This is the Title PhL 1.11.15

Note that page numbering is turned on in this template and view is 125%,
located in phil/roaming/microsoft/templates size about 219K.

Now let us ponder the three items in the definition of an embedding (page 67) , one at a time.

(i) The first says that $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$ is one-to-one. So each **t** in U maps into a *unique* **x** on the nD surface. This does not seem very mysterious. It would apply to the points of Fig 68 B or M in Fig 69 D.

(ii) The second says that (Dψ) is also one-to-one for any t in U. What exactly does that mean? If we use the reference point $(\mathbf{t}_0, \mathbf{x}_0)$, then $(D\psi)\mathbf{t}$ produces some point **x** in the tangent space associated with $\mathbf{t}_0$. If those columns are lin indep, then it seems reasonable that you would have one-to-one between general points **t** in U and general points **x** in the tangent space. But this paragraph requires a distinction between $\mathbf{t}_0$ and **t** and Sja never talks about such a distinction.

(ii)' What alternate meaning could this item have? We know (Dψ) maps U into R^N. It maps **t** in U into some **x** in R^N. This **x** is not a point on the nD surface. It is a point in the tangent space associated with **t**.

Contradiction: I can select $\mathbf{t} = \mathbf{e}_1$. Does this define a tangent space? There is some point $\mathbf{x} = \psi(\mathbf{e}_1)$ and that point would be the root point for a tangent space $T_{\mathbf{x}}\psi$. But then $\mathbf{t} = \mathbf{e}_2$ would define some completely different tangent space.

Conclusion: Sjamaar's discussion just falls apart for me on page 67. I need another source to learn what he is attempting to say (but he fails to get across to me with my knowledge environment).

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This means zero to me, so once again, I will try to draw a picture and babble words until something comes to mind.

I presume open set U exists within the k-dim manifold which in turn lies inside R^n . Think perhaps as the manifold as the torus surface so $R^k = R^2$, and this is inside $R^n = R^3$. So then U is a little patch on the surface of the torus which contains point **x** on the torus. I see then no distinction between U and U ∩ M.

So let's reinterpret. Perhaps U lies in $R^n = R^3$ and is then a spherical ball centered at point x on the torus. Then U is a spherical ball, but U ∩ M is a warped disk-like patch on the torus. So now U and U ∩ M are not the same. In this example n = 3 and k = 2.

So we are interested in doing a mapping of this warped disk h(U ∩ M) with some vector function h. The patch as noted has dimension k = 2.

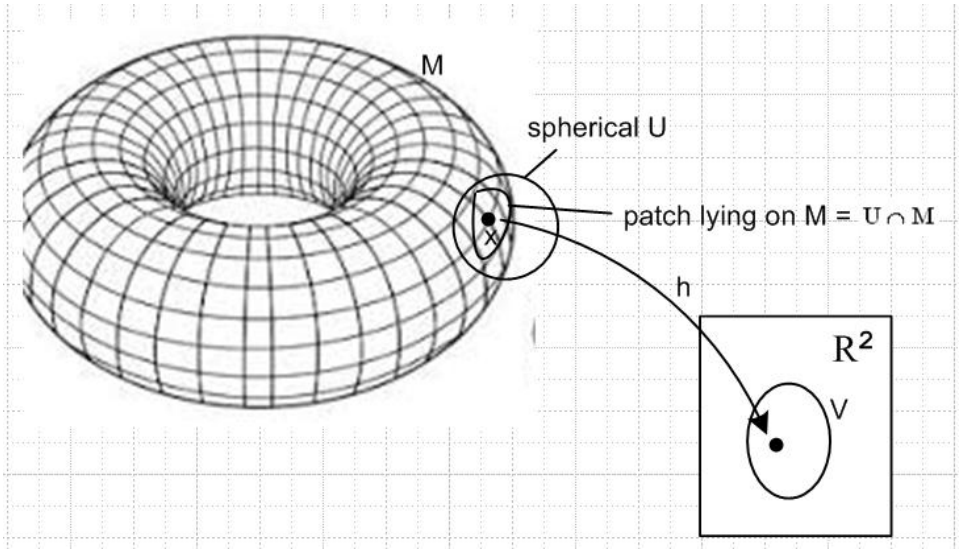
Now what is the meaning of the notation R^k x {0} ? Looking at the explanation line, I would say this is a space of n > k dimensions with points $\mathbf{y} = (y_1, y_2.....y_k, y_{k+1}.....y_n)$ where all the coordinates after y_k vanish. So you would say

