Notes on Sjamaar PDF PhL 6.29.15

Had to stop on this subject on 7.20.15 due to Cape Cod. After a gap of 20 days, on 8.10.15 I read through these notes but only up to the start of Chap 3. The Chap 3 notes are a bit confused, so I then switched over to the meta doc and read it all the way through. On 8.12.15 my review is up to the start of Chap 6.

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I will do mark up right in his PDF. // Had to give up on that, something wrong with PDF. I will mark up a hard copy instead.

Here is data on the other PDF I have which is 153 pages. Recently minted PhD, 1990!
(that is to say, a mere 25 years ago).

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Does he have updated notes on this site?

http://www.math.cornell.edu/~sjamaar/

Here is his recent teaching activity

Manifolds and differential forms lecture notes

Course home pages

Fall 2014

Math 3210 Manifolds and differential forms
Math 7530 Algebraic topology II

Fall 2013

Math 3210 Manifolds and differential forms
Math 6520 Differentiable manifolds I

Fall 2012

Math 3210 Manifolds and differential forms
Math 6520 Differentiable manifolds I

Fall 2011

Math 2210 Linear algebra

Fall 2010

Math 2210 Linear algebra

Fall 2009

Math 2210 Linear algebra
Math 3210 Manifolds and differential forms

so he is still doing his "manifolds and forms" classes. His notes on the site are the same 8.26.06 notes which I already have, so I know at least I am current. He does not seem to have notes on his other classes.

Preface.

Notes are for undergrads! Two more advanced books are **Spivak** and **Flanders** [on 2.11.16 I found Flanders as a djvu, and have long had Spivak.] . Says he has intended since 2006 to update these notes, but as of 2015 he has not done so. Maybe they are good and stable! [he did an update July 2015, but I continue to work with the older notes since I have invested heavily in them with markup.]

I notice that like my early tensor doc, this 153 page PDF has no equation or figure numbers.

Chapter 1: Introduction

1.1 Manifolds

He writes column and row vectors exactly as I do! E^n is called R^n. A manifold will get defined gradually, but the first fact is that a manifold is a subset of R^n. Recall that "set" is more general than "subspace" and I think this is a theme of manifold theory-- to keep things as general as possible.

A 1D manifold is a smooth non-intersecting curve. It can exist in E^n for $n \geq 2$ I would say. A trivial curve can exist in E^1, it would be just a piece of the x axis.

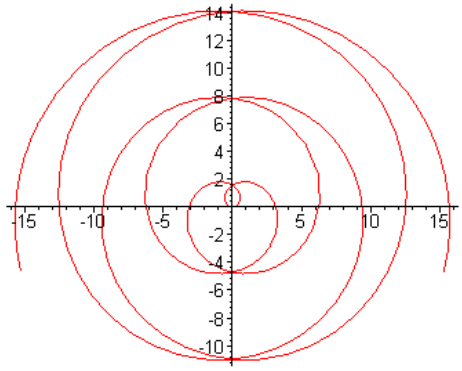
[2] He shows curves which are closed, infinite, open at one end and closed at the other, and a circle has a missing point and can be opened topologically as shown. All these **examples** are manifolds. I think one idea is that each point on the curve has to have a unique tangent, and this cannot be true for an intersection. The bow tie is treated as self-intersecting so is not a manifold.

[3] The spiral shown I think is the curve $r = k|\theta|$ in polar coordinates. I just made my own

```
r := 1*theta;
```

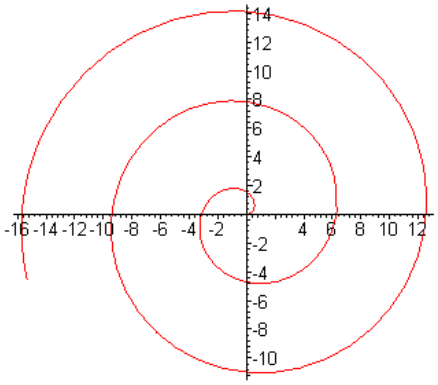
$$r := \theta$$

```
plot(r,theta = -16..16,coords =polar,scaling=constrained);
```



For negative θ, since $r < 0$ it gets drawn as $r>0$ with $\theta+\pi$. For only positive θ you get,

```
plot(r,theta = 0..16,coords =polar,scaling=constrained);
```



S shows how you can "pull the upper spiral apart" by adding a z coordinate to the curve so you then have a curve in 3D space. I just did this:

```
z := theta;
```

$$z := \theta$$

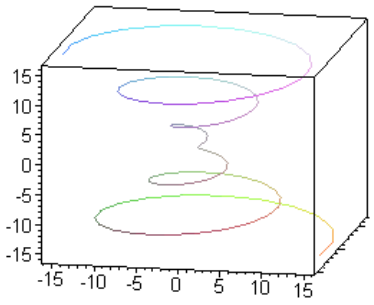
```
with(plots):
x := r*cos(theta);
```

$$x := \theta \cos(\theta)$$

```
y := r*sin(theta);
```

$$y := \theta \sin(\theta)$$

```
spacecurve([x,y,z],theta = -16..16,numpoints = 100);
```



The curve lies on a cone which has shape $(x^2 + y^2)^{1/2} = z$ which in our case is $r = z$ or $\theta = \theta$.

[4] Examples of a 2D manifold are shown. First we see cases where they lie in a plane R^2. One example is an open square with no boundary shown in dashes. Another example is closed and has a boundary and even has holes. The boundary here is finite. The closed square has non-smooth corners so is not a manifold. More interesting 2D manifolds are then shown in R^3 .

[5] Just as I would do if I were writing this and knew that the author knows, I would get into the Mobius strip. It is a manifold (though not orientable). He shows you would "make" a Mobius band (as he calls it) from a square piece of paper. The second picture shows how you would make a Klein bottle by first making a cylinder, then twisting it to get the bottle. The surface in this case has to pass through itself and therefore self-intersects, and thus is not a manifold.

See https://en.wikipedia.org/wiki/Real_projective_plane for some pictures. That site also deals with the three objects that S treats. They seem to give some plots of a "projective plane" object.

Claim is made that a Klein bottle in R^4 can have no intersections. I have not followed every detail of this section which I am sure is dear to S.

1.2 Equations

[7] You can usually describe a manifold by one or more equations. If there are fewer equations that there are unknowns, you get some kind of "surface" which could be a manifold. His symbol for surface is **φ** (bolded) whereas Bucks always use **Σ**. You can write $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{c}$ to represent a set of equations where the c_i are all constants. Equations in general are non-linear. If the equations are all linear, then you really have a matrix equation $\Phi\ \mathbf{x} = \mathbf{c}$ (generally Φ not square) and the manifold you get in this case is called an **affine subspace** and S teaches this in his linear algebra course. I think the solution of $\Phi\ \mathbf{x} = \mathbf{c}$ is a surface that is planar within E^n of some dimension ≤ n-1 depending on the rank of Φ. These planar "flats" generally don't pass through the origin, so they are not official linear subspaces (linear subspace must include the origin). It is said that an affine subspace in E^n is the set of all linear subspaces of E^n which are then translated an arbitrary distance away from the origin. [???]

Example 1.1 [7] $\Phi\ \mathbf{x} = \mathbf{c}$ where $E^n = E^3$ and there are m = 2 equations. The equations are a unit circle in the plane, and a tilted plane z = -y. You can extrude the circle into a cylinder (as I have done in Lagrange) and then you see that the intersection is a tilted plane with a cylinder, which turns out to be an ellipse. This ellipse is in fact a manifold of dimension 1, meaning a curve, but we have not really defined dimension yet in detail.

Aside: In Lines I claim that an ellipse in 3D looks like:

$$\begin{aligned} x &= A\cos(\omega t - a) \\ y &= B\cos(\omega t - b) \\ z &= C\cos(\omega t - c) \end{aligned}$$

Suppose I try that here. then I get

$$A^2\cos^2(\omega t-a) + B^2\cos^2(\omega t-b) = 1 \qquad \text{and} \qquad B\cos(\omega t-b) + C\cos(\omega t-c) = 0$$

The second equation says $B^2\cos^2(\omega t-b) = C^2\cos^2(\omega t-c)$ which suggests b = c and B = ±C, so put that equation aside for a while and look at the first equation. I know this is a rotated 2D ellipse, but the details elude me, so let it ride.

[8] **Example 1.2.** Defines ||x|| as norm and $\mathbf{x} \bullet \mathbf{y}$ as scalar product in R^n. In 3D, a sphere is a 2D surface which has the name S^2. So a sphere in R^n is then called S^{n-1}. The equation is just $\|\mathbf{x}\| = r$ which is one equation in n unknowns. Another example of an equation defining a manifold. If you call this manifold M, then you have

$$M = \{\mathbf{x} \text{ in } R^n \mid \|\mathbf{x}\| = r\} \qquad \text{// set notation}$$

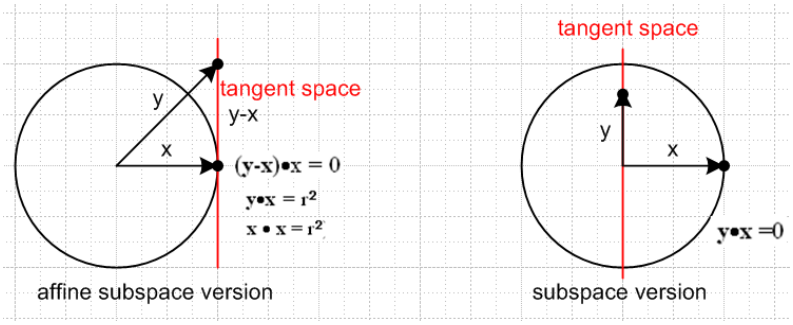
The tangent space concept

Consider the sphere situation, and pick a point **x** on the sphere. I know there is a tangent plane which has some basis vectors (two) all of which are perp to the normal at **x**. This plane does not pass through the

origin and can be described by plane(**x**) = {**y** in R^n | (**y**-**x**)•x = 0} = all points **y** such that the vector from **x** to that point is perp to the normal vector. For spherical our surface, you can also say (since **x** • **x** = r^2) plane(**x**) = {**y** in R^n | **y**•**x** = r^2} . This plane is an **affine subspace**. If you were to translate this plane to pass through the origin, it would be a **genuine subspace**.

The official "tangent space" for a point **x** is taken to be the actual tangent plane translated in this way, so then one has

tangent space at point x on manifold S^{n-1} = {**y** in R^n | **y**•**x** =0} = T_xM



Here we see a famous notation:

T_xM means "the tangent space at point **x** in M" // contains origin.

Now suppose M = circle in E^2. The tangent space would be a line through the origin, so in this case one would have dim(T_xM) = dim(M) = 1.

But suppose we had M = a circle in E^3. In this case I think T_xM would be any plane in E^3 which is tangent to the circle at point **x** on the circle. In this case, dim(T_xM) = n-1 of M has dimension M. Maybe it is true that dim(M) + dim(T_xM) = n ??

In any event, T_xM is a true linear subspace in E^n.

Passing note that a **closed ball** is a locus described by an **inequality** ≤ rather than an equation.

1.3 Parametrizations // several spellings are allowed

[9] Describe points on a circle by parameter θ, or points on a sphere by θ,φ (two parameters). S points out that typically you cannot arrange things so that every point is "covered only once" on the surface or manifold which is being described by the parameters. I am quite used to this problem.

1.4 Configuration Spaces

S is getting into some physics now. If you have a double pendulum of two connected math rods when any "state" of the system can be described by θ_1,φ_1, θ_2,φ_2 .The space spanned by these 4 parameters is called the configuration space for this system, and I suspect that space is a 4D manifold. Here we have 4 "degrees of freedom".

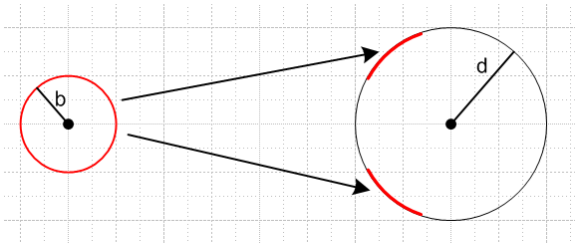
[10]. On this page he does the spherical pendulum θ,φ (**Example 1.3**) and then the double spherical pendulum (**Example 1.4**). This is really the kinematics of the system, whether or not there is gravity does not matter. He wants you to think of θ_1,φ_1, $\theta_2,\varphi_2 = (\theta_1,\varphi_1)$ x (θ_2,φ_2) = **Cartesian product**.

[11] A Rigid body has 6 degrees of freedom which are translation and Euler angles.

Example 1.6 is more detailed. Quadrilateral in E^2 with four fixed rods with universal joints! How many degrees of freedom are there, ignoring overall translations in E^2 ? It must be ≤ 4 since you could define 4 angles. Looking at page 11, if we keep edge AB fixed, you only have only one angle θ for BC as his drawing shows. You would then have 2 translation degrees of freedom for point A, and then one rotational degree φ to make edge AB. So I think the total is 4. If this were a rigid 2D object, you would only have a total of 3 degrees of freedom: 2 translation and one rotation.

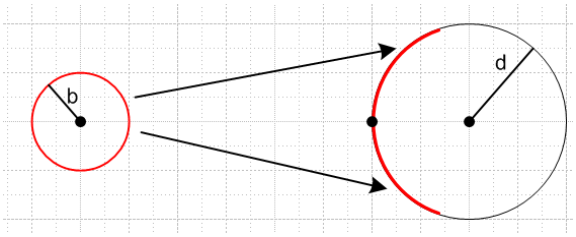
Imagine that edge b is very small relative to other edges. As C goes around in the circle shown, point D would do only a partial circle bounded by some angles. So D oscillates back and forth on a circular arc, and then C is the "wheel" and D is the "pump", like an oil well pump. Since rods can cross, there is then a separate arc locus below.

Now what is "the manifold" in this elaborate example?

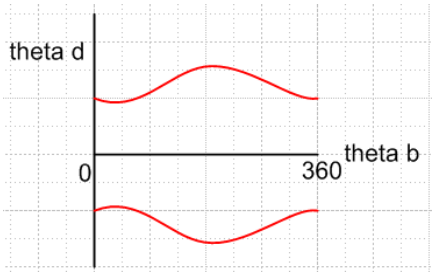


We have the two angles for configuration space, ignoring the other 3 degrees. If this were a mapping, there are two different functions being mapped, a 1:2 mapping. Each mapping has a circular domain, the ranges do not overlap. So S wants to think of there being two circles on the left, and they are disjoint in some sense that is not quite clear to me and which he does not clarify.

[12] Example 1.7 has a+b = c+d and I think this creates this situation



where now the two mappings meet at one point. He somehow gets this to be two circles touching at one point. I am afraid he has just lost me on these drawings. Well, think of two circles as two Riemann sheets that touch at that point. The actual configuration space would look something like this



and this shows all legal states of the system in angle parameter space.

Chapter 2: Differential Forms on R^n

2.1 Elementary Properties of differential forms on R^n

[17] Notice the word "on" in the title. He credits differential forms to Cartan in 1900, now sure that happened to Grassmann.

Differential form: $\alpha = \Sigma_I f_I(\mathbf{x})\, dx_I$ on R^n // multi-index, see my Buck notes.

He uses **Buck notation without the wedge symbols** so has $dx_1 dx_6$ for example. For a k-form, there are k factors of dx_i so there are then k indices i_1, i_2....i_k which must all be different. You should always say what space R^n you have in mind for your differential form. The single function $f_I(\mathbf{x})$ might only be defined on some *open* subset U of R^n. (not clear why open is required).

Example: $U = R^n$ - the z axis $= R^n - \{(x,y,z) \mid x^2+y^2 \neq 0\}$. // all cylinders except the line

Then $f_i(\mathbf{x}) = \ln(x^2+y^2)z$ is defined on U but not on R^n.

Note the traditional use of Greek letters for differential forms!

[18] We think of the $dx_1 dx_2 ....dx_k$ as describing a little differential cube within R^n and this product is the volume. S then just states that swapping any pair of dx_i negates the volume, and thus $dx_2 dx_2 = 0$. No justification is given at this point for this Minus Sign Rule.

S uses a hat notation to mark something that is missing, and I once used a slash notation for the same thing.

If n = k, then the k-form is the full volume form.

[19] S then does the ordered multi-index idea where there are then (n,k) unique terms, accounting for all the swap rules. I derive this in the Buck notes for Ch 7.

