrees with Sja page 24 A.

Example 3: $n = 4$ and $k = 2$, $dx_I = dx_1 dx_3$, $*dx_I = S\, dx_2 dx_4$.

Claim of the above is that

$$S = (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2} = (-1)^{1+3}(-1)^3 = (-1)^7 = -1$$

Let's manually check to verify that $dx_I\,(*dx_I) = dV$

$dx_I\ (*dx_I) = \ dx_1 dx_3(-dx_2 dx_4) = dV$

This example succeeds and agrees with Sja p 24B

So this is the best result I can get

$S = (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2}$ note that $S^2 = 1$ *(1.13)*

$dx_I = dx_a\, dx_b\, ...\, dx_q$ $a < b < ... < q$ a k-form *(2)*

Example 4 n = 4 and k = 1, $dx_I = dx_1$, $*dx_I = S\ dx_2 dx_3 dx_4$

Claim of the above is that

$S = (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2} = (-1)^1\ (-1)^{1(1+1)/2} = (-1)^{1+1} = +1$

Let's manually check to verify that $dx_I\ (*dx_I) = dV$

$dx_I\ (*dx_I) = \ dx_1 dx_2 dx_3 dx_4 = dV$

This example succeeds and agrees with Sja p 24C

2. What is the next step?

Recall that

$dx_I = dx_a\, dx_b\, ...\, dx_n$ *(2)*

$(*dx_I) = S\ [\ dx_1\ ....dx_a\ ....dx_b.....dx_q\ ....dx_n\ ] = S\ \Pi_h\ dx_i$ *(3)*

where S is the sign which makes this be true

$dx_I(*dx_I) \ = dV$ *(4)*

Now consider

$*(*dx_I) = *(\ S\ [\ dx_1\ ....dx_a\ ....dx_b.....dx_q\ ....dx_n\ ]\ )$ (2.1)

$= S\ *\{\ dx_1\ ....dx_a\ ....dx_b.....dx_q\ ....dx_n\ \}$ (2.2)

$= S\ *\{\ S'\ dx_a\, dx_b\, ...\, dx_q\ \}$

$= SS'\ \ dx_a\, dx_b\, ...\, dx_q$

$$= SS'\, dx_I \qquad (2.3)$$

where S' is the sign which causes *this* to be true

$$(*dx_I)*(*dx_I) = dV \qquad (2.4)$$

Our goal is to show that $SS' = (-1)^{kn+k}$ since then (2.3) gives the desired result (1). (2.4a)

Now write out (2.4) making use of (3) for $(*dx_I)$ and (2.3) for $*(*dx_I)$ to get

$$S\ [\ dx_1 \ldots dx_a \ldots dx_b \ldots\ldots dx_q \ldots dx_n\ ]\ S\ [\ S'\, dx_a\, dx_b \ldots .\, dx_q\ ] = dV \qquad (2.5)$$

Since S appears twice, we get

$$S'\ [\ dx_1 \ldots dx_a \ldots dx_b \ldots\ldots dx_q \ldots dx_n\ ]\ [\ dx_a\, dx_b \ldots .\, dx_q\ ] = [dx_1 \ldots\ldots dx_n]\ . \qquad (2.6)$$

Now let's start by moving dx_a all the way to the left and sits then after S'. It has then to pass through n-k dx_i factors, so we pick up $(-1)^{n-k}$. We then have

$$S'\,(-1)^{n-k}\ dx_a\ [\ dx_1 \ldots dx_a \ldots dx_b \ldots\ldots dx_q \ldots dx_n\ ]\ [dx_b \ldots .\, dx_q\ ] = dV \qquad (2.7)$$

Now do this next with dx_b and pick up another $(-1)^{n-k}$. Do this k times and we then have

$$S'\,(-1)^{kn-kk}\ [dx_a\, dx_b \ldots .\, dx_q\ ]\ [\ dx_1 \ldots dx_a \ldots dx_b \ldots\ldots dx_q \ldots dx_n\ ] = dV \qquad (2.8)$$

or

$$S'\,(-1)^{kn-kk}\ dx_a\, dx_b \ldots .\, dx_q\ \Pi_h\, dx_i\ = dV \qquad (2.9)$$

or

$$dx_a\, dx_b \ldots .\, dx_q\ \Pi_h\, dx_i\ = S'\,(-1)^{kn-kk}\, dV \qquad (2.10)$$

Finally, we use our previous result (6),

$$dx_a\, dx_b \ldots .\, dx_q\, \Pi_h\, dx_i\ = SdV \qquad \textit{(6)}$$

Comparing the last two results gives

$$S'\,(-1)^{kn-kk}\, dV = SdV \qquad (2.11)$$

so that

$$S'S = (-1)^{kn-kk} \qquad (2.12)$$

Now what can you say about

$$(-1)^{kk}\ ?$$

If k is even, then k^2 is certainly even $2^2 = 4 \quad 4^2 = 16$
If k is odd, then k^2 is odd. $1^2 = 1 \quad 3^2 = 9 \quad 5^2 = 25$

Therefore

$$(-1)^{kk} = (-1)^{k} \qquad ! \qquad (2.15)$$

and then

$$S'S = (-1)^{kn-k}$$

Now multiply by $1 = (-1)^{2k}$ to get

$$S'S = (-1)^{kn+k}$$

and we have then achieved our goal as stated in (2.4a) and thus (1) is proved.

Comments: Try a summary of the above derivation. S is defined by (4) and S' by (2.4). From (2.3) we find that we have solved the problem if we can show that $SS' = (-1)^{kn+k}$. We have this result

$$S' (\Pi_h dx_i) dx_I = dV \qquad (2.6)$$

After sliding dx_I to the left, this becomes

$$dx_I (\Pi_h dx_i) = S' (-1)^{kn-kk} dV \qquad (2.10)$$

But our earlier result (6) said

$$dx_I (\Pi_h dx_i) = SdV$$

and this then gives the solution $S'S = (-1)^{kn-kk}$. We never have to use the explicit formula for S which was developed above in Section 1.