$\mathbf{y} = (y_1, y_2.....y_k, 0.....0)$ = a general element of R^k x {**0**}

$\mathbf{y} = (y_1, y_2.....y_k, y_{k+1}.....y_n)$ = a general element of R^n or of V which lies within R^n

OK, next what is the meaning of $V \cap [R^k \text{ x } \{0\}]$? It seems that this would just force the higher components of vectors in V to be zero. Then $V \cap [R^k \text{ x } \{0\}]$ is a space which has only k variables.

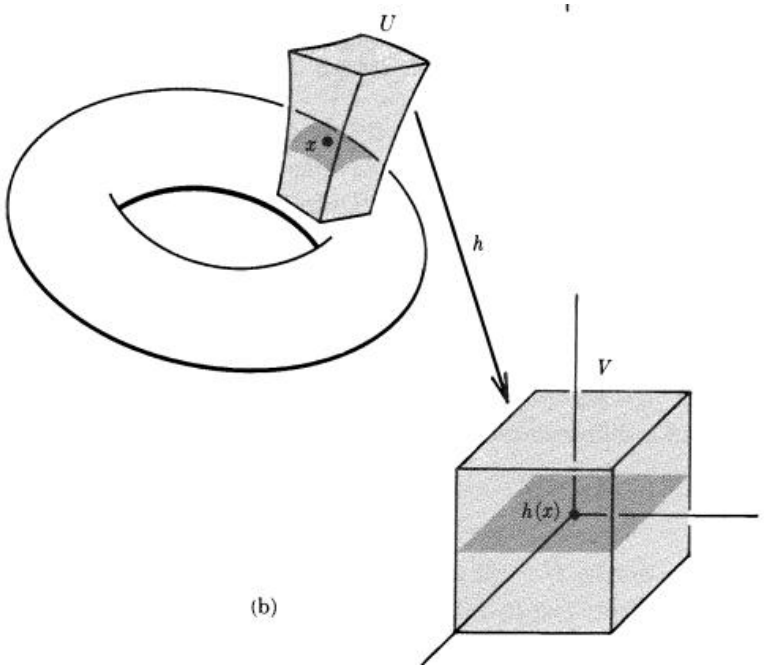
OK, here is my picture:



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We are just saying that locally you can map a patch on the manifold over to a patch in R^2 but we are also saying that the mapping is a diffeomorphism. That just means that if h is differentiable.

I then turn the Spivak page and he has my picture



He uses a cubic U and you see the patch U ∩ M and you see that patch mapping into a planar gray square, The reason you need the extra dimension(s) is so you can talk about differentiation in any direction. It is more than just the patch mapping into the square. You need to see the larger spatial features for the patch and the square. In this picture k = 2 and n = 3.

Comment: You will always see something like U ∩ M in this business if you want to be talking about differentiability in all directions at a point on a patch. You need a space in which your patch is embedded so you can differentiate around it.

Here U is still the cube of the previous picture, and M is the torus, and U ∩ M is the patch on the torus. This patch is the target of the mapping f (instead of the source as above) and that mapping comes from a region of R^2 called W, so that is why f(W) = U ∩ M.

The second item says that (Dy) has k independent columns, he uses my word **rank**, hurray! I think by having the column rank k, f'(y) = (Df) is invertible (even though not square).

The third item says that f^{-1} maps the patch back into W, and exists and its continuous.

So Spivak and Sjamaar are speaking the same language. I do get the deeper idea of a smooth mapping of any patch to a plane, but I am still confused by notation of Sjamaar.

My confusion is with regard to the tangent space I think. Maybe you are supposed to think of U as a very small square and the patch as also very small, and then there is only one tangent space involves.

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Here U is still the cube of the previous picture, and M is the torus, and $U \cap M$ is the patch on the torus. This patch is the target of the mapping f (instead of the source as above) and that mapping comes from a region of R^2 called W, so that is why $f(W) = U \cap M$.

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ok to here 8.13.15

Starting Over with Sjamaar Chapter 5 notes.