Definition: $\Omega^k(U)$ = **collection of all k-forms on U in R^n.** This set is a vector space because: (1) you can add two k-forms; (2) you can multiply a k-form by a constant; (3) there is an additive 0 and a multiplicative 1; (4) a few more basic axioms are met. $\alpha = 0$ thus exists as a differential form. I guess this

is a 0-form with f = 0. And β = 1 with f = 1 would also exist. Dim($\Omega^k(U)$) = ∞ because there are an infinite number of smooth coefficient functions you could choose.

[p 19] Suppose all f_I are **constants** by edict. This gives you some subset of $\Omega^k(U)$. You can still add these items and multiply them by a constant and there is still 0 and 1, so this is a linear subspace of $\Omega^k(U)$. Ignoring the constants, think of this space spanned by the (n,k) different dx_I. Thus, the dimension of this nameless subspace of $\Omega^k(U)$ is (n,k).

Note Added 2.11.16. Let's call the above subspace $\Omega^k(U)_c$. Elements are like 2 dx + 3 dy where the f_I functions are constants. Why is this subspace a vector space? Closed under addition. Has zero element which might be 0 dx for example. Scalar mult cleanly defined. So OK. Now what is meant by a basis for this subspace? If n = 2 and k = 1 then basis is {dx,dy} because can construct any element of $\Omega^1(U)_c$ as lincomb of these elements. If n=3 and k = 2, then basis is {dxdy,dxdz,dydz}, same idea. In general, for a given n and k, there will be (n,k) elements in ths basis which have the form dx_I. What is the dimensionality of this subspace $\Omega^k(U)_c$? It equals the number of elements in the basis, so the dimension of this space is (n,k).

Now go back to the larger vector space $\Omega^k(U)$. What is a basis here if n = 2 and k = 1?

dx, xdx, y^2dx , ... these are all elements of $\Omega^k(U)$

In the previous case, you could have said that the space has to include

dx, 2dx, 3dx, 7.1 dx, ...

but all these elements are "covered" by the one element α dx where α is a scalar, meaning a real number. in the $\Omega^k(U)$ case the items list above cannot be "covered" by a single element f(x,y)dx because in general f(x,y) is not a number. You cannot reach y^2dx by α dx where α is a scalar. So in my example, you have to include all possible elements in the basis having the form

f(x,y)dx, g(x,y)dy

and each group is an infinite set since there are an infinite number of functions f(x,y).

So the idea that you "span" the vector space $\Omega^k(U)_c$ with the dx_I does not apply to $\Omega^k(U)$.

Note Added 2.11.16. Above I just noted that the dimension of $\Omega^k(U)_c$ is (n,k). In wedge doc I note that the dimension of L^k (non-dual wedge product space) is (n,k), as is the dimension of Λ^k for the corresponding dual wedge space. The basis elements are $e^{\wedge I}$ and $\lambda^{\wedge I}$ for these two spaces. At this point in Sja, I have no reason to connect either of my wedge doc spaces with the $dx^{\wedge I}$ objects which are the basis elements of $\Omega^k(U)_c$.

[p 19] **Definition**: **Exterior product** of two differential forms. Assume one is a k-form and the other an m-form. The result is a k+m form.

α = k-form $= \Sigma_I f_I dx_I$

β = m-form $= \Sigma_J\, g_J\, dx_J$

$\alpha\beta = (\Sigma_I\, f_I\, dx_I)(\Sigma_J\, g_J\, dx_J) = \Sigma_{I,J}\, f_I\, g_J\, dx_I\, dx_J$ // exterior product of two diff forms
// page 19 B

where note the ordering of the dx_i stuff! As S points out, lots of terms will likely vanish in such an exterior product.

Example: Consider $\kappa \equiv \alpha^p$ = a differential form since products are allowed, where α is a k-form. The degree of this k form is then kp, meaning the number of factors of dx_i. If kp exceeds n, then you must have two the same and the form vanishes. Now, when is kp > n ?

k = 1 p > n
k = 2 p > n
...
k = k p > n

In any of these cases, if p > n, the form vanishes. Thus:

$\alpha^p = 0$ if p = n+1 regardless of k as long as k > 0

$\alpha^{n+1} = 0$ regardless of k as long as k > 0 page 19 C

Notes added 2.11.16. Here I will construct some text to be later added to wedge doc which involves the two ideas presented just above. The first affected area is Chapter 7 which does non-dual wedge k. I talk there about the wedge product of 3 or more tensors and this is the general result,

$$(T_1)\hat{}(T_2)\hat{}...\hat{}(T_N)\hat{} = \Sigma_I\, (T_1{}^{I_1} T_2{}^{I_2} .... T_N{}^{I_N})\, e\hat{}_I = \Sigma_I\, (T_1 \otimes T_2 .... \otimes T_N)^I\, e\hat{}_I \qquad (7.9.d.6)$$

where $e\hat{}_I = e_{i_1}\hat{}\, e_{i_2}\, ...\hat{}\, e_{i_{k_1+k_2+...+k_N}} = e_{i_1}\hat{}\, e_{i_2}\, ...\hat{}\, e_{i_\kappa}$

and $(T_1 \otimes T_2 .... \otimes T_N)^I = T_1{}^{I_1} T_2{}^{I_2} .... T_N{}^{I_N}$.

Now suppose all these tensors are the same and are just T of rank k. Then this becomes,

$$(T\hat{})^N \equiv T\hat{}T\hat{}...\hat{}T\hat{} = \Sigma_I\, (T^{I_1} T^{I_2} .... T^{I_N})\, e\hat{}_I = \Sigma_I\, (T \otimes T .... \otimes T)^I\, e\hat{}_I \qquad (7.9.d.6)$$

where $e\hat{}_I = e_{i_1}\hat{}\, e_{i_2}\, ...\hat{}\, e_{i_{k+k+..+k}} = e_{i_1}\hat{}\, e_{i_2}\, ...\hat{}\, e_{i_\kappa}$, $\kappa = Nk$

and $(T \otimes T .... \otimes T)^I = T^{I_1} T^{I_2} .... T^{I_N}$.

How many distinct e_i are there? Only n if dim(V) = n. So $e\hat{}_I = 0$ if $\kappa > n$ which is to say if Nk>n which is to say if N > (n/k). Thus we seen to arrive at this new rule;

$(T\wedge)^{N} = 0$ if $N > (n/k)$

Now as a special case, suppose $T\wedge$ is a rank-1 tensor (vector) so k = 1. We then conclude that

$(V\wedge)^{N} \equiv V \wedge V ... \wedge V = 0$ if $N > n$.

For example,

$(V\wedge)^{n+1} = 0$

and this is then the above result $\alpha^{n+1} = 0$. However, this will turn out to be in the dual space, so let's repeat the above discussion in that case.

$$(\mathcal{T}_1)\wedge(\mathcal{T}_2)\wedge ... \wedge(\mathcal{T}_N)\wedge = \Sigma_I (T_1)_{I_1}(T_2)_{I_2} .... (T_N)_{I_N} \lambda\wedge^I = \Sigma_I (T_1 \otimes T_2 .... \otimes T_N)_I \lambda\wedge^I \quad (8.9.d.6)$$

where $\lambda\wedge^I = \lambda^{i_1}\wedge \lambda^{i_2} .....\wedge \lambda^{i_{k_1+k_2+...+k_N}} = \lambda^{i_1}\wedge \lambda^{i_2} .....\wedge \lambda^{\kappa_N}$

and $(T_1 \otimes T_2 .... \otimes T_N)_I = (T_1)_{I_1}(T_2)_{I_2} .... (T_N)_{I_N}$.

Now suppose all these tensors are the same and are just T of rank k. Then this becomes,

$$(\mathcal{T}\wedge)^{N} \equiv \mathcal{T}\wedge\mathcal{T}\wedge ... \wedge\mathcal{T}\wedge = \Sigma_I (T)_{I_1}(T)_{I_2} .... (T)_{I_N} \lambda\wedge^I = \Sigma_I (T \otimes T .... \otimes T)_I \lambda\wedge^I \quad (8.9.d.6)$$

where $\lambda\wedge^I = \lambda^{i_1}\wedge \lambda^{i_2} .....\wedge \lambda^{i_{k_1+k_2+...+k_N}} = \lambda^{i_1}\wedge \lambda^{i_2} .....\wedge \lambda^{\kappa_N}$

and $(T \otimes T .... \otimes T)_I = (T)_{I_1}(T)_{I_2} .... (T)_{I_N}$.

How many distinct λ^i are there? Only n if dim(V*) = n. So $\lambda\wedge^I = 0$ if $\kappa > n$ which is to say if Nk>n which is to say if N > (n/k). Thus we seen to arrive at this new rule;

$(\mathcal{T}\wedge)^{N} = 0$ if $N > (n/k)$

Now as a special case, suppose $\mathcal{T}\wedge$ is a rank-1 tensor (vector) so k = 1. We then conclude that

$(\alpha)^{N} \equiv \alpha \wedge \alpha ... \wedge \alpha = 0$ if $N > n$.

For example,

$(\alpha)^{n+1} = 0$

and this is then the above result $\alpha^{n+1} = 0$.

Theorem 2.1. $\beta\alpha = (-1)^{km}\alpha\beta$ α = k-form β = m-form

We know that

$\alpha\beta = \Sigma_{I,J} f_I \; g_J \; dx_I \, dx_J$

$\beta\alpha = \Sigma_{I,J} f_I \; g_J \; dx_J \, dx_I$

So the question is: what happens when you slide all of dx_I through dx_J ?

Slice first dx_i through and pick up $(-1)^k$ since need to do k swaps
Then slide second dx_i through, get another factor $(-1)^k$
Do this m times and you then have $(-1)^{km}$ QED.

He shows this in p 19 D. Fine.

Note that $\beta\alpha = \pm \alpha\beta$, so you *almost* have the two forms commuting, but you might pick up a sign. This relation is then called **"graded" commutivity**.

Corollary 2.2: $\alpha^2 = (-1)^{k*k} \alpha^2$ for α = any k-form so that $\alpha^2 = 0$ for k = odd.

Notice that S is sneaking in some numbers that are like equation numbers, but instead are the numbers of theorems (propositions) and corollaries [I will add this idea as well to wedge doc ***]

2.2 The Exterior Derivative [20]

This phrase refers to applying operator "d" onto a differential form "α" of any order. Only the coefficient functions are "hit" by the d operator, the dx_i are constants. S starts on page 20 applying d to a 0-form to get a 1-form. The d operator does this to any coefficient: $d(f) = (\partial_i f)\, dx_i$ and this adds a dx_i factor raising the order of the form you acted upon by one.

Question added 2.11.16. Why are the dx_i treated as constants? You could also ask in wedge doc why $\mathbf{e}_i$ and λ^i might be treated as constants. This entire Sja Chapter 2 has "Euclidean space" in its title, which I call Cartesian space. In that case, I would argue that yes, $\mathbf{e}_i$ and $\mathbf{e}^i$ are constants in space and so λ^i is a constant and eventually when we set $\lambda^i = dx^i$ we also have dx^i = constants in space. So that is my temporary answer to this question. Then all the math of this section is justified.

Acting on a **general 1-form in $\mathbf{R}^n$** is treated bottom of page 20. Here is the result

$$d\alpha = d(\, df_i dx_i) = d(df_i)dx_i = [\, (\partial_j f_i)\, dx_j]dx_i = \Sigma_{i<j}\, (\partial_j f_i)\, dx_j dx_i \; + \; \Sigma_{j<i}\; (\partial_j f_i)\, dx_j dx_i$$

$$= \Sigma_{i<j}\, (\partial_j f_i)\, dx_j dx_i \;\text{ (as was) }\; + \; \Sigma_{i<j}\; (\partial_i f_j)\, dx_i dx_j \;\text{(swap dummy sum indices)}$$

$$= \Sigma_{i<j}\, [\; (\partial_i f_j) - \; (\partial_j f_i)]\; dx_i dx_j \,. \qquad (2.2)$$

Here you see the order increase by 1, and you see a very "curl like" structure formed in the bracket, but is true only for R^3. In that case,

$$(\partial_i f_j) - (\partial_j f_i) = \varepsilon_{ijk}\varepsilon_{kbc}\partial_b f_c = \varepsilon_{ijk}[\text{curl}(\mathbf{f})]_k = \varepsilon_{ijk}[\nabla \times \mathbf{f}]_k$$

So for R^3 I guess we have this result

$$d\alpha = \Sigma_{i<j} [\ \varepsilon_{ijk}[\nabla \times \mathbf{f}]_k\ ]\ dx_i dx_j \qquad (2.2a)$$

I think in tensor doc I already claim that the form shown in (2.2) is the generalization of the curl. Yes, except it is all covariant derivatives. I can see that we will have to redo all this work in the covariant world at some point!

[21] Now try a **general 2-form in $\mathbf{R}^n$** : It is important to start out right!!! Here is α:

$$\alpha = \Sigma_{i<j}\ f_{ij} dx_i dx_j$$

$$d\alpha = \Sigma_{i<j}\ [df_{ij}]dx_i dx_j = \Sigma_{i<j}\ [\Sigma_k(\partial_k f_{ij})dx_k]dx_i dx_j = \Sigma_{i<j}\Sigma_k\ (\partial_k f_{ij})\ dx_k dx_i dx_j$$

Here Σ_k is over all k initially, but to get a non-zero term it cannot equal i or j. There are three possible positions k can therefore take relative to i and j:

$$\Sigma_{i<j}\Sigma_k = \Sigma_{k<i<j} + \Sigma_{i<k<j} + \Sigma_{i<j<k}$$

where I don't bother to show the extremal limits 1 and n for R^n. So write the three terms

$$d\alpha = [\ \Sigma_{i<j<k} + \Sigma_{i<k<j} + \Sigma_{k<i<j}\ ]\ (\partial_k f_{ij})\ dx_k dx_i dx_j$$

$$= \Sigma_{i<j<k}(\partial_k f_{ij})\ dx_k dx_i dx_j + \Sigma_{i<k<j}\ (\partial_k f_{ij})\ dx_k dx_i dx_j + \Sigma_{k<i<j}\ (\partial_k f_{ij})\ dx_k dx_i dx_j\ .$$

Now the plan is to do dummy index swaps to get the last terms to look like the first. The second term requires only a single swap k↔j to give

$$\Sigma_{i<k<j}\ (\partial_k f_{ij})\ dx_k dx_i dx_j = \Sigma_{i<j<k}\ (\partial_j f_{ik})\ dx_j dx_i dx_k\ .$$

But the last term requires three swaps which in effect do a backwards cyclic, so we get kij→ijk

$$\Sigma_{k<i<j}\ (\partial_k f_{ij})\ dx_k dx_i dx_j = \Sigma_{i<j<k}\ (\partial_i f_{jk})\ dx_i dx_j dx_k\ .$$

Now all three terms are $\Sigma_{i<j<k}$ and we then have

$$d\alpha = \Sigma_{i<j<k}\ \{\ (\partial_k f_{ij})\ dx_k dx_i dx_j + (\partial_j f_{ik})\ dx_j dx_i dx_k + (\partial_i f_{jk})\ dx_i dx_j dx_k\ \}$$

The next step is to get the dx_i into ijk order in each term, so

$$d\alpha = \Sigma_{i<j<k} \{+ (\partial_k f_{ij})\, dx_i dx_j dx_k - (\partial_j f_{ik})\, dx_i dx_j dx_k + (\partial_i f_{jk})\, dx_i dx_j dx_k \}$$

$$= \Sigma_{i<j<k} \{ (\partial_k f_{ij}) + (\partial_i f_{jk}) - (\partial_j f_{ik}) \}\, dx_i dx_j dx_k \qquad \text{// agrees with (2.4)}$$

The object in {..} is then some higher tensor form object.

