Proof of a Lemma PhL 7.15.15

Show that for a k-form: $*(*dx_I) = (-1)^{kn+k}\, dx_I$ Sjamaar p 28 Ex 2.12 (CH2)
(*)

[If α = any k-form, this shows that $**\alpha = (-1)^{kn+k}\, \alpha$.]

1. Definitions:

$dx_I \equiv dx_a\, dx_b \ldots dx_q$ $a < b < \ldots < q$ a k-form (1.1)

$\Pi_h\, dx_i \equiv dx_1 \ldots dx_a \ldots dx_b \ldots\ldots dx_q \ldots dx_n$ red = holes (1.2)

$*dx_I \equiv S\, \Pi_h\, dx_i$ (1.3)

2. Calibration for S:

$dx_I(*dx_I) = dV$ or $dx_I\, S\, \Pi_h\, dx_i = dV$

$\Rightarrow dx_I(\Pi_h\, dx_i) = S\, dV$ (2.1)

3. Compute $*(\Pi_h\, dx_i)$:

$(\Pi_h\, dx_i) = dx_1 \ldots dx_a \ldots dx_b \ldots\ldots dx_q \ldots dx_n$

$*(\Pi_h\, dx_i) = S'\, dx_a\, dx_b \ldots dx_q = S' dx_I$ (3.1)

4. Calibration for S' :

$(\Pi_h\, dx_i)*(\Pi_h\, dx_i) = dV$

$\Rightarrow (\Pi_h\, dx_i) S' dx_I = dV$

$\Rightarrow (\Pi_h\, dx_i) dx_I = S'\, dV$ (4.1)

5. **Sublemma**: (this will be proved in a later item below)

$(\Pi_h\, dx_i) dx_I = (-1)^{kn+k}\, dx_I (\Pi_h\, dx_i)$. (5.1)

6. Therefore using (4.1) on the LHS of sublemma, and (2.1) for RHS of sublemma, we have

$S'\, dV = (-1)^{kn+k} S\, dV$

or

$SS' = (-1)^{kn+k}$ (6.1)

7. Now consider

$*(*dx_I) = *(S \Pi_h dx_i)$ // using (1.3) for $*dx_I$

$= S *(\Pi_h dx_i)$ // factor out constant

$= SS'dx_I$ // using (3.1) for $*(\Pi_h dx_i)$

$= (-1)^{kn+k} dx$ // using (6.1) (7.1)

This concludes the proof since we have shown (*).

8. **Proof of sublemma**: $(\Pi_h dx_i)dx_I = (-1)^{kn+k} dx_I(\Pi_h dx_i)$

Start with the LHS

$(\Pi_h dx_i)dx_I = dx_1 ....dx_a ....dx_b.....dx_q ....dx_n \quad dx_a dx_b .... dx_q$

Now let's start by moving dx_a all the way to the left.. It has then to pass through n-k dx_i factors, so we pick up $(-1)^{n-k}$. We then have

$(\Pi_h dx_i)dx_I = (-1)^{n-k}dx_a \ dx_1 ....dx_a ....dx_b.....dx_q ....dx_n \ dx_b .... dx_q$

Now move dx_b in this same manner and pick up another factor $(-1)^{n-k}$, so

$(\Pi_h dx_i)dx_I = (-1)^{2n-2k} \ dx_a \ dx_b \ dx_1 ....dx_a ....dx_b.....dx_q ....dx_n \ .... dx_q$

Do this for all k factors up through dx_q and end up with

$(\Pi_h dx_i)dx_I = (-1)^{kn-kk} dx_a dx_b .... dx_q \ dx_1 ....dx_a ....dx_b.....dx_q ....dx_n$

$= (-1)^{kn-kk} dx_I (\Pi_h dx_i)$

Finally, note that $(-1)^{kk} = (-1)^k$ so

$(-1)^{kn-kk} = (-1)^{kn-k} = (-1)^{kn+k}$ since $(-1)^{2k} = 1$

Therefore

$(\Pi_h dx_i)dx_I = (-1)^{kn+k}dx_I (\Pi_h dx_i)$ (8.1)

and the sublemma is proved.

Comments: Although the derivation involved signs S and S', we never had to obtain an expression for S or S' separately, we only needed an expression for the product SS'. In my original document I showed that

$S = (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2}$ note that $S^2 = 1$ *(1.13)*

for

$dx_I = dx_a\ dx_b\ ....\ dx_q$ $a < b < ... < q$ a k-form *(2)*

Since $SS' = (-1)^{kn+k,}$ we could then deduce an expression for S'

$S' = (-1)^{a+b+..+q}\ (-1)^{k(k+1)/2}\ (-1)^{kn+k}$

$(-1)^{a+b+..+q}\ (-1)^{-k(k+1)/2}\ (-1)^{kn+k}$

$(-1)^{a+b+..+q}\ (-1)^{kn-k(k-1)/2}$