Page 67

Opens with world map discussion, fine. Then comes the idea of "an embedding". I will discuss this entirely in terms of the toroid picture. U (points **t**) is a rectangle of R^2 within R^3. Although you can't see it, this rectangle U has some R^3 above and below it, so you can talk about (Dψ)(t) which Spivak just calls ψ'(t) in a very dense notation. So you can discuss $\partial_i \psi_j$ for i in the third dimension out of paper as well as in paper. This concept that the rectangle U is itself embedded in R^3 is clear in the above Spivak picture. In our embedding situation, item (i) says $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$ is one-to-one, no problem with that. The rectangle maps over to the open section of a torus. The function $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$ is general and non-linear. I see how the requirement of one-to-one rules out **self-intersection** of the embedded surface $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$

Now we come to item (ii) and the (Dψ) object. This is a matrix, $(\partial_i \psi_j)$. It is my R matrix for non-square situations. You can apply this matrix to vectors in U, such as $\xi_i = (D\psi)\mathbf{e}_i$. So whereas ψ: U→ R^3 is a general non-linear mapping, (Dψ): U →R^3 is I guess a linear mapping! Just the way $d\mathbf{x}' = R d\mathbf{x}$ is a linear mapping. Maybe this is a point I have not absorbed. The matrix (Dψ) has columns, and I showed that these columns are the basis of the tangent plane at a point **t**, no problem with that. I can see that you would always want the tangent base vectors to be linearly independent over in that tangent space, so that means that the matrix (Dψ) has to have full column rank which in this example is 2.

I think the correct statement is to write $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$ and $d\mathbf{x} = (\mathbf{D}\boldsymbol{\psi})d\mathbf{t}$ where (Dψ) is a matrix evaluated at the point **t** and **x**. As **t** moves over U, $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$ moves *on* the toroid surface. So if you go to $\mathbf{t}+d\mathbf{t} = \mathbf{t}'$ in U, then you go to $\mathbf{x}' + d\mathbf{x}$ on that surface. You have $(\mathbf{x}+d\mathbf{x}) = \boldsymbol{\psi}(\mathbf{t}+d\mathbf{t})$ where both **x** and $\mathbf{x}+d\mathbf{x}$ are ON THE SURFACE. That is why $d\mathbf{x}$ must be ALONG THE SURFACE. Once again, $d\mathbf{x} = (\mathbf{D}\boldsymbol{\psi})\, d\mathbf{t}$ and (Dψ) is

then just a matrix like my R matrix. However, unlike my R matrix, d**t** lies in R^2 whereas d**x** lies in R^3 but also lies on the toroid's 2D surface in R^3. It is this relation d**x** = (**Dψ**) d**t** that we want to be invertible! That then means the mapping between d**t** ↔ d**x** is "one-to-one".

OK, once again write d**x** = (**Dψ**) d**t** where recall d**t** is in R^2 whereas d**x** is in R^3. This means of course that the matrix (Dψ) is not square, so you cannot talk about det(Dψ). But you can still talk about one-to-one. Once again: recall that I had d**x**' = R(**x**)d**x** in tensor doc. Here that is written d**x** = [(**Dψ**)(**t**)] d**t.** So if we are in a tiny neighborhood of **t** and **x**, then we have this matrix relationship d**x** = [(**Dψ**)(**t**)] d**t.** We can now apply this matrix to non-differential basis vectors and say $\boldsymbol{\xi}_i = [(\mathbf{D\psi})(\mathbf{t})]\ \mathbf{e}_i$ and these then give certain non-differential (and non-unit) basis vectors of the tangent space at **t** and **x**.

If you want the tangent space basis vectors $\boldsymbol{\xi}_i$ to be linearly independent, you must have the columns $\mathbf{c}_i$ of the matrix (Dψ) be linearly independent, since in fact $\boldsymbol{\xi}_i = \mathbf{c}_i$ exactly.

I see that **x** = **ψ**(**t**) is one-to-one, but I have trouble with (Dψ) in this one-to-one regard. I think this is what it really means. First, we have **x** = (**Dψ**)(**t**) = **ψ**'(**t**) in Spivak. Going the other way, **t** = $\boldsymbol{\psi}^{-1}$(**x**) which is clear, but we also have **t** = ($\mathbf{D\psi}^{-1}$)(**x**) as an inverse derivative mapping. ALL derivatives exist in either direction. That is the Spivak idea of **diffeomorphism**. In tensor doc language, we have d**x**' = Rd**x** but R is invertible as long as det(S) ≠0 and we then can write d**x** = $\mathbf{R}^{-1}$d**x**'. The only catch is that tensor doc assumes dx and dx' are in R^n spaces of same dimension. I will fix this below.