Some Rules 2.5

1. $d(A\alpha + B\beta) = A(d\alpha) + B(d\beta)$ α,β are any diff forms of any order, A,B constants

2. $d(\alpha\beta) = (d\alpha)\,\beta + (-1)^k\, \alpha\, (d\beta)$ α = k-form $\beta = \ell$-form

Proof of Rule 1:

$$d(A\alpha + B\beta) = d(A \Sigma_I f_I\, dx_I + B \Sigma_J g_J\, dx_J) = A \Sigma_I (df_I)\, dx_I + B \Sigma_J (dg_J)\, dx_J$$

$$= A(d\alpha) + B(d\beta) \qquad \text{QED}$$

Proof of Rule 2

$$d(\alpha\beta) = d[\ (\Sigma_I f_I\, dx_I)(\Sigma_J g_J\, dx_J)\ ] = \Sigma_I \Sigma_J\, d(f_I g_J)\, dx_I\, dx_J$$

$$= \Sigma_I \Sigma_J\, [\ (df_I) g_J + f_I (dg_J)]\ dx_I\, dx_J$$

$$= \Sigma_I \Sigma_J\ (df_I) g_J\ dx_I\, dx_J + \Sigma_I \Sigma_J\, f_I (dg_J)\, dx_I\, dx_J$$

Now comes a key point. The things (df_I) and (dg_J) each contain a dx_i factor, so you cannot slide things around willy nilly. So note that in the second term,

$$(dg_J)\, dx_I = (\text{sign})\ dx_I (dg_J)$$

where the sign arises from sliding the single dx_i through all the dx_I guys. But the dx_I goes with α which is a k-form, so there are k factors we have to slide dx_s through, so you really do get $(-1)^k$ = sign. So

$$= \Sigma_I \Sigma_J\ (df_I) g_J\ dx_I\, dx_J + (-1)^k \Sigma_I \Sigma_J\, dx_I\ f_I (dg_J)\, dx_J$$

$$= [\Sigma_I (df_I) dx_I\ ][\Sigma_J\, g_J dx_J] + (-1)^k [\Sigma_I\, f_I\, dx_I][\ \Sigma_J (dg_J)\, dx_J]$$

$$= (d\alpha)\,\beta + (-1)^k \alpha\, (d\beta)\ . \qquad \text{QED}$$

Note that for a 0-form (function) you get the "usual" expected result $d(\alpha\beta) = (d\alpha)\beta + \alpha(d\beta)$. Remember that $\beta\alpha = (-1)^{k\ell}\, \alpha\beta$ from **2.1**-- forms do not trivially "commute".

So page 21 was chock full of stuff.

[22] Rule 2.6 that $d^2\alpha = 0$

Proof:

$\alpha = \Sigma_I f_I dx_I$
$d\alpha = \Sigma_I (df_I)dx_I = \Sigma_I\Sigma_k (\partial_k f_I)\, dx_k dx_I$

$d(d\alpha) = \Sigma_I\Sigma_k [d(\partial_k f_I)]dx_k dx_I = \Sigma_I\Sigma_k\Sigma_j [\partial_{jk} f_I]dx_j dx_k dx_I = 0$ by symmetry!

His proof is a little different:

$\alpha = \Sigma_I f_I dx_I$
$d\alpha = \Sigma_I (df_I)dx_I = \Sigma_I\Sigma_k (\partial_k f_I)\, dx_k dx_I = \Sigma_I \{ \Sigma_k (\partial_k f_I)dx_k\} dx_I$

Then

$d(d\alpha) = \Sigma_I\, d\{ \Sigma_k (\partial_k f_I)dx_k\}\, dx_I$

Notice that {...} is a 1-form. Recall from earlier that when α is a 1-form,.

$d\alpha' = \Sigma_{i<j} [(\partial_i f'_j) - (\partial_j f'_i)]\, dx_i dx_j$ where $\alpha' = \Sigma_k f'_k dx_k$.

So now let $\alpha' = \Sigma_k (\partial_k f_I)dx_k$ so that the role of f'_k is played by $(\partial_k f_I)$. Then

$d[\Sigma_k (\partial_k f_I)dx_k] = \Sigma_{i<j} [(\partial_i[\partial_j f_I]) - (\partial_j[\partial_i f_I])\, dx_i dx_j$

$= \Sigma_{i<j} [0]\, dx_i dx_j = 0$

and therefore above earlier we have

$d(d\alpha) = \Sigma_I\, d\{ \Sigma_k (\partial_k f_I)dx_k\}\, dx_I = \Sigma_I\, 0\, dx_I = 0$

OK, but I think my proof is a lot simpler and does not need to draw on other theorems.

Comment: How interpret $d^2\alpha = 0$? For a function in regular calculus you would get

$df = \partial_i dx_i$ $d^2 f = \partial_{ij} dx_i dx_j \neq 0$

So the whole reason $d^2\alpha = 0$ in "form world" is that asymmetry $dx_i dx_j = - dx_j dx_i$ and this is really a property of the wedge product which Sjamaar is suppressing throughout his doc so far. You have to keep in mind that in a form the object dxdy is not what it appears to be.

2.3 Closed and Exact Forms [22]

If $d\alpha = 0$, the form α is closed.
If $d\alpha = \beta$, the form β is exact. It is like an "exact differential" in regular calculus.
β exact $\Rightarrow$ β closed: $d\beta = d(d\alpha) = d^2\alpha = 0$ so β is closed // β not closed $\Rightarrow$ β not exact **

If α = 0-form, $\alpha = f$ is closed if all $f_i = 0$, so f is a constant. Only constant 0-forms are closed.
If β is closed, it may be exact, but it depends on topology! Coming later.

When is a 1-form α exact? Must find g such that $dg = \alpha$. Gives system of n 1st order PDE's in (2.5):

$$\partial_i g = f_i \quad \text{for } i = 1.2..n \quad . \qquad (2.5)$$

If α is not closed, there can be no solution by **. Now recall that for a general 1-form in R^n,

$$d\alpha = \Sigma_{i<j} [\ (\partial_i f_j) - (\partial_j f_i)]\ dx_i dx_j . \qquad (2.2)$$

To make sure α is closed before you test for exactness, you need to show $(\partial_i f_j) = (\partial_j f_i)$ which is p 22C. Or you just show directly that $d\alpha = 0$. So don't waste time trying to solve (2.5) until you show $(\partial_i f_j) = (\partial_j f_i)$.

Example 2.9. We are given a messy 1-form α in R^3 We find that $d\alpha = 0$, so closed, so can maybe find g with $\alpha = dg$. S shows in detail how you solve the PDE system (2.5) ["integrating the system"], and the solution g is stated where $dg = \alpha$. I remember this kind of PDE solution method I think from Sneddon.

[23] **Theorem without proof**: Every *closed* 1-form α on R^n is exact! You are supposed to show this by mimicking the example just done. Exercise 2.7 addresses this proof. Note that α must be defined on all of R^n for this theorem to be true.

Example 2.10: The Angle Form (a 1-form) on R^2 with origin removed. They (with me) show that $d\alpha = 0$ so this angle form is in fact closed, but we will later see it is not exact. This does not contradict the above theorem because the angle form is not defined on all of R^2.

Question: What is the integrability condition and the PDE system for an arbitrary 2-form?

$\alpha = \Sigma_I f_I\ dx_I$ ordered multi-index

$= \Sigma_{i<j} f_{ij} dx_i dx_j$. specific

We know from (2.4) above that

$$d\alpha = = \Sigma_{i<j<k} \{\ (\partial_k f_{ij}) + (\partial_i f_{jk}) - (\partial_j f_{ik})\ \} dx_i dx_j dx_k$$

so the integrability condition in this case is

$(\partial_k f_{ij}) + (\partial_i f_{jk}) - (\partial_j f_{ik}) = 0$.

If we want $\alpha = dg$, where g is a 1-form $g = \Sigma_k g_k dx_k$ then we have

$dg = \Sigma_{i<j} [(\partial_i g_j) - (\partial_j g_i)] dx_i dx_j$ // from (2.2)

$= \Sigma_{i<j} f_{ij} dx_i dx_j$.

So the system of equations you need to solve to find g is given by

$(\partial_i g_j) - (\partial_j g_i) = f_{ij}$. // agrees with S (2.6).

At this point, the reader has no feel whatsoever for the significance of a form being closed and being exact. S is just getting the reader used to the machinery.

2.4 The Hodge star operator [23]

The first concept here is this:

dx_I = some multi-index set of dx_i in R^n

$(*dx_I) = (\text{sign})\, dx_{Ic}$ = the ones that are missing in dx_I within R^n (c means complement)

What is the sign? Well, when you write the combination $dx_I\, dx_{Ic}$ you will get ALL the dx_i . When you adjust this list to put them into standard order 1,2,3... you may get a minus sign if it takes an odd number of swaps. With this sign adjustment, you always get $dx_I(*dx_I) = dx_1....dx_n$ = the positive volume element.

Now using this idea, we define

$\alpha = \Sigma_I f_I\, dx_I$

$*\alpha = \Sigma_I f_I\, (*dx_I)$ this then is the "Hodge starred α" // Hodge circa 1930

If α is a k-form, then *α is in (n-k) form [sometimes called dual form] . Lots of examples are given.

They note that there are (n,k) k-forms α on R^n and there are (n,n-k) (n-k)-forms = *α forms = (n,k). The connection is not yet stated.

Exercise 2.12: Show that $**\alpha = (-1)^{kn+k}\alpha$ if α is a k-form.

$\alpha = \Sigma_I f_I\, dx_I$ k-form

$*\alpha = \Sigma_I f_I\, (*dx_I)$

$*(*\alpha) = \Sigma_I \, f_I \, *(*dx_I)$

Lemma: Show that

$*(*dx_I) = (-1)^{kn+k} \, dx_I$.

If we can show this Lemma, then we continue the previous proof

$*(*\alpha) = \Sigma_I \, f_I \, *(*dx_I) = \Sigma_I \, f_I \, (-1)^{kn+k} \, dx_I = (-1)^{kn+k} \, \Sigma_I \, f_I dx_I = (-1)^{kn+k} \, \alpha$.

Proof of Lemma: $*(*dx_I) = (-1)^{kn+k} \, dx_I$

Do an example first

$dx_I = dx_2 dx_4$ in R^6

$*dx_I = S \, dx_1 dx_3 dx_5 dx_6$

$dx_I(*dx_I) = dx_2 dx_4 \, S \, dx_1 dx_3 dx_5 dx_6 = S \, (-1)^2 dx_1 dx_2 dx_4 dx_3 dx_5 dx_6 = S(-1)^3 \, dV$

Think of this in terms of permutations. We have

$dx_I = 24$ $*dx_I = S \, 1356$ $k = 2$ $n = 6$

$dx_I(*dx_I) = S \; 241356 = S \, (-1)^2 \, 124356 = S(-1)^2 \, (-1)^1 \, 123456$

OK, now go back to

$*dx_I = S \, dx_1 dx_3 dx_5 dx_6$

$*(*dx_I) = S \, S' \, dx_2 dx_4$

Now S' is set by this requirement

$*(*dx_I) \, (*dx_I) = 123456$

So we then have

$S \, S' \, 24 \;\; S \, 1356 = 123456$

The sign S cancels out and we are left with

$S' \; 241356 = 123456$

and as before, $S' = (-1)^3 = -1$. In this example then we have

$$*(*dx_I) = S\, S'\, dx_2 dx_4 = SS'\, dX_I$$

and $SS' = +1$. The rule was $(-1)^{kn+k}$ with k = 2 and n = 6 so we get the right answer. But I don't have much insight on how to do the general case. But I later proved this in a separate doc "proof of a lemma v 2.doc".

2.5 div, grad and curl [24]

You can write any 1-form as $\alpha = \mathbf{F}\bullet d\mathbf{x}$ on R^n. Thus, there is a direct 1-to-1 connection between α and **F**. Probably there is a simple connection between a rank-2 tensor and a 2 form in that $\alpha = F_{ij}\, dx^i dx^j$ and F would be antisymmetric. In this way, I guess every tensor is related to a form. This is definitely a new appendix for **tensor doc** at some point and maybe a new whole chapter! I suspect all the forms are tensorial scalars.

Note added 2.11.15 re last comment above. When I write $T = \Sigma_{ij} T^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$, under a Type A transformation the basis vectors do not move, and this becomes $T' = \Sigma_{ij} T'^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$ where the T'^{ij} is the usual thing, so "this is how a tensor transforms". In the dual space $\mathcal{T} = \Sigma_{ij} T_{ij}\, \boldsymbol{\lambda}^i \otimes \lambda^j$ and again the basis vectors do not move under a transformation, so I would then say $\mathcal{T}' = \Sigma_{ij} T'_{ij}\, \boldsymbol{\lambda}^i \otimes \lambda^j$. I think the above is just the same when you replace ⊗ with ^, so $T = \Sigma_{ij} T^{ij}\, \mathbf{e}_i \wedge \mathbf{e}_j$, and then $T' = \Sigma_{ij} T'^{ij}\, \mathbf{e}_i \wedge \mathbf{e}_j$ and you are here just dealing with tensors which are in a certain subspace of the original space. So when we eventually write $\mathcal{T} = \Sigma_{ij} T_{ij}\, dx^i \otimes dx^j$, I would say $\mathcal{T}' = \Sigma_{ij} T'_{ij}\, dx^i \otimes dx^j$. Thus, my statement above that "I suspect all forms are tensorial scalars" is incorrect. However, all tensor *functions* are scalars, but these don't exist at this point in Sjamaar World.

Let f be a 0-form and then $\nabla f \equiv \mathbf{F}$ is a directly related vector field. Then consider $\alpha = \nabla f \bullet d\mathbf{x} = df$. But I don't see the big deal here. If f is a 0-form, then yes, grad f appears in df. Write

$\alpha = f$	0-form	$\alpha \leftrightarrow f$	
$d\alpha = df$	1-form	$d\alpha \leftrightarrow \nabla f$	// namely, $d\alpha = \nabla f \bullet d\mathbf{x}$

I think I now better understand Buck's ↔ symbol. It means there is a 1-to-1 mapping between the items on the two sides.

Now comes a longer deal. Write

$\alpha = \mathbf{F} \bullet d\mathbf{x}$	1-form	
$(*\alpha) = \mathbf{F} \bullet (*d\mathbf{x})$	where $(*d\mathbf{x})_1 = dx_2 dx_3 .... dx_n$, for example	n-1 form

Later we will claim that d**x** and (*d**x**) are related to orthogonal subspaces in R^n, the way dx_1 and dx_2 are in R^2. This idea is just mentioned at this point. Now consider

$d(*\alpha) = d(\Sigma_k F_k\ (*dx)_k = \Sigma_k d(F_k)\ (*dx)_k = \Sigma_k\Sigma_j \partial_j F_k\ dx_j (*dx)_k$. n-form

Now consider $dx_j(*dx)_k$. The combination $(*dx)_k$ is missing dx_k .

If j = k, we then have $dx_k(*dx)_k$ which according to p 24 B is + $dx_1....dx_n$. = dV

If j≠ k, we then have $dx_j(*dx)_k = 0$ because two dx_s will appear together, killing off the form.

Thus,

$d(*\alpha) = \Sigma_k\Sigma_j \partial_j F_k\ dx_j(*dx)_k = \Sigma_k\Sigma_j \partial_j F_k\ \delta_{j,k} V = \Sigma_k\ (\partial_k F_k) V = (\text{div } \mathbf{F})\ V$. n-form

This diagonal effect brings out the divergence! Now do one more step

$*[d(*\alpha)] = *[\ (\text{div } \mathbf{F})\ V] = (\text{div } \mathbf{F})(*V) = (\text{div } \mathbf{F})(1) = \text{div } \mathbf{F}$. 0-form

So what does that really tell me? I can write (div **F**) in terms of the 1-form associated with **F**. :

$\alpha = \mathbf{F} \bullet d\mathbf{x}$ 1-form $\alpha \leftrightarrow \mathbf{F}$

$*[d(*\alpha)] = (\text{div } \mathbf{F})$ 0-form $*[d(*\alpha)] = \text{div } \mathbf{F}$

Finally, let's repeat the above, but do things in a different order

$\alpha = \mathbf{F} \bullet d\mathbf{x}$ = same starting 1-form as above

$d\alpha = \Sigma_{i<j}\ [\ (\partial_i F_j) - (\partial_j F_i)]\ dx_i dx_j$ // our usual formula (2.2)

$*(d\alpha) = \Sigma_{i<j}\ [\ (\partial_i F_j) - (\partial_j F_i)]\ *(dx_i dx_j)$ // agrees with p 27A

He writes

$*(dx_i dx_j) = (-1)^{i+j+1}\ dx_1 dx_2 .....dx_i ....dx_j .....dx_n$ // red ones are missing

To verify that sign, consider

$(dx_i dx_j)\ (-1)^{i+j+1}\ dx_1 dx_2 .....dx_i ....dx_j .....dx_n = (-1)^{i+j+1}\ [\ dx_i dx_j dx_1 dx_2 .....dx_i ....dx_j .....dx_n]$

$= (-1)^{i+j+1}[\ (-1)^{j-1}\ dx_i dx_1 dx_2 .....dx_i ....dx_j .....dx_n\ ]$ // did not pass through dx_i since missing

$= (-1)^{i+j+1}[\ (-1)^{j-1}\ (-1)^{i} dx_1 dx_2 .....dx_i ....dx_j .....dx_n\ ]$

$= dx_1 dx_2 .....dx_n = dV$

So OK, go back to

$*(d\alpha) = \Sigma_{i<j}\ (-1)^{i+j+1}\ [\ (\partial_i F_j) - (\partial_j F_i)]\ dx_1 dx_2 .....dx_i ....dx_j .....dx_n$ // red missing

This result is a little hard to interpret in general R^n , is just a fact.

Now what happens if you are in R^3 ? Then write out the terms :

i=1 j=2 (+) [$(\partial_1 F_2) - (\partial_2 F_1)$] dx_3

i=1 j=3 (-) [$(\partial_1 F_3) - (\partial_3 F_1)$] dx_2

i=2 j=3 (+) [$(\partial_2 F_3) - (\partial_3 F_2)$ dx_1

Then

$*(d\alpha) = [(\partial_1 F_2) - (\partial_2 F_1)] dx_3 + [(\partial_2 F_3) - (\partial_3 F_2)] dx_1 + [(\partial_3 F_1) - (\partial_1 F_3)] dx_2$

$= [(\partial_2 F_3) - (\partial_3 F_2)] dx_1 + [(\partial_3 F_1) - (\partial_1 F_3)] dx_2 + [(\partial_1 F_2) - (\partial_2 F_1)] dx_3$

$= [\text{curl } \mathbf{F}]_1 dx_1 + [\text{curl } \mathbf{F}]_2 dx_2 + [\text{curl } \mathbf{F}]_3 dx_3$

$= [\text{curl } \mathbf{F}] \bullet d\mathbf{x}$

Note that α = 1-form, dα = 2-form, *(dα) = n-2 = 1-form again for R^3.

Therefore we have this correspondence idea:

$\alpha = \mathbf{F} \bullet d\mathbf{x}$ $\alpha \leftrightarrow \mathbf{F}$ one-to-one

$*(d\alpha) = [\text{curl } \mathbf{F}] \bullet d\mathbf{x}$ $*(d\alpha) \leftrightarrow \text{curl } \mathbf{F}$

How about the **Laplacian** from Ex 2.14. Start again with a 0-form

$\alpha = f$ 0-form $\alpha \leftrightarrow f$

$d\alpha = df = \partial_i f\, dx_i$ 1-form

$*(d\alpha) = \partial_i f\, (*dx_i) = (-1)^i \partial_i f\ dx_1 dx_2 .... dx_i .... dx_n$ n-1 form

$d[*(d\alpha)] = (-1)^i \partial_j \partial_i f\ dx_j dx_1 dx_2 ..... dx_i .... dx_n$ n-form

Implied sum on i and j. If $j \neq i$, then dx_j appears twice so get 0. If j = i, we just slide dx_j into position and kill the sign

$d[*(d\alpha)] = \partial_i^2 f\ dx_j dx_1 dx_2 ..... dx_i .... dx_n$ n-form

Then finally

$*d[*(d\alpha)] = \partial_i{}^2 f = \nabla^2 f$ // scalar Laplacian. 0-form

OK, lets recap these amazing little results:

$\alpha = f$	0-form in R^n	$\alpha \leftrightarrow f$	PL1
$d\alpha = df$	1-form in R^n	$d\alpha \leftrightarrow \nabla f$	// namely, $d\alpha = \nabla f \bullet d\mathbf{x}$
$\alpha = f$	0-form in R^n	$\alpha \leftrightarrow f$	
$*d[*(d\alpha)] = \partial_i{}^2 f = \nabla^2 f$	0-form on R^n	$*d[*(d\alpha)] \leftrightarrow \nabla^2 f$	PL2
$\alpha = \mathbf{F} \bullet d\mathbf{x}$	1-form on R^n	$\alpha \leftrightarrow \mathbf{F}$	
$*[d(*\alpha)] = (\text{div } \mathbf{F})$	0-form on R^n	$*[d(*\alpha)] = \text{div } \mathbf{F}$	PL3
$\alpha = \mathbf{F} \bullet d\mathbf{x}$	1-form in R^3	$\alpha \leftrightarrow \mathbf{F}$	one-to-one
$*(d\alpha) = [\text{curl } \mathbf{F}] \bullet d\mathbf{x}$	1-form in R^3	$*(d\alpha) \leftrightarrow \text{curl } \mathbf{F}$	PL4

Now imagine this correspondence idea

$\alpha = f$	$\leftrightarrow f$	α = 0-form	R^n
$d\alpha$	$\leftrightarrow \nabla f$		
$*d[*(d\alpha)]$	$\leftrightarrow \nabla^2 f$		
$\alpha = \mathbf{F} \bullet d\mathbf{x}$	$\leftrightarrow \mathbf{F}$	α = 1-form	R^3 only
$*[d(*\alpha)]$	$\leftrightarrow \text{div } \mathbf{F}$		
$*(d\alpha)$	$\leftrightarrow \text{curl } \mathbf{F}$		

If you are willing to "work in the form universe", you can construct fancy differential operators from the symbols α, d and * and they certainly looks simple.

At this point I did Exercise 2.12 (**α) and part of Exercise 2.18 (Maxwell). I am ready to continue on. See separate docs for those exercises "Sjamaar Ex 2_18 on Maxwells v2.doc".

I do Exercise 2.6 in the Chapter 4.1 notes below, to show that a closed 1-form on R^n is exact. Recall that exact $\Rightarrow$ closed, so this is going the other way. The key idea is that the 1-form is closed on ALL of R^n.

Chapter 3: Pulling Back Forms

3.1 Determinants

This is a long digression which I can only assume he will make use of later on.

Note added 2.11.16. I think the multilinear idea is very simple. The determinant of a square matrix having columns $\mathbf{v}_1$, $\mathbf{v}_2$...$\mathbf{v}_k$ is k-multilinear in those arguments. So f($\mathbf{v}_1$, $\mathbf{v}_2$ .. $\mathbf{v}_k$) = det($\mathbf{v}_1$, $\mathbf{v}_2$ .. $\mathbf{v}_k$) is a k-multilinear function of its k vector arguments. It is also a totally antisymmetric function of the arguments. When I first read this stuff here, I did not understand the significance of these two properties. In wedge doc I show that

$$(\lambda^{j1} \wedge \lambda^{j2} \wedge .... \wedge \lambda^{ik})(v_{i_1},v_{i_2}....v_{i_k}) = (1/k!)\ \det[\ (v_{i_*})^{j*}] \qquad (8.3.9a)$$

$$= (1/k!)\ \det\ (v_{i_1},v_{i_2}....v_{i_k})$$

so this shows that the degree-k basis function shown here, closed to be a tensor function, is in fact both k-multilinear and totally antisymmetric and in this case this follows directly from the properties of the determinant. Later still I show that $\mathcal{T}^{\wedge}(v_{j_1},v_{j_2}....v_{j_k}) \equiv [\mathrm{Alt}(\mathcal{T})]\ (v_{j_1},v_{j_2}....v_{j_k})$ where things are then less obvious.

Author reviews classic definitions of detA and comes up with his own: Three axioms which are the **multilinear** rule, the **minus sign** on swap columns, and **det($\mathbf{e}_i$) = 1**. The multi rule needs a clearer statement:

$$\det(\ a_1,\ a_2..... \underset{\text{column j}}{[ca_i + c'a'_i]},.... a_n)$$

What is confusing here is that the column marked as column j is a linear combination of any two columns of A which two are denoted by a_i and a'_i. The rule is then

$$\det(\ a_1,\ a_2..... [ca_i + c'a'_i],.... a_n)$$

$$= c\ \det(\ a_1,\ a_2..... \underset{\text{col j}}{a_i}.... a_n)\ + c'\ \det(\ a_1,\ a_2..... \underset{\text{col j}}{a'_i},.... a_n)$$

I might have presented this differently. Consider determinant

$$\det(\ \mathbf{a}_1,\ \mathbf{a}_2..... \mathbf{a}_j\ .....\mathbf{a}_n)\ .$$

Now let $\mathbf{a}_\alpha$ and $\mathbf{a}_\beta$ be any two columns of the matrix. Then make this replacement

$$\mathbf{a}_j = c_\alpha\mathbf{a}_\alpha + c_\beta\mathbf{a}_\beta\ .$$

The result is det of a new matrix

$$Q = \det(\ \mathbf{a}_1,\ \mathbf{a}_2.....\ c_\alpha\mathbf{a}_\alpha + c_\beta\mathbf{a}_\beta,\ .....\mathbf{a}_n)$$

The axiom is then this

$$\det(\ \mathbf{a}_1,\ \mathbf{a}_2.....\ c_\alpha\mathbf{a}_\alpha + c_\beta\mathbf{a}_\beta,\ .....\mathbf{a}_n) = c_\alpha\det(\ \mathbf{a}_1,\ \mathbf{a}_2..... \mathbf{a}_\alpha\ .....\mathbf{a}_n) + c_\beta\det(\ \mathbf{a}_1,\ \mathbf{a}_2..... \mathbf{a}_\beta\ .....\mathbf{a}_n)\ .$$

If we take $c_\alpha = 1$ and $a_\alpha = a_j$, then we are adding a multiple of a column to a column, so the above rule is more general. How about breaking this down. First, here is what happens if you replace a_j with a sum of any two columns,

$$\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_\alpha + \mathbf{a}_\beta,\ \ldots\ldots \mathbf{a}_n) = \det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_\alpha \ldots\ldots \mathbf{a}_n) + \det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_\beta \ldots\ldots \mathbf{a}_n) \qquad \#1$$

Then another rule would be

$$\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ c\,\mathbf{a}_j \ldots\ldots \mathbf{a}_n) = c\,\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_j \ldots\ldots \mathbf{a}_n) \qquad \#2$$

Then his rule would follow as follows

$$\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ c_\alpha\mathbf{a}_\alpha + c_\beta\mathbf{a}_\beta,\ \ldots\ldots \mathbf{a}_n) = \det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ c_\alpha\mathbf{a}_\alpha \ldots\ldots \mathbf{a}_n) + \det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ c_\beta\mathbf{a}_\beta \ldots\ldots \mathbf{a}_n)$$

$$= c_\alpha\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_\alpha \ldots\ldots \mathbf{a}_n) + c_\beta\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_\beta \ldots\ldots \mathbf{a}_n) \qquad \text{Sj (i)}$$

So I think he is just trying to keep his axiom count low by combining things into a single axiom.

I. Add multiple of column to a *different* column:

$$\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_j + c_\beta\mathbf{a}_\beta,\ \ldots\ldots \mathbf{a}_n) =$$

$$\det(A) + c_\beta\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_\beta,\ \ldots\ldots \mathbf{a}_n) = 1\ \det(A) \quad . \qquad \text{mult} = 1$$

The detA does not change because two columns are the same and swap rule says zero.

II. Mult column by constant

$$\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ c\,\mathbf{a}_j \ldots\ldots \mathbf{a}_n) = c\,\det(\mathbf{a}_1, \mathbf{a}_2 \ldots\ldots\ \mathbf{a}_j \ldots\ldots \mathbf{a}_n) = c\,\det(A) \qquad \text{mult} = c$$

III. Swap columns

$$\det(A) \rightarrow (-1)\ \det A \qquad \text{mult} = -1$$

So this is the content of his 3.2 Lemma page 32.

3.4 Det determined by the three axioms is unique. Fine, I skip it in these notes.

3.5 Comment that you can normalize differently if you want. Then det' I = c for that different third axiom.

3.6 This is my expansion with a permutation sum. I write it this way

$$\det(M) = \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ M_{a_1 1} M_{a_2 2} \ldots\ldots M_{a_n n} \qquad (B.2.6)$$

$= \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ M_{1a_1}M_{2a_2} \ldots\ldots M_{na_n}$.

In his notation, he has

$a_1 = \sigma(1)$
$a_2 = \sigma(2)$
...
$a_n = \sigma(n)$

so he has

3.6. THEOREM (existence of determinants). *Every $n \times n$-matrix A has a well-defined determinant. It is given by the formula*

$$\det A = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

which I like to write as

$\mathbf{a} = A\ \mathbf{z_0}$ $\qquad\qquad$ $\det(M) = \Sigma_{\mathbf{a}}\ \text{sign(a)}\ M_{\mathbf{z0\ a}}$

where A is a matrix,

$$\mathbf{a} \equiv \begin{pmatrix} a_1 \\ a_2 \\ \ldots \\ a_n \end{pmatrix} = A \begin{pmatrix} 1 \\ 2 \\ \ldots \\ n \end{pmatrix} = A\ \mathbf{z_0} \qquad \text{(B.1.2)}$$

I can think of this as a mapping $P_{\mathbf{a}}$ in which

$1 \rightarrow a_1$
$2 \rightarrow a_2$
...

which you could write as

$a_1 = P_{\mathbf{a}}(1)$
$a_2 = P_{\mathbf{a}}(2)$

and so on. He refers to the mapping $P_{\mathbf{a}}$ as σ. It's just another way to do it, I like my way better.

[34] Comparison

Me	Sjamaar	
$\text{Parity}(\mathbf{a}) = (-1)^{S\mathbf{a}}$	$\text{sign}(\sigma) = (-1)^{\ell(\sigma)}$	parity of permutation **a** relative to $\mathbf{z_0}$
swap	inversion	
$S_{\mathbf{a}}$	$\ell(\sigma)$	number of swaps, length of σ

He then does a proof of his Theorem 3.6 which I skip since I have my own world on this.

[35] 3.7 Theorem is a collection of standard det rules. Only the triangular rule is not in my Appendix B of Lagrange.

He then claims that | det(A) | is a "volume change factor" where A maps the n-cube into an n-piped. It is true that the entire interior of the n-cube maps into the interior of the n-piped. But interior→interior does not convince me that the volume changes by this factor! I had to write a whole Appendix B in tensor doc to prove this fact, where volume is not defined by this process. I think he has pulled a fast one here.

This concludes the long determinant digression, and we now get on with the main chapter topic:

3.2 Pulling back forms

[36] Sja is now going to treat a subject I am quite interested in. What happens to a form under a transformation of the variables? So start with a form

$$\alpha = \Sigma_I f_I(\mathbf{y})\, dy_I \qquad \text{p 36 A}$$

and do this transformation on it

$$\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x}) \qquad \text{p 36 B} \qquad // \quad \mathbf{x} = \boldsymbol{\varphi}^{-1}(\mathbf{y}) \text{ if it exists}$$

Note: x-space has dimension E^n but y space has dimension E^m as shown p 36B, so in general an inverse is not going to exist. So the number of variables in the two spaces is not generally the same. Thus, you cannot talk about dy_I and dx_I as if they had the same sets of dy_i and dx_i factors!

Now define the action of operator φ* on a scalar function in this manner

$$\varphi^* f_I(\mathbf{x}) \equiv [f_I * \boldsymbol{\varphi}]\, f_I(\mathbf{x}) = f_I(\varphi(\mathbf{x})) = f_I(\mathbf{y}) \qquad \mathbf{x} \xrightarrow{\varphi} \mathbf{y}$$

Digression on the words Concatenation and Composition and use of the circle operator

Comment: for a function, we are just taking $f(\mathbf{y}) \rightarrow F(\mathbf{x}) = f(\varphi(\mathbf{x}))$ so you could say that the new functional form $F = f * \varphi$ which for me is a concatenation. Buck and Ahlfors don't seem to use this word and I could find no mention of the subject.

Wiki https://en.wikipedia.org/wiki/Function_composition speaks of "composition":

In mathematics, **function composition** is the pointwise application of one function to the result of another to produce a third function. For instance, the functions $f : X \to Y$ and $g : Y \to Z$ can be *composed* to yield a function which maps x in X to $g(f(x))$ in Z. Intuitively, if z is a function of y, and y is a function of x, then z is a function of x. The resulting *composite* function is denoted $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all x in X.[note 1] The notation $g \circ f$ is read as "g circle f", or "g round f", or "g composed with f", "g after f", "g following f", or "g of f", or "g on f". Intuitively, composing two functions is a chaining process in which the output of the first function becomes the input of the second function.

So they put things in "my" normal ordering which is that (g∘f)(x) = g(f(x)) so that the functions are in the same order on both sides. They use the circle, and even state "g circle f" above.

I see from the web that my word "concatenation" is never used for this meaning! (I think I used the word in bipolar doc and elsewhere). I look in my doc notes and see concat used in reference to combining matrices. I see things like this

- Concatenation of rotations around x, y, z axes

$$\mathbf{R}_{x,y,z}(\theta_x, \theta_y, \theta_z) = \mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z)$$

where again you could say this is matrices.

Conclusion: Concatenation is a good word for composing two linear transformations with matrices, but the correct word in general is composition, not concatenation. I stand corrected and will fix up docs at some point perhaps.