OK, here is an idea, staying with our example We have

$$dx_i = (D\psi)_{ij}dt_j = \partial_j\psi_i\ dt_j = (\partial\psi_i/\partial t_j)dt_j$$

where note ij reversal as in tensor doc on **∇ψ** object. In our example, i = 1,2,3 and j = 1,2 so (Dψ) has three rows and 2 columns.

$$(D\psi) = \begin{matrix} R_{11} & R_{12} \\ R_{21} & R_{22} \\ R_{31} & R_{32} \end{matrix}$$

Suppose the 2 columns are linearly independent. Then I know I can find a third column $(a,b,c)^T$ that is also linearly independent, and then I could create an "extended" (Dψ) matrix that was 3x3

$$(D\psi)_{ex} = \begin{matrix} R_{11} & R_{12} & a \\ R_{21} & R_{22} & b \\ R_{31} & R_{32} & c \end{matrix}$$

If I apply this extended matrix to points **t** = $(t_1,t_2,0)$ in the R^3 that contains rectangle U, I get

$$d\mathbf{x} = (D\psi)_{ex}\ d\mathbf{t}$$

and this creates exactly the same d**x** as in the regular d**x** = (Dψ) d**t** with **t** = (t_1,t_2). But now, since the columns are linearly independent, the 3x3 $(D\psi)_{\mathbf{ex}}$ is full rank, and is invertible, and it is ONE-TO-ONE ! We can write the inverse as

$$d\mathbf{t} = [(D\psi)_{\mathbf{ex}}]^{-1} d\mathbf{x} \qquad\qquad d\mathbf{t} = (dt_1, dt_2, 0)$$

This them makes a connection between these two notions:
(1) the columns of (Dψ) are linearly independent
(2) (Dψ) is one-to-one

Sja assumes you already know this in his statement page 67 C.

Then everything is good. Furthermore, I would say det $[(D\psi)_{\mathbf{ex}}] \neq 0$. So I think this may explain the use of the term one-to-one in item 2.

Item (iii): We already know that ψ^{-1} exists since **x** = **ψ**(**t**) was declared one-to-one in item (i). We can then write this as **t** = $\boldsymbol{\psi}^{-1}(\mathbf{x})$. You could say that $\boldsymbol{\psi}^{-1} : \boldsymbol{\psi}(U) \rightarrow U$. That is to say, the image of U maps back into U in a clean one-to-one manner. So all item (iii) adds is to say that **t** = $\boldsymbol{\psi}^{-1}(\mathbf{x})$ is *continuous*.

So in our torus picture, we have **t** = $\boldsymbol{\psi}^{-1}(\mathbf{x})$ mapping the surface of the torus over to rectangle U. This is like mapping a portion of the earth to a flat piece of paper map, and this is why mapping **t** = $\boldsymbol{\psi}^{-1}(\mathbf{x})$ is called a **chart**, also called a **coordinate map**. I of course think of the actual chart as the *image* of this mapping in R^2, and that image is just U.

So let's review again item 6.1

ψ : U → R^3 describes an **embedding** of U into R^3 if
(i) ψ is one-to-one
(ii) at every point, the tangent-space basis vectors are linearly independent
(iii) ψ^{-1} is continuous

Item (ii) is the same as saying that $(D\psi)_{\mathbf{ex}}$ is one-to-one and therefore invertible

Finally, for each point **t** in U which maps to **x**, we get a tangent space we call $T_{\mathbf{x0}}(U)$. For any **t'** in U we then have $T_{\mathbf{x}}(\mathbf{t}')$ = [(Dψ)(**t**)] **t'** (matrix times a vector). If we let **t'** run over all of U, the right side runs over all of the tangent space for **t**. This tangent space is an n-dimensional (2 in our case) affine space, but you are suppose to slide it to the origin, and then it is basically R^n (perhaps rotated). So I think THIS is what Sj means in page 67 B..