But here is somebody using my word:

An image-space distortion uses a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, which converts a 2D image into another 2D image. That is, each point p in the input image is mapped onto a point p' in the output image. While it is possible to define such functions and to work with them, recent work has shown that it is computationally effective to think of a distortion in 2D as a concatenation of two functions. The first function g maps every point in 2D onto a surface in 3D:

$$g : \mathbb{R}^2 \to \mathbb{R}^3$$

Subsequently, the resultant 3D surface is rendered to produce a new image in 2D using a rendering function

$$r : \mathbb{R}^3 \to \mathbb{R}^2$$

Thus, the image-space distortion is regarded as the concatenation of the two preceding functions, yielding $f = r \circ g$. In the following, we will see how this concatenation can be interpreted geometrically, which also leads to a possible way of implementing such image-space distortions in an efficient manner.

Now, regarding composition, I note above that wiki uses (g∘f)(x) = g(f(x)) so the functions stay in the same order. That is the same for person above using the concat word. All sites use the circle thing. So where did I see the * symbol used? Some sources say (fg)(x) = f(g(x)) so there is no operator.

Now on page 36 we get this (A courier new letter o works well)

$\varphi^* f = f \circ \varphi = f(\varphi(x))$

So Sja is using the standard $f \circ \varphi$ ordering, but for some reason he wants to use φ^* as if it were an operator which changes the function of f from whatever it is to φ(whatever it is). The operator φ^* acts on whatever is to its right. OK, this has no name.

Now he wants to do a similar thing to $dy_{\mathtt{I}}$:

$\varphi^* \, dy_{\mathtt{I}} = \varphi^* (dy_{\mathtt{a}} dy_{\mathtt{b}} ....) \equiv (d\varphi_{\mathtt{a}} d\varphi_{\mathtt{b}} ...)$ $\qquad \mathbf{y} = \boldsymbol{\varphi}(x)$

What exactly is this saying? Since $y_{\mathtt{a}} = \varphi_{\mathtt{a}}(\mathbf{x})$ you would think that

$dy_{\mathtt{I}} = dy_{\mathtt{a}} dy_{\mathtt{b}} .... = d\varphi_{\mathtt{a}} d\varphi_{\mathtt{b}} ...$ $\qquad$ without any special operator

The intention is really this

$dy_{\mathtt{I}} = dy_{\mathtt{a}} dy_{\mathtt{b}} .... = d\varphi_{\mathtt{a}}(\mathbf{x}) \, d\varphi_{\mathtt{b}}(\mathbf{x}) ...$ $\qquad$ [wrong!]

so he wants to emphasize the idea using the φ^* operator [wrong]. So his meaning is simply this

$\varphi^{*}\, dy_{I} = \varphi^{*}\,(dy_{a}dy_{b}....) \equiv d\varphi_{a}(\mathbf{x})\, d\varphi_{b}(\mathbf{x})...$

Nothing more and nothing less. Recall that

$\varphi^{*}\, f_{I} = f_{I}(\varphi(\mathbf{x}))$

and this is a true function composition, whereas $\varphi^{*}\, dy_{I}$ is just a cosmetic notation IMHO.

Now look at the action of φ^{*} on a form

$\alpha(\mathbf{y}) = \Sigma_{I}\, f_{I}(\mathbf{y})\, dy_{I} \qquad\qquad \mathbf{y} = \varphi(\mathbf{x})$

Here I have made explicit that $\alpha(\mathbf{y})$ is defined on y-space, including the y_{i} coordinates and the dy_{i} differentials. I could simply replace everywhere $\mathbf{y} = \varphi(\mathbf{x})$ to get

$\alpha(\varphi(\mathbf{x})) = \Sigma_{I}\, f_{I}(\varphi(\mathbf{x}))\, d\varphi_{I}(\mathbf{x}) \qquad\qquad \mathbf{y} = \varphi(\mathbf{x})$

and then I could account for the "change of functional form by writing

$\alpha'(\mathbf{x}) = \Sigma_{I}\, f_{I}'(\mathbf{x})\, d\varphi_{I}(\mathbf{x})$

$d\varphi_{i}(\mathbf{x}) = \Sigma_{j=1}^{m}(\partial\varphi_{i}/\partial x_{j})\, dx_{j} \qquad\qquad i = 1,2..n$

Of course if α was a k-form, there would be only four $d\varphi_{i}$ factors. He will talk more on this below.

So OK, instead of using primes to show a new functional form, just write things like this

$\alpha(\mathbf{y}) = \Sigma_{I}\, f_{I}(\mathbf{y})\, dy_{I}$

$\alpha(\varphi(\mathbf{x})) = \Sigma_{I}\, f_{I}(\varphi(\mathbf{x}))\, d\varphi_{I}(\mathbf{x})$

$(\alpha \circ \varphi)(\mathbf{x}) = \Sigma_{I}\, (f_{I} \circ \varphi)(\mathbf{x})\, (dy_{I} \circ \varphi)(\mathbf{x})$

Here the idea is $(dy_{I} \circ \varphi)(\mathbf{x}) = dy_{I}(\boldsymbol{\varphi}(x)) = d\varphi_{I}(\mathbf{x})$. It is a little stretched, but Sja is free to define things the way he wants. Then you translate the above bo

$(\varphi^{*}\alpha)(\mathbf{x}) = \Sigma_{I}\, (\varphi^{*}f_{I})(\mathbf{x})\, d\varphi_{I}(\mathbf{x})$

You can then think of

$\alpha'(\mathbf{x}) = (\varphi^{*}\alpha)(\mathbf{x})$

so the operator φ^{*} is in effect creating a function having your new functional form of x. I am happy.
You can then just say $\alpha' = \varphi^{*}\alpha$ where φ^{*} acts right on the form itself.

You are starting with a form $\alpha(\mathbf{y}) = \Sigma_I\, f_I(\mathbf{y})\, dy_I$ defined in y-space and using $\mathbf{y} = \varphi(\mathbf{x})$ and all the fancy notation above you end up with $(\varphi^*\alpha)(\mathbf{x}) = \Sigma_I\, (\varphi^* f_I)(\mathbf{x})\, d\varphi_I(\mathbf{x})$ being a form in x-space. I might have named the regions X and Y instead of U and V. Then we have

$\varphi: X \to Y$ $\qquad$ $\varphi(\mathbf{x}) = \mathbf{y}$ $\qquad$ a forward map going left x-space to right y-space

$\varphi^*: \Omega^k(Y) \to \Omega^k(X)$ $\qquad$ $\varphi^*\alpha = \Sigma_I\, (\varphi^* f_I)(\mathbf{x})\, d\varphi_I(\mathbf{x}) = \beta(\mathbf{x})$

$\alpha(\mathbf{y}) \quad \to \quad \beta(\mathbf{x})$

Words:

You have a *forward* mapping $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ which takes points in x-space to points in y-space.
You have a *backward* mapping $\beta(\mathbf{x}) = \varphi^*\, \alpha(\mathbf{y})$ which takes a k-form defined on y-space
and gives you a k-form defined on x-space.

You are *pulling the form α back* from y-space to x-space where it is $\beta(\mathbf{x})$.

The picture on page 37 is then just fine by me.

3.9 Example. Suppose we have some 2-form like $\alpha = f(y)\, dy_1 dy_2$ in R^2.
Then the pullback is doing to be $\varphi^*\alpha = (\varphi^* f)\, \varphi^*(dy_1 dy_2)$. So here n = 2 and k = 2.
The equation $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ is square in that m = n = 2. the purpose of this example is merely to compute the quantity $\varphi^*(dy_1 dy_2)$ since he has not done that yet. We have

$y_1(\mathbf{x}) = x_1{}^3 x_2$
$y_2(\mathbf{x}) = \log(x_1 + x_2)$

You then just use the chain rule to find that

$\varphi^*(dy_1 dy_2) = d\varphi_1 d\varphi_2 = F(x_1,x_2)\, dx_1 dx_2$ as shown p 38 A

In general terms we would get

$d\varphi_1 = (\partial\varphi_1/\partial x_i) dx_i$
$d\varphi_2 = (\partial\varphi_2/\partial x_j) dx_j$

$d\varphi_1 d\varphi_2 = \Sigma_{ij}\, [\, (\partial\varphi_1/\partial x_i)(\partial\varphi_2/\partial x_j)\,]\, dx_i dx_j = \Sigma_{ij}\, [\, (\partial_i\varphi_1)(\partial_j\varphi_2)\,]\, dx_i dx_j$

Now while we are here, consider

$\Sigma_{ij}\, f_{ij}\, dx_i dx_j = \Sigma_{i<j}\, f_{ij}\, dx_i dx_j + \Sigma_{j<i}\, f_{ij}\, dx_i dx_j$

$= \Sigma_{i<j}\, f_{ij}\, dx_i dx_j + \Sigma_{i<j}\, f_{ji}\, dx_j dx_i = \Sigma_{i<j}\, (\, f_{ij} - f_{ji})\, dx_i dx_j$

So in our example we get

$$d\varphi_1 d\varphi_2 = \Sigma_{ij}\,[\,(\partial_i\varphi_1)(\partial_j\varphi_2)\,]\,dx_i dx_j = \Sigma_{i<j}\,[\,(\partial_i\varphi_1)(\partial_j\varphi_2) - (\partial_j\varphi_1)(\partial_i\varphi_2)]\,dx_i dx_j$$

and now you see a Jacobian-like object appearing. More generally

$$d\varphi_a d\varphi_b = \Sigma_{i<j}\,[\,(\partial_i\varphi_a)(\partial_j\varphi_b) - (\partial_j\varphi_a)(\partial_i\varphi_b)]\,dx_i dx_j = \Sigma_{i<j}\,\frac{\partial(\varphi_a,\varphi_b)}{\partial(x_i,y_j)}\,dx_i dx_j$$

Even if $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ is not "square", the above line makes sense. There may be no overall Jacobian due to non-square, but this little 2x2 Jacobian chunk will always exist.

Proposition 3.10

Here we have three "rules" for the φ* operator called (i), (ii) and (iii).

(i) linear: $\varphi^*(a\alpha+b\beta) = a\varphi^*\alpha + b\varphi^*\beta$

My proof:

show that $(a\alpha+b\beta)\circ\varphi = a(\alpha\circ\varphi) + b(\beta\circ\varphi)$

or

show that $(a\alpha+b\beta)(\varphi(\mathbf{x})) = a\alpha(\varphi(\mathbf{x})) + b\beta(\varphi(\mathbf{x}))$

Well just write φ(x) as y and then

show that $(a\alpha+b\beta)(\mathbf{y}) = a\alpha(\mathbf{y}) + b\beta(\mathbf{y})$

Sja never really talked about "adding two forms" so maybe I will address that right here:

adding two forms (must be of same degree)

$\alpha + \beta = \Sigma_I f_I dx_I + \Sigma_I g_I dx_I = \Sigma_I (f_I + g_I) dx_I = \Sigma_I h_I dx_I = \gamma$ (some new form)
$c\alpha = c\,\Sigma_I f_I dx_I = \Sigma_I (c f_I) dx_I = \delta$ (some new form)

So given this, there is nothing more to show.

(ii) multiplicative: $\varphi^*(\alpha\beta) = (\varphi^*\alpha)(\varphi^*\beta)$

Proof: show $(\alpha\beta)(\varphi(\mathbf{x})) = \alpha(\varphi(\mathbf{x}))\,\beta(\varphi(\mathbf{x}))$

show $(\alpha\beta)(\mathbf{y}) = \alpha(\mathbf{y})\,\beta(\mathbf{y})$

The thing on the left is an "exterior product" as defined on page 19 B. so

$(\alpha\beta) = \Sigma_{I,J} f_I\, g_J\, dy_I\, dy_J$ = obvious

(ii) natural: $\varphi^*(\psi^*\alpha) = (\psi \circ \varphi)^*\alpha$

Proof: LHS = $\varphi^*(\psi^*\alpha) = \varphi^*(\alpha \circ \psi) = (\alpha \circ \psi) \circ \varphi$

RHS = $(\psi \circ \varphi)^*\alpha = \alpha \circ (\psi \circ \varphi)$

But I am sure the circle operator is associative. Wiki

> The composition of functions is always associative—a property inherited from the composition of relations.[1] That is, if f, g, and h are three functions with suitably chosen domains and codomains, then $f \circ (g \circ h) = (f \circ g) \circ h$, where the parentheses serve to indicate that composition is to be performed first for the parenthesized functions. Since there is no distinction between the choices of placement of parentheses, they may be left off without causing any ambiguity.

BUT, this is for functions, and we need it for form α with ψ and φ being functions. Then

LHS = $\varphi^*(\psi^*\alpha) = \varphi^*(\psi^*\Sigma_I f_I\, dy_I) = \varphi^*\,[\,\Sigma_I(\psi^*f_I)\,(\psi^*dy_I)]$

$= \varphi^*\,[\,\Sigma_I(f_I(\psi(\mathbf{x}))d\psi_I] = \Sigma_I(f_I(\varphi(\psi(\mathbf{x})))\,\varphi^*(d\psi_I)$

So you can see that some WORK is needed in this proof, and so I just go with his work as shown. I am not sure about that (B.6) chain rule. Maybe look at that now:

(B.6) Chain Rule: $\dfrac{\partial(\varphi^*\psi_i)}{\partial x_i} = \Sigma_j\, \varphi^*\left(\dfrac{\partial\psi_i}{\partial y_j}\right)\dfrac{\partial\varphi_j}{\partial x_i}$

Show that

$$\frac{\partial\psi_i(\varphi(x))}{\partial x_i} = \Sigma_j\, \varphi^*\left(\frac{\partial\psi_i}{\partial y_j}\right)\frac{\partial\varphi_j}{\partial x_i} = \Sigma_j \frac{\partial\psi_i}{\partial\varphi_j}\frac{\partial\varphi_j}{\partial x_i} \qquad \text{// seems OK}$$

I have scribbled on the left facing page the last steps of his proof, let's not dwell on it now.

Theorem 3.11. $\varphi^*d = d\varphi^*$ these operators commute.

I have read the proof twice, and it uses the same (B.6) above. How about a naive proof

$\varphi^*(d\alpha) = \varphi^*\,[\Sigma_I\,(df_I)\,dy_i] = [\Sigma_I\,\varphi^*(df_I)\,\varphi^*(dy_i)\,] = \Sigma_I\,\varphi^*(df_I)\,d\varphi_i$

$d(\varphi^*\alpha) = d\,[\Sigma_I\,(\varphi^*f_I)\,\varphi^*(dy_i)] = d\,[\Sigma_I\,(\varphi^*f_I)d\varphi_i\,] = \Sigma_I\,d(\varphi^*f_I)d\varphi_i$

At this point we must use the rule for functions, then continue

$= \Sigma_I\,\varphi^*d(f_I)d\varphi_i$ = QED

His proof of the "rule for functions" is straightforward, using his B.6. But I will try a naive on that now:

$$\varphi^*(df) = (df)(\varphi(x)) = [\Sigma_j\ (\partial f/\partial y_j)dy_j\ ](\varphi(x)) = \Sigma_j\ (\partial f/\partial y_j)d\varphi_j$$

$$d(\varphi^* f) = d(f(\varphi(x)) = \Sigma_i\ (\partial/\partial x_i)\ f(\varphi(x))\ dx_i = \Sigma_{ij}(\partial f/\partial y_j)(\partial \varphi_i/\partial x_i)\ dx_i = \Sigma_j(\partial f/\partial y_j)\ d\varphi_j$$

OK, good enough.

[40] The Determinant Connection.

I am happy here down through line E which I will rewrite this way

$$d\varphi_a d\varphi_b....d\varphi_k = \Sigma_{ABC}....\ R_{aA}R_{bB}....R_{kK}\ dx_A dx_B.....dx_K$$

k factors k factors

Now how would I convert this to a determinant? The sums on A,B,C are full range 1 to n. The only contributions in the sum are when all the dx_i are different!

Let's try a simple example first

$$d\varphi_a d\varphi_b = \Sigma_{AB}R_{aA}R_{bB}dx_A dx_B = \Sigma_{A<B}\ (R_{aA}R_{bB} - R_{aB}R_{bA})dx_A dx_B$$

You MUST get the thing into an ordered index sum in order to reveal the determinant! Let's try it for the case k = 3:

$$d\varphi_a d\varphi_b d\varphi_c = \Sigma_{ABC}R_{aA}R_{bB}R_{cC}dx_A dx_B dx_C$$

$$= \Sigma_{A<B<C}[R_{aA}R_{bB}R_{cC} + 5 \text{ signed permutations}]\ dx_A dx_B dx_C$$

So here is the general question: When you write the thing as an *ordered* sum A<B<C...<K, what exactly is the factor which multiplies $dx_A dx_B.....dx_K$? The claim is that this factor is a determinant.

OK, consider this even more general situation

$$\Sigma_{ABC}\ T_{ABC} = \Sigma_{A<B<C}\ T_{ABC} + \Sigma_P\ \ \Sigma_{P(A<B<C)}\ T_{ABC}$$

where T_{ABC} is assumed to vanish when any two indices are the same. What the above says is that you can break up the original triple sum into 6 different summation sectors. The first has A<B<C. The remaining sums have P(A<B<C) ≡ A'<B'<C' where (A',B',C') = P(A,B,C) and we sum over all remaining permutations. We might as well include the first term and write

$$\Sigma_{ABC}\ T_{ABC} = \Sigma_P\ \Sigma_{P(A<B<C)}\ T_{ABC}$$

where we now include the identity permutation. The indices on T_{ABC} are always the same, they do not feel the permutation. Now if (A'B'C') = P(ABC), then (ABC) = P^{-1}(A'B'C'). But stop, let's try to get this into the language of my (Lagrange doc) Appendix B which has some very powerful tools!

$$\mathbf{A} \equiv \begin{pmatrix} A \\ B \\ ... \\ K \end{pmatrix}, \text{ a vector}$$

$$\mathbf{A'} = P\mathbf{A} \qquad \text{for example } \mathbf{A'} = \begin{pmatrix} B \\ A \\ ... \\ K \end{pmatrix} = P \begin{pmatrix} A \\ B \\ ... \\ K \end{pmatrix} = P\mathbf{A}$$

I am changing notation relative to App B to be closer to what fits this current need. Now **A** and **A'** are vectors, whereas P is a permutation matrix.

The set of all matrices P which does all shuffles forms a group (permutation group of k objects). Now consider:

$$\Sigma_{ABC}\, T_{ABC} = \Sigma_P\ \Sigma_{P(A<B<C)}\ T_{ABC} = \Sigma_P\ \Sigma_{A'<B'<C'}\ T_{ABC}$$

Now for each term in the sum, we want to rename variables so that

$$(A'B'C') \rightarrow (ABC)$$

I *think* (will show later) that this means

$$\Sigma_P\ \Sigma_{A'<B'<C'}\ T_{ABC} = \Sigma_{A<B<C}\, \Sigma_P T_{P(ABC)}$$

For the moment, assume this is correct and let's see where it leads when

$$T_{ABC} = R_{aA}R_{bB}R_{cC}dx_A dx_B dx_C$$

$$T_{P(ABC)} = P_2(R_{aA}R_{bB}R_{cC})\, P(dx_A dx_B dx_C) = P_2(R_{aA}R_{bB}R_{cC})\, (-1)^S\, dx_A dx_B dx_C$$

where S is the number of swaps need to rearrange P(ABC) back to order (ABC). We then have

$$\Sigma_{ABC}\, T_{ABC} = \Sigma_{A<B<C}\, \Sigma_P T_{P(ABC)} = \Sigma_{A<B<C}\, [\, \Sigma_P\, (-1)^S\, P_2(R_{aA}R_{bB}R_{cC})]\ dx_A dx_B dx_C$$

$$= \Sigma_{A<B<C}\ \det(R_{abc,ABC})\, dx_A dx_B dx_C = d\varphi_a d\varphi_b d\varphi_c$$

where det(R) in this case is the determinant of a 3x3 submatrix of the full R matrix. The row labels are given by a,b,c while the column labels are given by A,B,C. Each ordered term in the sum thus involves different columns A,B,C of the full R matrix (but the same rows). Now using ordered notation, you might say,

$$d\varphi_a d\varphi_b d\varphi_c = d\varphi_I = \Sigma_{A<B<C}\ \det(R_{abc,ABC})\, dx_A dx_B dx_C = \Sigma_J \det(R_{I,J})\, dx_J$$

and this explains the notation appearing in Sja p 41 (3.3).

Now my R matrix elements are

$R_{aA} = \partial\varphi_a/\partial x_A = \partial_A\varphi_a \equiv (D\varphi)_{A,a}$

$R_{aA}R_{bB} = (D\varphi)_{A,a}(D\varphi)_{B,b}$

If we define an ordered multi index I = AB and J = ab we might abbreviate

$R_{I,J} = (D\varphi)_{I,J}$

So then my result above is

$d\varphi_I = \Sigma_J \det(R_{I,J})\, dx_J = \Sigma_J \det[(D\varphi)_{I,J}]\, dx_J$ // matches (3.3)

where everything is ordered!!

What happens if k = n, the max value? Then I think the entire R matrix is involved, and we get

$d\varphi_1 d\varphi_2 \ldots . d\varphi_n = dV_\varphi = \det(R)\, dx_1 dx_2 ... dx_n = \det(D\varphi)\, dx_1 dx_2 ... dx_n$

This is because in the sum $\Sigma_{A<B<C}$ there is only one term plus those permutations, so Σ_J goes away. The only ordered term is 1<2<3..<n .

Comments:

1. This entire discussion involves *forms* like $d\varphi_a d\varphi_b d\varphi_c$ and $dx_A dx_B dx_C$. These are not the same as products of differentials. One example is that there is no sign swap rule for a product of differentials.

2. The maximal case (only) does seem to align with the usual integration Jacobian rule. The author first addresses integration in the next chapter. Right now we have no idea how to do integration.

3. In general, the det is only of a square portion of the full R matrix.

4. These smaller determinants only show up if you use ordered summation, or (which is the same thing) ordered multi-indices.

5. The reason the author had that long opening section on determinants was to get these little form differentials into a determinant form!

Note that in φ^* the asterisk is a superscript. The Hodge operator is a main line * character.
Seems unfortunate to use the same symbol for both.

Chapter 4: Integration of 1-forms

4.1 Definition of the integral and its elementary properties [47]

[47] We now have to make a translation of Chapter 3 concepts to this particular application:

Chapter 3	Chapter 4	
x_i in x-space = R^n and U (p37)	t in t-space = R^1 and [a,b] (p47)	left
y_i in y-space = R^m and V (p37)	$\mathbf{x}$ in R^m and U	right
$\mathbf{y} = \boldsymbol{\varphi}(x)$	$\mathbf{x} = \mathbf{c}(t)$	
α	α	
$\varphi^*\alpha$	$c^*\alpha$	

The idea is that "the space on the right" holds the trace of a curve $\mathbf{c}(t)$ in R^m while "the space on the left" is just R^1 with t in [a,b]. This is of course the parameter of the curve in Buck language.

Now how do our big gun results apply here?

$\alpha = \Sigma_i f_i(\mathbf{x})dx_i$ = 1-form associated with space on the right

$c^*\alpha = \Sigma_i f_i(\mathbf{c}(t))dc_i = \Sigma_i f_i(\mathbf{c}(t))(\partial_t c_i)\, dt \equiv g(t)\, dt$ // page 42 A $dx_J \rightarrow dt$

where $g(t) \equiv \Sigma_i f_i(\mathbf{c}(t))(\partial_t c_i)$

This is the pullback, now associated with the space on the left which is just t-space.

Now Sja *defines* the integral of a 1-form in this manner

$$\int_c \alpha \equiv \int_{[a,b]} c^*\alpha = \int_a^b g(t)\, dt = \int_a^b \Sigma_i f_i(\mathbf{c}(t))(\partial_t c_i)\, dt$$

In other words, the integral over the curve c in R^m of the form α is defined as the integral of the pullback $c^*\alpha$ in R^1, the space on the left. This certainly is an integral we know how to do. [Bucks make this same definition in their book, see p 376 (7-7); of course notation is different.]

Example 4.1. (the angle form). Here we have

$\alpha = \Sigma_i f_i(\mathbf{x})dx_i = f_1(\mathbf{x})\, dx + f_2(\mathbf{y})\, dy = -(y/r^2)dx + (x/r^2)dy$ $r^2 \equiv x^2 + y^2$

and

$\mathbf{x} = \mathbf{c}(t)$: x = cost [a,b] = [0,2π] curve is circle radius 1 in $R^m = R^2$

y = sint r = 1

If t is time, then point (x,y) moves around the unit circle with ω = 1 radians/sec.

$g(t) = \Sigma_i f_i(\mathbf{c}(t))(\partial_t c_i) = f_1(\text{cost,sint})\partial_t(\text{cost}) + f_2(\text{cost,sint})\partial_t(\text{sint})$

$= -(y/r^2)\,[-\text{sint}] + (x/r^2)[\text{cost}] = \text{ysint} + \text{xcost} = \sin^2 t + \cos^2 t = 1$

$c^*\alpha = g(t)\, dt = dt$ //extremely simple example!

Then

$$\int_{\mathbf{c}} \alpha \;=\; \int_{\mathbf{a}}^{\mathbf{b}} g(t)\,dt \;=\; \int_{\mathbf{0}}^{\mathbf{2\pi}} 1\,dt \;=2\pi$$

We might interpret this as the "length" of the circumference of the unit circle, but Sja says nothing yet about such length.

Respeeding a Curve. [bottom p 47]

We had above that $\mathbf{x} = \mathbf{c}(t)$ and now we reparametrize by saying $t = p(s)$ so

$\mathbf{x} = \mathbf{c}(t) = \mathbf{c}(p(s) = (\mathbf{c} \circ p)(s)$ interval is $[\bar{a},\bar{b}]$ in s-space

Require $p'(s) \neq 0$ on this interval so the motion does not reverse itself! Compare:

$\mathbf{c}$: [a,b] in R^1 t-space$\rightarrow$ curve $\mathbf{c}$ in U within R^m

$(\mathbf{c} \circ p)$: $[\bar{a},\bar{b}]$ in R^1 s-space $\rightarrow$ curve $\mathbf{c}$ in U within R^m

If $p'(s) > 0$ then orientation of the curve is preserved, otherwise it is reversed.

Example 4.2. Let $t = 2s$ in previous example so then

$\mathbf{x} = \mathbf{c}(2s)$:	$x = \cos(2s)$	$[a,b] = [0,\pi]$	curve is circle radius 1 in $R^m = R^2$
	$y = \sin(2s)$	$r = 1$	

If s is the new time, then point rotates around circle at $\omega = 2$ radians/sec and one loop uses $s = [0,\pi]$.

Suppose $t = 2\pi s^2$ so s runs on [0,1]. Then $x = \cos(2\pi s^2) = \cos(\omega s)$ so $\omega = 2\pi s$. Then the point has an angular velocity which starts at 0 and linearly increases to 2π at the end point.

Theorem 4.3. Apart from a possible orientation minus sign, reparametrization does not alter the integral of a 1 form over a curve!

Proof:

$$\int_{\mathbf{c}} \alpha \;\equiv\; \int_{[a,b]} c^*\alpha$$

$$\int_{\mathbf{c\,o\,p}} \alpha \;\equiv\; \int_{[\bar{a},\bar{b}]} (c \circ p)^* \alpha$$

Now we get to use our earlier painful "naturality" theorem from p 38 which says [$t = p(s)$]

$(c \circ p)^* \alpha = p^*(c^*\alpha) = p^*(g(t)dt) = g(p(s))\, dp$ // recall $f_I(\varphi(\mathbf{x}))d\varphi_I$

$= (p^*g)\, dp$ // really $(p^*g(t)) = g(p(s))$

$= (p^*g)(\partial p/\partial s)\, ds$

Then we have

$$\int_{c \circ p} \alpha \equiv \int_{[\bar{a},\bar{b}]} (c \circ p)^* \alpha = \int_{[\bar{a},\bar{b}]} (p^*g)(\partial p/\partial s)\, ds = \int_{[\bar{a},\bar{b}]} g(p(s)(\partial p/\partial s)\, ds$$

Recall that

$$\int_c \alpha \equiv \int_{[a,b]} c^*\alpha = \int_a^b g(t)\, dt$$

If we now make a normal calculus substitution t = p(s) where dt = dp = (dp/ds)ds we get

$$\int_c \alpha = \int_{p^{-1}(a)}^{p^{-1}(b)} g(p(s))\, |\, (dp/ds)\, |ds = \text{sign}(dp/ds) \int_{[\bar{a},\bar{b}]} g(p(s)(\partial p/\partial s)\, ds$$

Thus we have shown that

$$\int_{c \circ p} \alpha = \text{sign}(p'(s)) \int_c \alpha$$

Sub-example. Consider

$$\int_0^1 t\, dt = (1/2)$$

$t = \pm 2s$

$dt = \pm 2ds$ *but* $dt = J\, ds$ for integration $J = |\pm 2| = +2$

$$\int_0^1 t\, dt = \int_0^{\pm 2} [\pm 2s]\, [+2ds] = \pm 4 \int_0^{\pm 2} s\, ds \ .$$

There are two cases to consider:

+ sign: $= +4 \int_0^2 s\, ds = 4 * (1/2)2^2 = 8$

- sign: $= -4 [\int_0^{-2} s\, ds] = -4[-\int_{-2}^0 s\, ds] = 4\,(1/2)\,(0 - (-2)^2) = -8$

So the key fact is that the Jacobian for integration has the |....| sitting there. In this example, the integral over the right triangle is positive because s is positive in the entire range. The integral over the left triangle is the sum of rectangle positive widths times the function which is negative there, so the triangle on the left integrates to -8.

So reparametrization does not alter the integral of a 1-form over a curve, apart from minus sign if you have reversal. A fact familiar to me from Buck. [And on 2.12.16 I firmed that idea up by adding notes going through the above proof without the need for the formal "naturality" rule.]

PL Corollary: If you are asked to compute the 1-form integral

$$\int_{c\,o\,p} \alpha \;=\; \int_{[\bar{a},\bar{b}]} (c \circ p)^{*} \alpha \;=\; \int_{[\bar{a},\bar{b}]} g(p(s))\,(\partial p/\partial s)\,ds$$

you might save time by instead computing the exact same number using

$$\int_{c} \alpha \;\equiv\; \int_{[a,b]} c^{*}\alpha \;=\; \int_{a}^{b} g(t)\,dt \qquad \text{where} \quad g(t) \equiv \; \Sigma_i f_i(\mathbf{c}(t))(\partial_t c_i)$$

where t runs through [a..b] and s runs through $[\bar{a}...\bar{b}]$. We have s = p(t) so $\bar{a}$= p(a) and $\bar{b}$ = p(b).

Interpretation of $\int_c \alpha$. Recall $\alpha = \mathbf{F}\bullet d\mathbf{x}$ is associated (one to one!) with vector **F**. If d**x** is a physical distance in physical space, and if **F** is the force on a particle, then α is the work done on that particle as it moves d**x**. For example if **F** and d**x** are aligned, you are doing positive work on the particle, you are accelerating it or maybe increasing its potential energy. So our integral is then specifically "the total work done on the particle" as it traverses the curve in the presence of this force field **F** .

[49] Interesting conclusion: If particle doubles its speed along the path, the field still does exactly the same work on the particle. This is true for any variation of the speed during motion, as long as it does not go backwards.

Suppose **F** is such that you can write $\mathbf{F} = -\nabla\varphi$ where φ is then a potential.

$$\alpha = \mathbf{F} \bullet d\mathbf{x} \qquad d\alpha = (d\mathbf{F}) \bullet d\mathbf{x} \;=\; dF_i\, dx_i \;= (\partial_j F_i) dx_j dx_i = -\,\partial_{ij}\varphi\, dx_j dx_i = 0$$

So in this case dα = 0 so α is **closed**. Moreover, $d\varphi = (\partial_i\varphi)\, dx_i = -\,F_i dx_i = -\,\mathbf{F}\bullet d\mathbf{x} = -\alpha$, so we find that in this case α = -dφ, so α is also **exact**. Consistent with "every exact form is closed".

I now repeat the above paragraph with the other sign:

Suppose **F** is such that you can write $\mathbf{F} = \nabla\varphi$ where φ is then a potential.

$$\alpha = \mathbf{F} \bullet d\mathbf{x} \qquad d\alpha = (d\mathbf{F}) \bullet d\mathbf{x} \;=\; dF_i\, dx_i \;= (\partial_j F_i) dx_j dx_i = \partial_{ij}\varphi\, dx_j dx_i = 0$$

So in this case $d\alpha = 0$ so α is **closed**. Moreover, $d\varphi = (\partial_i \varphi)\, dx_i = F_i dx_i = \mathbf{F} \bullet d\mathbf{x} = \alpha$, so we find that in this case $\alpha = d\varphi$, so α is also **exact**. Consistent with "every exact form is closed".