$T_{\mathbf{x}}(\mathbf{t}')$ = [(Dψ)(**t**)] **t'** = a point in the tangent space for (**t**,**x**) which space is ≈ R^n

Using set notation, you could then say

$T_{\mathbf{x}}(U)$ = [(Dψ)(**t**)] {U} ≈ R^n

On the other hand, U itself lies in its own R^n .

Conclusion: I STILL don't like the equation p 67 B. I think I understand that

$T_{\mathbf{x}}(\mathbf{t}') = [(D\psi)(\mathbf{t})]\,\mathbf{t}'$

$T_{\mathbf{x}}(U) = [(D\psi)(\mathbf{t})]\ \{U \subset R^n\} \approx R^n$ = the tangent space at $(\mathbf{t},\mathbf{x})$

Equation 67 B is simply illogical!

This concludes my "start over", for page 67 only.

Page 68

I don't understand the "doubles back on itself" example, though I do understand the sequence comments.

Example 6.2. Everything makes sense in this torus example.

Example 6.3. Let ψ: U→ R^m (used to be n). Write $\Psi(\mathbf{t}) = (\mathbf{t}, \psi(\mathbf{t}))$ = n + m.
We then have a map

$\Psi : U \rightarrow R^N$

The set of points $(\mathbf{t}, \psi(\mathbf{t}))$ as $\mathbf{t}$ runs over U is called a **graph**.
[The set of points (x, f(x)) as x runs over [a,b] is called a graph.]

Looking at how Ψ is defined, the image of Ψ $= (\mathbf{t}, \psi(\mathbf{t}))$ = that graph, I agree.

Question: is Ψ an embedding of U into R^N ??

Check the three clauses

(i) yes, $\mathbf{X} = \Psi(\mathbf{t})$ is one-to-one regardless of whether ψ is one-to-one:

$\Psi(\mathbf{t}_2) = (\mathbf{t}_2, \psi(\mathbf{t}_2))$
$\Psi(\mathbf{t}_1) = (\mathbf{t}_1, \psi(\mathbf{t}_1))$
$\Rightarrow \quad \Psi(\mathbf{t}_2) - \Psi(\mathbf{t}_1) = (\mathbf{t}_2 - \mathbf{t}_1, , \psi(\mathbf{t}_2) - \psi(\mathbf{t}_1))$

So even if $\psi(\mathbf{t}_1) = \psi(\mathbf{t}_2)$, we still have $\Psi(\mathbf{t}_2) \neq \Psi(\mathbf{t}_1)$ so we are one-to-one.

(ii) $(D\Psi) = (D\mathbf{t}, D\psi) = (I_n, (D\psi))$, so right off the bat $(D\Psi)$ has n lin indep columns.

(iii) $(\Psi^{-1}\Psi)(\mathbf{t}) = I\,\mathbf{t} = \mathbf{t}$ $\quad \Psi^{-1}(\mathbf{t}, \psi(\mathbf{t})) = \mathbf{t}$ and $\mathbf{t}$ is continuous

So yes, Ψ an embedding of U into R^N .

Page 69

Let's apply **manifold** Definition 6.4 to the torus just to get started.

Let V = open ball in R^3 which contains point **x** on the torus.

Let U be as before, open subset of R^2.

Let ψ be an **embedding** where ψ: U $\rightarrow$ R^3, as discussed for the torus.
The torus surface is our candidate manifold M.

$V \cap M$ is the intersection of (the ball around **x)** with (the manifold torus surface M) = a patch

We require that $\psi(U)$ [ie, the image of U] = $V \cap M$ = the patch.

PL Discussion: If you have some horrible self-intersecting surface like the torus would be for a larger U than shown, then ψ would not be "an embedding". So the we have to think of the patch as being small enough so that the patch itself is non-intersecting and has no other strange topo issues.

The idea is that **at each x** on M, you are allowed to pick V and U *as small as necessary* so that the intersection $V \cap M$ = the patch = a smooth patch so that ψ: U$\rightarrow$$R^N$ is then an embedding, as per the definition of an embedding.

I might write it this way:

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Puzzle Question: Why do you need **multiple charts** for Fig p 69 D?

Suppose we take as our "patch" the entire curve M shown, rather than the short piece of curve shown inside V. I don't see the problem. Wiki on manifolds might have the answer:

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// this will be redone!