4.2 Integration of exact 1-forms [49]

Theorem 4.4. Suppose $\alpha = dg$ so α is exact. Then consider

$c^*\alpha = c^*(dg) = d(c^*g)$ // from earlier theorem

Now g must be a 0-form since $\alpha = dg$ is assumed to be a 1-form, so g is then a function. But what is it a function of? Well, α and g are both associated with the space of the curve which is R^m. If g is a 0-form on this space, then it is $g(\mathbf{x})$. But then $c^*g = g(\mathbf{c}(t)) = (g \circ \mathbf{c})(t) = (\mathbf{c}^*g)(t)$, lots of ways to write it. So yes, I agree that you could define $h(t) \equiv (c^*g)(t)$. Then $dh = d(c^*g) = c^*\alpha$ from above. Then we have

$$\int_c \alpha \equiv \int_{[a,b]} c^*\alpha = \int_{[a,b]} d(c^*g) = \int_{[a,b]} dh = h(b) - h(a) = g(\mathbf{c}(b)) - g(\mathbf{c}(a)) \qquad \text{QED}$$

Obviously if c(a) = c(b) which would be the case for a closed curve, we get $\int_c \alpha = 0$, So we have shown that the integral of an exact 1-form around a closed curve vanishes. This all applies when $\alpha = dg$ is exact, and that in turn applies when $\mathbf{F} = -\nabla\varphi$ and then the work done going around closed path is 0 if the force can be written in this manner (a **conservative force**).

Note added: In my integral theorems doc I have this theorem,

$$\int_c \alpha = \int_A^B \nabla\varphi \bullet d\boldsymbol{\ell} = \varphi(B) - \varphi(A) \qquad (9) \qquad \text{MM 4-36 p 156}$$

This is the case just described where $\mathbf{F} = +\nabla\varphi$ so we identify $\alpha = d\varphi$ which is then exact.

Theorem 4.5. Three ways to say the same thing, with fairly long proofs.

(i) α is exact, meaning there exists g such that $\alpha = dg$

(ii) $\int_c \alpha = 0$ for any closed curve c

(iii) $\int_c \alpha$ depends only on the endpoints of the curve c.

I think I could roll my own proofs, but I read through the first two proofs, both simple. The third has many steps. Part of the third proof is this idea: Let $\mathbf{c_x}$ be any curve from $\mathbf{x_o}$ to $\mathbf{x}$ and define

$g(\mathbf{x}) = \int_{\mathbf{cx}} \alpha$ // integral of a 1-form is a 0-form or function.

This turns out to be an appropriate g such that $dg = \alpha$ and, if shown true, this concludes the proof that (iii)$\Rightarrow$(i). Naively that certain seems reasonable. You have constructed an explicit g(x) that works! As I

show elsewhere, what you need to prove is that $\partial_i g = f_i$ where $\alpha = \Sigma_i f_i(x)\, dx_i$. Sja gives a very long proof that $\partial_i g = f_i$ and therefore $dg = \alpha$. I skip the details shown by my pencil mark on pages 50,51.

Example 4.6. We have shown that for $\alpha = -(y/r)dx + (x/r)dy$ that $d\alpha = 0$ so α is closed. If it were also exact, we could write $\alpha = dg$ for some function g. In that case, $\int_c \alpha = \int_c dg = g(a) - g(b)$ and then we would have that $\int_c \alpha = 0$ for any closed path. But by direct calculation we found that this integral = 2π when path c is a circle (a closed path), so it must be that α is *not* exact, even though it is closed.

Exercise 2.6.

Theorem: Any closed 1-form on (all of) R^n is exact. Given α being a closed 1-form (meaning $d\alpha = 0$), you want to show that $\alpha = dg$ where then g would be a 0-form, that is to say, g is a function. This is demonstrated on page 27 Exercise 2.6 which I am now reading. You assume that $\alpha = \Sigma_i f_i dx_i$ and g is expressed in terms of the f_i in this fancy way: [proof by construction of the solution]

$$g(\mathbf{x}) = \int_0^{x_1} f_1(t, x_2, x_3, \ldots, x_n)\, dt + \int_0^{x_2} f_2(0, t, x_3, x_4, \ldots, x_n)\, dt + \int_0^{x_3} f_3(0, 0, t, x_4, x_5, \ldots, x_n)\, dt + \cdots + \int_0^{x_n} f_n(0, 0, \ldots, 0, t)\, dt.$$

The proof that $\alpha = dg$ requires showing that $\Sigma_i f_i dx_i = dg = \Sigma_i \partial_i g\, dx_i$ which is to say $f_i = \partial_i g$. Note that

$\partial_1 g(\mathbf{x}) = f_1$ (from first term) + no other terms since none contain x_1

Next want to show that $\partial_2 g(\mathbf{x}) = f_2$. First, look at the contribution to $\partial_2 g$ from the first term in the above expansion of g(x),

$\partial_2[\int_0^{x_1} f_1(t,x_2,x_3....)\, dt = \int_0^{x_1} \partial_2[f_1(t,x_2,x_3....)]$ // assuming this is allowed

Now pause to look at $d\alpha = 0$ which we know is true (closed). This says $d[\Sigma_i f_i dx_i] = 0$ or $\Sigma_i (df_i) dx_i = 0$. Write this as

$\Sigma_{ij} (\partial_j f_i) dx_j dx_i = 0$ or $\Sigma_{i<j} [(\partial_j f_i) - (\partial_i f_j)]\, dx_j dx_i = 0.$ // as in p 20 (2.2)

I suppose this says that

$\partial_j f_i = \partial_i f_j$ and in particular $\partial_2 f_1 = \partial_1 f_2$

Thus my first term contribution above can also be written as

$$\int_0^{x_1} \partial_1[f_2(t,x_2,x_3....)]\, dt \;=\; [\, f_2(x_1,x_2,x_3.....) - \; f_2(0,x_2,x_3.....)]$$

Therefore, including the contribution of the second term I am getting

$$\partial_2 g(x) = [f_2(x_1,x_2,x_3.....) - \; f_2(0,x_2,x_3.....)] \; + f_2(0,x_2,x_3....) \; = f_2(x_1,x_2,x_3.....)$$

So OK, I accept that this tricky sum of integrals shown in Ex 2.6 does do $\partial_i g = f_i$ and therefore we have shown that dg = α.

Now where does the requirement that α be defined on all of R^n come in ? Well, in order to say $\partial_j f_i(\mathbf{x}) = \partial_i f_j(\mathbf{x})$ [α is closed at **x**] ... Suppose this is only true for **x** in some *subset* of R^n. If that subset has a closed boundary somewhere (or is fully closed), then $\partial_j f_i(\mathbf{x})$ is not going to be defined at the boundary so the proof fails at any point on a boundary.

More specifically: It seems that we are requiring $\partial_j f_i(\mathbf{x}) = \partial_i f_j(\mathbf{x})$ to be true in the following hypercube: $[0,x_1] \times [0,x_2] \times ....x\ [0,x_n]$ and this includes the origin point! Our thing has to also be true at the origin.

But couldn't you fix up this problem by changing the 0 lower endpoint to .01, say? I would have to change all 0's to .01's in the various integrals and their endpoints. But then all x_i have to be > .01 for all the integrals to make sense. Then g(**x**) is only defined for $x_i > .01$ so g(**x**) not defined at the origin.

But the form as shown is not defined for $x_i < 0$? But I suppose each integral is in fact defined. So I guess I am hazy on this theorem's proof?

Again, assume expression shown is valid for all x_i. Then we have g(**x**) for all **x**. But now back to the issue that $\partial_j f_i(\mathbf{x}) = \partial_i f_j(\mathbf{x})$ has then to be true at the origin if things are OK for all R^n. But as we shall see, the angle form α is not OK at the origin because the f_i blow up there and so do $\partial_j f_i$.

So the punctured plane is missing the origin, so g(**x**) won't be defined at the origin.

Again, suppose in E^2 we try to restrict to U = a square centered on the origin, upper right corner of the square at (x,y) = (a,a). Then we just restrict $|x_i| < a$, say, so we have an open square. I think for this kind of special subset of R^2, we might be OK in Ex 2.6. For every point in U we have $\partial_j f_i(\mathbf{x}) = \partial_i f_j(\mathbf{x})$, and for every point we presume that f_i is finite (continuous). Then excluding the origin means g(x) not defined at the origin, and then you can't really say that α = dg in the punctured square since g does not exist at the origin.

So, although the angle form is a 1-form, since it is only defined in the plane punctured at the origin and not over all R^2 or a square centered at the origin, we conclude that the proof that α = dg fails, and it is likely that the angle form is NOT exact. We know it is not exact since the integral around a closed curve (circle) gives 2π. So I see why the Exercise 2.6 proof fails for this case.

PL Theorem: Any 1-form defined on an open cube centered at the origin is exact within that cube.

2.13.16. The above notes are pretty good on this Ex 2.6 thing. My trademark "splainin'".

Pause to do some comparisons between Sjamaar and Buck.

I added some notes in Buck Ch 7 and in doing so finally learned that Buck's ω* is the pullback written without showing the name of the function φ.

Sjamaar	Buck	
α	ω	a generic differential form
c	γ	a curve
$\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$	$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{u})$	a transformation
φ*α	ω*	a pullback

Just one other item, then I will end this pause:

$$\int_{\gamma} \omega = \int_{\gamma} [\, A(\mathbf{r})dx + B(\mathbf{r})dy \,] = \int_{t_1}^{t_2} [\, A(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{x}}(t) + B(\boldsymbol{\gamma}(t))\gamma'_{\mathbf{y}}(t)] \; dt \quad \text{// Buck}$$

$$\int_{\mathbf{c}} \alpha \equiv \int_{[a,b]} c^*\alpha = \int_{a}^{b} g(t)\, dt = \int_{a}^{b} \Sigma_i f_i(\mathbf{c}(t))(\partial_t c_i)\, dt \quad \text{// Sjamaar}$$

4.3 The global angle function and the winding number [51]

Now back to bottom of page 51. We find that for the **angle form**, dα = dθ where we regard this as an angle in physical space. Not only do we have a problem at the origin, we also have a problem where θ jumps by 2π. If we define θ = 0 on the positive real axis in the usual manner, there is a discontinuity at 2π. We really needs some Riemann sheets or something to get winding numbers and understand this disc.

[52-53] I think what Sjamaar is doing here is forming a curve which wraps around the origin a few times in terms of pieces which do not do so. For each piece we don't have a problem, then we just take the final curve to be the union of all those pieces. [similar I think to the manifold concept]

Well look at p 52 A where he defines

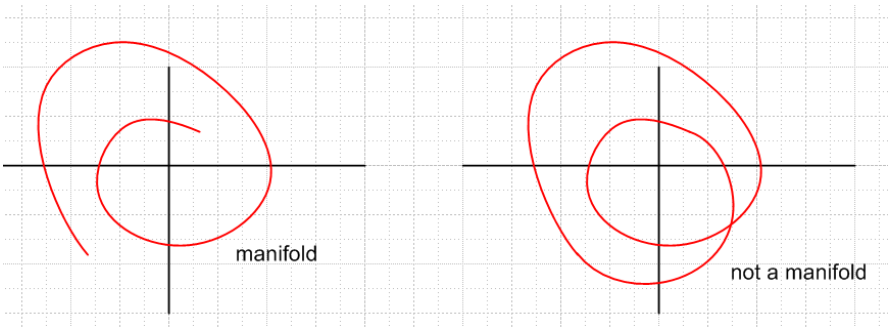
$$\theta(t) = \theta(0) + \int_{0}^{t} c^*\alpha \; = \text{the pull back of } \alpha \text{ relative to c'}$$

The mapping starts out on t in [0,2π) say, but then we just extend it to [0, 42π) or whatever. The pullback integral is perfectly well defined for any range of t. For t in (0,2π) we have $\theta = \theta$ but then we have provided an extension of θ beyond this range by means of θ.

Note that if you draw a vector from origin to point **c** on the curve **c**(t), the direction of that vector is given by (x,y) and if you make that a unit vector, by (ξ,η) on page 51. So as you move on the curve, we can regard this little unit vector $\hat{\mathbf{d}} = (\xi,\eta)$ as a "dial" which measures the angle θ . The curve on page 53 could wind around the origin 50 times, and that dial would go around 50 or so times. The pictures are quite clear on this issue.

Corollary: If curve is closed after winding around the origin k times, then $\Delta\theta = k(2\pi)$ and we can call k the **winding number** of the closed curve c. Recall that such a curve is not a manifold because it self-

intersects. The curve on the left below exceeds 2π but is still a manifold and can be mapped with an atlas, but the curve on the right with winding number 2 intersects itself so is not a manifold.



OK, I think I am now happy with Chapter 4. I guess I should update the meta notes for Ch 4.

Chapter 5: Integration and Stokes' theorem

I argue for Stokes's theorem but both forms are OK. Sja never mentions Gauss. [well he says Gauβ]

5.1 Integration of forms over chains

In the pullback chapter 3 we had $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ as the transformation going from left to right.
In the 1-form int chapter 4 we had $\mathbf{x} = \mathbf{c}(t)$ as the transformation going from left to right.

In this Stokes's chapter 5 we have $\mathbf{x} = \mathbf{c}(\mathbf{t})$ with $\mathbf{t}$ a vector from left to right. Although for a k-form we are not dealing with a curve, the letter $\mathbf{c}$ is still used. Bucks used $\mathbf{x} = \boldsymbol{\Sigma}(u,v)$ for their 2-form surface work. But Sja is going to keep it fully general in this chapter for k-forms, with examples of various k. In other words, Sjamaar beefs up his Ch 4 notation for 1-forms to apply to k-forms. A bit odd that he did it this way.

So as an example, consider $\mathbf{x} = \mathbf{c}(t_1,t_2)$. This is a surface say in R^3 and there is no reason such a surface cannot penetrate itself, as shown Buck p 331.

In the curve chapter, our left-space domain was [a,b] in R^1. For a k-form, this left-space domain becomes a "rectangular block R" as described in p 57 A and shown there in direct product notation as well. No problem for me.

Now let's translate the pullback from Chapter 3. One ingredient there was this (from meta notes)

$$d\varphi_a d\varphi_b d\varphi_c \ldots d\varphi_q = \Sigma_{A,B,C\ldots Q}\, R_{abc\ldots q,ABC\ldots Q}\, dx_A dx_B dx_C \ldots dx_Q \qquad \text{// non-ordered sum}$$
(k factors) (k factors)

$$d\varphi_I = \Sigma_J \det(R_{I,J})\, dx_J = \Sigma_J \left[\det[(D\varphi)_{I,J}]\, dx_J \right. \qquad \text{// ordered sum on J, } \Sigma_{A<B<C\ldots<Q}$$

$$\alpha = \Sigma_I f_I(\mathbf{y})\, dy_I \qquad\qquad \mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$$

$\varphi^*\alpha = \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{x}))\ dy_I = \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{x}))\ d\varphi_I(\mathbf{x})$

$= \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{x}))\ [\ \Sigma_J \det[(D\varphi)_{I,J}]\ dx_J$

$= \Sigma_J\ \{\ \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{x}))\ \det[(D\varphi)_{I,J}]\ \}\ dx_J$

$= \Sigma_J\ \{g_J(\mathbf{x})\ \}\ dx_J$

$= \Sigma_J\, g_J(\mathbf{x})\, dx_J \qquad g_J(\mathbf{x}) \equiv \Sigma_I\, f_I(\boldsymbol{\varphi}(\mathbf{x}))\ \det[(D\varphi)_{I,J}]$ // p 42 A and B

$\equiv \beta$

I will now translate all the above into Chapter 5 notation:
Now let's translate the pullback from Chapter 3. Space on the left has **t**, space on the right has **x**.