My first effort here is to show that, if a parameterized surface is defined by $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$, then

1. If you write $d\mathbf{t} = dt_i\ \mathbf{e}_i$ and then $d\mathbf{x} = (D\psi)(\mathbf{t})\ d\mathbf{t} = (D\psi)[dt_i\ \mathbf{e}_i] = dt_i\{\ (D\psi)\mathbf{e}_i\ \}$.

2. But d**x** lies on the surface $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$, and therefore $\boldsymbol{\xi}_i \equiv (D\psi)\mathbf{e}_i$ are the tangent vectors at point x and t.

3. But easy to show that if $(D\psi) = (\mathbf{c}_1, \mathbf{c}_2 ..... \mathbf{c}_n)$, then $\boldsymbol{\xi}_i = \mathbf{c}_i$

4. Therefore, the columns of $(D\psi)$ are tangent vectors in the tangent space at point x.

5. If the columns are all linearly independent, then the columns form a basis of that tangent space.

Note that the matrix $(D\psi)$ is like my R matrix and has its natural setting in $d\mathbf{x} = (D\psi)(\mathbf{t})\, d\mathbf{t}$.

6. The tangent base vectors = columns of $(D\psi)(\mathbf{t})$ are specific to the point $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$ on the surface.

7. $\mathbf{t}$ is in R^n while $\mathbf{x}$ is in R^N

After this long initial set of notes, I "start over" in red text at page 67. Our first task is to understand the meaning of an **embedding** of U into R^N. This thing defines a surface based on parameter space U which has elements $\mathbf{t}$. For example, you map the rectangle into a toroidal surface shown p 68.

(i) the mapping $\mathbf{t} = \boldsymbol{\psi}(\mathbf{x})$ is one-to-one so surface will not be self-intersecting.

STOP. I don't really have time to do this right before going to Cod. It needs to be digested a bit more so I will just pick up on July 30.

****

Recall the idea of a pullback from chapter 3 :

$$\varphi^*\alpha = \varphi^*[\ \Sigma_I\ f_I\, dx_I\ ] = \Sigma_I\ \varphi^*[f_I]\ \varphi^*[dx_I] = \Sigma_I\, f_I[\varphi(x)]\, d\varphi_I = \Sigma_I\, f_I[\varphi(x)]\, \Sigma_J\ \det(D\varphi)_{I,J}\, dx_J$$

$$= \Sigma_{I,J}\ f_I[\varphi(x)]\, \det(D\varphi)_{I,J}\, dx_J = \Sigma_J\, g_J(\mathbf{x})\, dx_J\ , \quad g_J(\mathbf{x}) \equiv \Sigma_I\ f_I[\varphi(x)]\, \det(D\varphi)_{I,J}$$

where φ was a set of functions $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{t})$ mapping $\varphi: U \to R^N$ or similar. We could write

$$g_J(\mathbf{x}) = \Sigma_I\ \varphi^* f_I(\mathbf{x})\, \det(D\varphi)_{I,J}$$

Now here is the RHS of 88A:

$$\alpha_{\boldsymbol{\varphi}(\mathbf{x})}(\ D\varphi(\mathbf{x})\mathbf{v}_1,\ D\varphi(\mathbf{x})\mathbf{v}_1, \ldots D\varphi(\mathbf{x})\mathbf{v}_1)$$

and now you see the k-multilinear function nature of things much better.

Now I will attempt to compute the pullback $\varphi^*\alpha_{\mathbf{x}}$:

$$\varphi^*[\alpha_{\mathbf{x}}(\mathbf{v}_1,\mathbf{v}_2 \ldots \mathbf{v}_k)\ ] = \varphi^*[\Sigma_I\ \alpha_{\mathbf{x}}(\mathbf{e}_I)\, dx_I(\mathbf{v}_1,\mathbf{v}_2 \ldots \mathbf{v}_k)]$$

$$= \varphi^*[\ \Sigma_I\ f_I(\mathbf{x})\, dx_I(\mathbf{v}_1,\mathbf{v}_2 \ldots \mathbf{v}_k)\ ]$$

$$= \Sigma_I\ \varphi^*[f_I(\mathbf{x})]\ \varphi^*\ [\ dx_I(\mathbf{v}_1,\mathbf{v}_2 \ldots \mathbf{v}_k)]$$

$$= \Sigma_I \, [f_I(\varphi(\mathbf{x}))] \;\; \varphi^* \, [\, dx_I(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k})]$$

$$= \Sigma_I \, \alpha_{\varphi(\mathbf{x})}(\mathbf{e}_I) \;\; \varphi^* \, [\, dx_I(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k})]$$

Here then is a *conjecture* that I will try soon to validate:

$$\varphi^*(\, dx_I[\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}]) \;= dx_I[(D\varphi)\mathbf{v_1},(D\varphi)\mathbf{v_2}.....(D\varphi)\mathbf{v_k}] \; .$$

Suppose this were true. Then we would have

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}) \;\equiv\; \varphi^*[\alpha_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k})\,] = \;\; \Sigma_I \, \alpha_{\varphi(\mathbf{x})}(\mathbf{e}_I) \, dx_I[(D\varphi)\mathbf{v_1},(D\varphi)\mathbf{v_2}.....(D\varphi)\mathbf{v_k}]$$

But now look back at

$$\alpha_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}) \;=\; \Sigma_I \, \alpha_{\mathbf{x}}(\mathbf{e}_I) \, dx_I(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}) \;\; .$$

Using this format, we can then write

$$(\varphi^*\alpha)_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}) \;=\; \Sigma_I \, \alpha_{\varphi(\mathbf{x})}(\mathbf{e}_I) \;\; dx_I[(D\varphi)\mathbf{v_1},(D\varphi)\mathbf{v_2}.....(D\varphi)\mathbf{v_k}]$$

$$= \alpha_{\varphi(\mathbf{x})}[(D\varphi)\mathbf{v_1},(D\varphi)\mathbf{v_2}.....(D\varphi)\mathbf{v_k}]$$

which is the claim of p 88A. So to make this work, I would like to show that this is true:

$$\varphi^*(\, dx_I[\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}]) \;= dx_I[(D\varphi)\mathbf{v_1},(D\varphi)\mathbf{v_2}.....(D\varphi)\mathbf{v_k}] \; . \qquad (*)$$

Now recall that dx_I is the *name* of a certain function as given by page 85 C. Lets do an example first

$$"dx_1dx_2 \;...dx_k"(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}) \;= \det[\, dx_i(v_j)]$$

where the matrix is this:

$$\begin{array}{llll} dx_1(\mathbf{v_1}) & dx_1(\mathbf{v_2}) & ... & dx_1(\mathbf{v_k}) \\ dx_2(\mathbf{v_1}) & dx_2(\mathbf{v_2}) & ... & dx_2(\mathbf{v_k}) \\ ... & & & \\ dx_k(\mathbf{v_1}) & dx_k(\mathbf{v_2}) & ... & dx_k(\mathbf{v_k}) \end{array}$$

and were the $\mathbf{v}_i$ are arbitrary vectors in V. Now here would be a general case for I:

$$dx_I(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}) \;=\; "dx_{i1}dx_{i2} \;...dx_{ik}"(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}) \;= \det[\, dx_i(v_j)]$$

where the matrix now is this

$$\begin{array}{llll} dx_{i1}(\mathbf{v_1}) & dx_{i1}(\mathbf{v_2}) & ... & dx_{i1}(\mathbf{v_k}) \\ dx_{i2}(\mathbf{v_1}) & dx_{i2}(\mathbf{v_2}) & ... & dx_{i2}(\mathbf{v_k}) \end{array}$$

...

$dx_{ik}(\mathbf{v}_1)$ $dx_{ik}(\mathbf{v}_2)$ ... $dx_{ik}(\mathbf{v}_k)$

Then we could look at the RHS of

$$dx_I((D\varphi)\mathbf{v}_1,(D\varphi)\mathbf{v}_2.....(D\varphi)\mathbf{v}_k) = "dx_{i1}dx_{i2} \ ...dx_{ik}"((D\varphi)\mathbf{v}_1,(D\varphi)\mathbf{v}_2.....(D\varphi)\mathbf{v}_k) = \det[\ dx_i((D\varphi)v_j)]$$

where the matrix now is this

$$\begin{array}{llll} dx_{i1}((D\varphi)\mathbf{v}_1) & dx_{i1}((D\varphi)\mathbf{v}_2) & ... & dx_{i1}((D\varphi)\mathbf{v}_k) \\ dx_{i2}((D\varphi)\mathbf{v}_1) & dx_{i2}((D\varphi)\mathbf{v}_2) & ... & dx_{i2}((D\varphi)\mathbf{v}_k) \\ ... & & & \\ dx_{ik}((D\varphi)\mathbf{v}_1) & dx_{ik}((D\varphi)\mathbf{v}_2) & ... & dx_{ik}((D\varphi)\mathbf{v}_k) \end{array} \qquad (**)$$

So I how have the object of interest $dx_I[(D\varphi)\mathbf{v}_1,(D\varphi)\mathbf{v}_2.....(D\varphi)\mathbf{v}_k]$ expressed as a determinant. That seems like a step forward. Recall from Chapter 3 that we had

$$\varphi^*(dx_I) = d\varphi_I = \Sigma_J \det[(D\varphi)_{I,J}]\ dx_J$$

where there is also a determinant.

Now <u>just for fun,</u> suppose we were to select $(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) = (\mathbf{e}_{j1},\mathbf{e}_{j2}.....\mathbf{e}_{jk}) = \mathbf{e}_J$ which is then associated with a different multi-index J. And we also use $dx_{i2} = e^T_{i2}$ for example. Then what does the above matrix look like? Before answering that question, consider from above:

$$(\varphi^*\alpha)_x(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k) \equiv \varphi^*[\alpha_x(\mathbf{v}_1,\mathbf{v}_2.....\mathbf{v}_k)\ ] = \ \Sigma_I\ \alpha_{\varphi(x)}(\mathbf{e}_I)\ dx_I[(D\varphi)\mathbf{v}_1,(D\varphi)\mathbf{v}_2.....(D\varphi)\mathbf{v}_k]$$

$$(\varphi^*\alpha)_x(\mathbf{e}_{j1},\mathbf{e}_{j2}.....\mathbf{e}_{jk}) \equiv \varphi^*[\alpha_x(\mathbf{e}_{j1},\mathbf{e}_{j2}.....\mathbf{e}_{jk})\ ] = \ \Sigma_I\ \alpha_{\varphi(x)}(\mathbf{e}_I)\ dx_I[(D\varphi)\mathbf{e}_{j1},(D\varphi)\mathbf{e}_{j2}.....(D\varphi)\mathbf{e}_{jk}]$$

$$(\varphi^*\alpha)_x(\mathbf{e}_J) \equiv \varphi^*[\alpha_x(\mathbf{e}_J)\ ] = \ \Sigma_I\ \alpha_{\varphi(x)}(\mathbf{e}_I)\ dx_I[(D\varphi)\mathbf{e}_{j1},(D\varphi)\mathbf{e}_{j2}.....(D\varphi)\mathbf{e}_{jk}]$$

For example

$$dx_{i1}((D\varphi)\mathbf{v}_2) \ \rightarrow\ dx_{i1}((D\varphi)\mathbf{e}_{j2} \ = \ e_{i1}{}^T\,(D\varphi)\ e_{j2} \ = \ (D\varphi)_{i1,j2}$$

Hmmm. Then the above messy matrix (**) reduces to just (Dφ) and we then have

$$dx_I((D\varphi)\mathbf{e}_{j1},(D\varphi)\mathbf{e}_{j2}.....(D\varphi)\mathbf{e}_{jk}) \ = \ \det(\ (D\varphi)_{I,J})$$

and then

$$\varphi^*(\ dx_I[(\mathbf{e}_{j1},\mathbf{e}_{j2}.....\mathbf{e}_{jk})]) \ = \ \det(\ (D\varphi)_{I,J})$$

or

$$\varphi^*(dx_I(\mathbf{e}_J)) = \det((D\varphi)_{I,J})$$

As I just noted above, in Chapter 3 we had

$$\varphi^*(dx_I) = d\varphi_I = \Sigma_J \det[(D\varphi)_{I,J}]\, dx_J \qquad \text{page 41 (3.3)}$$

These are not TOO far apart, However, in page 86 B I showed that

$$dx_I(\mathbf{e}_J) = \delta_{I,J}$$

so that seems to say

$$\varphi^*(\delta_{I,J}) = \det((D\varphi)_{I,J})$$

which seems rather strange. I am s bit to fuzzy right now to get this cleared up, but seems close. If I = J this then goes on to say

$$\varphi^*(1) = \det((D\varphi)_{I,I})$$

But this makes no sense. If 1 is a 0-form, what does k have to do with this?