We had x on the left and y on the right. So we want to do $\mathbf{x} \rightarrow \mathbf{t}$ and $\mathbf{y} \rightarrow \mathbf{x}$ and $\boldsymbol{\varphi} \rightarrow \mathbf{c}$:

$dc_a dc_b dc_c ....dc_q = \Sigma_{A,B,C...Q}\ R_{abc...q,ABC...Q}\ dt_A dt_B dt_C ...dt_Q$ // non-ordered sum
k factors k factors

$dc_I = \Sigma_J \det(R_{I,J})\, dt_J = \Sigma_J\, [\det[(\mathbf{Dc})_{I,J}]\, dt_J$ // ordered sum on J, $\Sigma_{A<B<C...<Q}$

$\alpha = \Sigma_I\, f_I(\mathbf{x})\, dx_I \qquad \mathbf{x} = \mathbf{c}(\mathbf{t})$

$c^*\alpha = \Sigma_I\, f_I(\mathbf{c}(\mathbf{t}))\, dx_I = \Sigma_I\, f_I(\mathbf{c}(\mathbf{t}))\, dc_I(\mathbf{t})$

$= \Sigma_I\, f_I(\mathbf{c}(\mathbf{t}))\ [\ \Sigma_J \det[(Dc)_{I,J}]\ dt_J$

$= \Sigma_J\ \{\ \Sigma_I\, f_I(\mathbf{c}(\mathbf{t}))\ \det[(Dc)_{I,J}]\ \}\ dt_J$

$= \Sigma_J\ \{g_J(\mathbf{t})\ \}\ dt_J$

$= \Sigma_J\, g_J(\mathbf{t})\, dt_J \qquad g_J(\mathbf{t}) \equiv \Sigma_I\, f_I(\mathbf{c}(\mathbf{t}))\ \det[(Dc)_{I,J}]$ // p 42 A and B

$\equiv \beta$

Spaces here and previous: Notice how Sjamaar wantonly redefines m,n and V and U.

page 37:	$U \subset R^n$ on the left (**x**)	$V \subset R^m$ on the right (**y**)	general pullback
chap 4:	$[a,b] \subset R^1$ on the left (t)	$U \subset R^n$ on the right (**x**)	pullback of 1-forms
chap 5:	$[0,1]^k$ or $R \subset R^k$ on the left (**t**)	$\overline{R} \subset U \subset R^n$ on the right (**x**)	pullback of k-forms

Now the notation shown above $\Sigma_J\, g_J(\mathbf{t})\, dt_J$ is general in that it would allow something like $dt_1 dt_3$ as a differential included in the sum. But in our current context, we want ALL the dt factors in the form

$dt_1dt_2.....dt_k$. So we can further simplify the above equations for this special case: (first the general rule, then the special case)

$dc_a dc_b dc_c....dc_q = \Sigma_{A,B,C...Q} R_{abc...q,ABC...Q} dt_A dt_B dt_C...dt_Q$ // non-ordered sum
k factors k factors

$dc_1 dc_2 dc_3....dc_k = \Sigma_{A,B,C...Q} R_{123...k,ABC...Q} dt_A dt_B dt_C...dt_Q$ // non-ordered sum
k factors k factors

$= \Sigma_{A,B,C...Q} R_{123...k,ABC...Q} [P_{ABC..Q} dt_1dt_2dt_3...dt_k]$ // parity operator

$= [\Sigma_{A,B,C...Q} P_{ABC..Q} R_{123...k,ABC...Q}] dt_1dt_2dt_3...dt_k$

$= [\det(R_{I,I})] dt_1dt_2dt_3...dt_k$

Remember that in general $k < n$, so $R_{I,I}$ is a the upper left square kxk submatrix of the full matrix R. We indicate the idea that this is submatrix by writing $R_{I,I}$ where I indicates that we only involve the first k coordinates of the space R^n. In $R_{abc...q,ABC...Q}$ we suppress indices i for $k < i < n$. Continuing:

$dc_I = \det(R_{I,I}) dt_I = \det[(\mathbf{Dc})_{I,I}] dt_I$ $dc_I \equiv dc_1dc_2....dc_k$

$\alpha = f_I(\mathbf{x}) dx_I$

$c*\alpha = f_I(\mathbf{c}(\mathbf{t})) dc_I$

$= f_I(\mathbf{c}(\mathbf{t})) \det[(Dc)_{I,I}]dt_I$

$= f_I(\mathbf{c}(\mathbf{t})) \det[(Dc)_{I,I}] dt_I$

$= g_I(\mathbf{t}) dt_I$ $g_I(\mathbf{x}) \equiv f_I(\mathbf{c}(\mathbf{x})) \det[(\mathbf{Dc})_{I,I}]$ // p 42 A and B

$\equiv \beta$

Now again since we know that I means the entire set (the volume differential) we write again

$dc_1dc_2dc_3....dc_k = \Sigma_{A,B,C...Q} R_{123...k,ABC...Q} P_{ABC..Q} dt_1dt_2dt_3...dt_k$
k factors k factors

$d\mathbf{c} = \det(R) d\mathbf{t} = \det[(\mathbf{Dc})] d\mathbf{t}$ // ordered sum on J

$\alpha = f(\mathbf{x}) dx$ $dc \equiv \Pi_{i=1}^{k} dc_k$ $dx \equiv \Pi_{i=1}^{k} dx_k$ $dt \equiv \Pi_{i=1}^{k} dt_k$

$c*\alpha = f(\mathbf{c}(\mathbf{t})) dc$

$= f(\mathbf{c}(\mathbf{t}))\ \det[(Dc)]dt$

$= \ f(\mathbf{c}(\mathbf{t}))\ \det(Dc)\ dt$

$= g(\mathbf{t})\ dt$

$= g(\mathbf{t})\ dt \qquad\qquad g(\mathbf{t}) \equiv f(\mathbf{c}(\mathbf{x}))\ \det(\mathbf{Dc}) \qquad\qquad$ // p 42 A and B

$\equiv \beta$

So finally we arrive at p 57 B

$$c^*\alpha = g(\mathbf{t})\ dt\ =\ g(\mathbf{t})\ dt_1dt_2dt_3...dt_k$$

$$\int_c \alpha \equiv \int_R c^*\alpha = \int_R g(\mathbf{t})\ dt\ =\ \int\int\int...\int g(\mathbf{t})\ dt_1dt_2dt_3...dt_k \qquad \text{p 57 B}$$

where as just noted $g(\mathbf{t}) \equiv f(\mathbf{c}(\mathbf{t}))\ \det(\mathbf{Dc})$.

Now back up just a little. We used to have $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x})$ and now we have $\mathbf{x} = \mathbf{c}(\mathbf{t})$ as our underlying mapping. The object **c** is the mapping (just as Bucks say the curve is the mapping, not the image of the mapping). In the current case, what is the image? It is some kind of k-dimensional surface within R^m (page 37). Normally one takes m = k+1, but I don't think that would be required.

For k = 1, the image could be a self-intersecting curve.
For k = 2 it could be a self intersecting 2D surface as shown Buck p *.
For k = 3, it could be some self-intersecting 3D surface displayable in R^4.

What is the situation when k = 0? Chapter 4 was k = 1, so maybe this was not addressed.

1. the differential form α must be a 0-form if k = 0, so α is a function of **x**. We have $\alpha = f(\mathbf{x})$

2. The only point in t space is $\mathbf{t} = \mathbf{0}$. This space is R^0.

2. The mapping c does this : $\mathbf{c}$: ($\mathbf{t}=\mathbf{0}$) → image $\mathbf{c}(0)$ in R^m $\qquad\qquad \mathbf{c}(0) = \mathbf{x}$

So I guess we imagine that the domain has only the point **0** and this maps to point **x** in R^m

3. $c^*\alpha = g(\mathbf{t})$ which is a function (pullback does not change degree of a form)

$$\int_c \alpha \equiv \int_R c^*\alpha = \int_R g(\mathbf{t})\ \delta(\mathbf{t}=\mathbf{0})dt\ = g(\mathbf{0}) \qquad \text{below p 57 B}$$

4. I guess we set $\det(\mathbf{Dc}) = 1$ and write $g(\mathbf{t}) \equiv f(\mathbf{c}(\mathbf{t}))\ \det(\mathbf{Dc})$ and $g(0) = f(\mathbf{c}(0)) * 1 = f(\mathbf{x})$

and I dimly arrive at result p 57 C. They call it a definition:

$\int_{\mathbf{c}} \alpha \equiv f(\mathbf{x})$ when $\alpha = f(\mathbf{x})$ a 0-form $k = 0$

Reparametrization

We start with $\mathbf{x} = \mathbf{c}(\mathbf{t})$ = some surface in the range of $\mathbf{c}$: $R^{\mathbf{k}} \rightarrow R^{\mathbf{m}}$. I will mimic my own proof from earlier in these notes for the case k = 1, but now k = k. We now have $\mathbf{t} = \mathbf{p}(\mathbf{s})$ as the reparam:

$(\mathbf{c} \circ \mathbf{p})^{\star} \alpha = \mathbf{p}^{*}(\mathbf{c}^{*}\alpha) = p^{*}(\mathbf{g}(\mathbf{t})dt) = g(\mathbf{p}(\mathbf{s}))\, dp$

$= (p^{*}g)\, dp = (p^{*}g)dp_1 dp_2 ....dp_{\mathbf{k}} = (p^{*}g)\, \det(\mathbf{Dp})\, ds_1 ds_2 .....ds_{\mathbf{n}}$

$= (p^{*}g)\, \det(\mathbf{Dp})\, dt = g(\mathbf{p}(\mathbf{s}))\, \det(\mathbf{Dp})\, ds$ // agrees with p 58 B

We assume that $\mathbf{t} = \mathbf{p}(\mathbf{s})$ is one-to-one and onto (bijective) and that $\det(\mathbf{Dp}) \neq 0$. Then

$\int_{\mathbf{c} \circ \mathbf{p}} \alpha \equiv \int_{\overline{\mathbf{R}}} (\mathbf{c} \circ \mathbf{p})^{\star} \alpha = \int_{\overline{\mathbf{R}}} g(\mathbf{p}(\mathbf{s}))\, \det(\mathbf{Dp})\, ds_1 ds_2 .....ds_{\mathbf{n}}$ // agrees p 58 C

What is $\overline{R}$? It is the mapping of the t-cube by $\mathbf{t} = \mathbf{p}(\mathbf{s})$. If p were linear, $\overline{R}$ might be a block, but in general it is just some unknown shape, the image of the original block under p^{-1}: $\mathbf{t} \rightarrow \mathbf{s}$. But now we just do a multiple change of variables and we can write

$\int_{\overline{\mathbf{R}}} g(\mathbf{p}(\mathbf{s}))\, \det(\mathbf{Dp})\, ds_1 ds_2 .....ds_{\mathbf{n}} = \text{sign}[\det(Dp)] \int_{\mathbf{R}} g(t)\, dt$

But we earlier had from 57 B,

$\int_{\mathbf{c}} \alpha \equiv \int_{\mathbf{R}} c^{*}\alpha = \int_{\mathbf{R}} g(\mathbf{t})\, dt$

Therefore I have shown right here that

$\int_{\mathbf{c} \circ \mathbf{p}} \alpha = \text{sign}[\det(Dp)] \int_{\mathbf{c}} \alpha$ // agrees with p 58 E

This says that any reparametrization leaves the k-form integral unchanged except for a possible sign! This seems to me to be a fairly powerful result. [Thus we have generalized the 1-form result of Ch 4.]

Comment: Recall from Buck that in respeeding we went from [a,b] to [α,β] as the interval change with I think with f = f(t). We didn't use a separate name for the function f(t). In Sjamaar page 48 we went from p: $[\overline{a}, \overline{b}]$ to [a,b] under t = p(s) so went from t = $[\overline{a}, \overline{b}]$ to s = [a,b]. Now in k dimensions on page 57 we have $\mathbf{t} = \mathbf{p}(\mathbf{s})$ and this respeeding (reparam) takes $\mathbf{t} = \overline{\mathbf{R}}$ to $\mathbf{s} = R$ where p: $\overline{R}$ to R. So Sja is trying to use the same kind of notation, I appreciate that concession. So the reparam $\mathbf{t} = \mathbf{p}(\mathbf{s})$ maps one cube into the other cube in a bijective manner probably with some extra rules. In the Buck 1D case we had conditions at the endpoints, so perhaps here there are conditions on the cube boundaries.

[58] Remark. In (5.1) Sja identifies $dt_1 \wedge dt_2$ with normal calculus $dt_1 dt_2$ with no comment. His reconciliation of the minus sign issues is OK, but I won't be happy until I understand the wedge product. [I suspect Sja would say the wedges *are* present in page 57 B, I will deal with this later.]

About k-cubes and k-chains [59]

Definition of a **k-cube**: It is a mapping **c**: $[0,1]^k \to U$. Seems to me it is whatever you get in this mapping, and the image is NOT a cube. It is the image of a cube. So seems an odd name. Certain mappings would in fact map the source cube into a block in U space. [the word "cube" refers to the shape of the domain in the mapping, generalization of an "interval" in R^1]

Example: A 0-cube is c:$\mathbf{0} \to U$ and we decided $\int_c \alpha = \int_c f = f(\mathbf{x})$. The point 0 has to map to some point x, so the claim is that 0-cube = $\{\mathbf{x}\}$. Again, this seems to be the image of a 0-cube mapping.

Example: A 1-cube is c: $[0,1] \to U$ and this gives some curve in U. The 1-cube image is not a line segment.

Example: A 2-cube is c:$[0,1]^2 \to U$ and this gives some surface say in R^3. Again the image of the 2-cube is not a block in R^3.

Definition of a **k-chain:** It is a mapping formed as a linear combination of k-cube mappings. This is clearly shown on page 59 C. Somewhere we must have shown that

$$\int_{c+d} \alpha = \int_c \alpha + \int_d \alpha \qquad \text{// probably page 38 i}$$

It then follows that (mappings are linear)

$$\int_c \alpha = \int_{a_1c_1+a_2c_2+\ldots} \alpha = \Sigma_i a_i \int_{c_i} \alpha \qquad \text{which is p 59 D}$$

This last line is "an integration over a k-chain" which is based "on the integration over a k-cube".

Note: a k-cube is a surface of dimension m-1 in R^m let's say, so we are integrating over a sum of surfaces, or maybe it is the union of those surfaces, when integrating over a k-chain. Each surface is of the same dimension and is a k-cube.

Example: A 0-chain would be $\Sigma_i a_i \{\mathbf{x}_i\}$. He wants us to think of this the way we think of

$$\rho(\mathbf{x}) = \Sigma_i \, a_i \delta(\mathbf{x}-\mathbf{x}_i) \qquad a_i = \text{charges at points } x_i$$

Example: A 1-chain would be $c = \Sigma_i a_i c_i$. Perhaps this is a sum of curves in the image space, each with some weight a_i. For a 1-chain remember that a 1-cube image is a curve in space, it is not a cube. For a 0-chain replace curve with point.

Idea: Maybe we are going to "chain" some curves together to get a Stokesian boundary or something like that.

Things are a little hazy to me at this point in terms of those charge analogies.

2.13.16. Alright, for k = 2 you could string together a bunch of patches on a donut to cover the donut. Each of these patches could from a mapping on $[0,1]^2$ so each donut's curved patch is a 2-cube and then you cover the entire donut with a chain of these 2-cubes. I think that is the Manifold idea starting here.

5.2 The boundary of a chain

The boundary of a curve is stated as $\partial c = \{c(1)\} - \{c(0)\}$. What does this mean? Is this the difference between two sets? It seems to be one set containing 1 point minus a second set containing a different point. Sja notation is hazy, he never uses the word set. Since this is a sum of points, it is a 0-chain, so the ∂ operator lowered you one level of chain space.

For a 2-cube (a finite surface patch) we get this

$\partial c = c_1 + c_2 - c_3 - c_3$ = a *chain* of 1-cubes.

In this little sum, c is a 2-cube while each of the c_i is a 1-cube. ∂ lowers the degree by one.

The boundary is the sum or union of the four curved segments shown (a 1-chain of 1-cubes), but they have signs based on the direction for each segment. I recall something from Buck on this but don't digress now.

In the above case we have

$c_1(t) = c(t,0)$
$c_2(t) = c(1,t)$
$c_3(t) = c(t,1)$
$c_4(t) = c(0,t)$

Again each c here is a 1-cube! Note that 0,1 specify location on the four sided patch but in $[0,1]^2$.

For a **3-cube** boundary we might have 2k = 6 faces as on a normal cube. Perhaps

$c_1(a,b) = c(a,b,0)$
$c_2(a,b) = c(a,b,1)$ // an opposite side pair
$c_3(a,b) = c(0,a,b)$
$c_4(a,b) = c(1,a,b)$ // an opposite side pair

$c_5(a,b) = c(a,0,b)$
$c_6(a,b) = c(a,1,b)$ // an opposite side pair

We can rewrite this as

$c_1(t_1,t_2) = c(t_1,t_2,0)$
$c_2(t_1,t_2) = c(t_1,t_2,1)$ // an opposite side pair
$c_3(t_1,t_2) = c(0,t_1,t_2)$
$c_4(t_1,t_2) = c(1,t_1,t_2)$ // an opposite side pair
$c_5(t_1,t_2) = c(t_1,0,t_2)$
$c_6(t_1,t_2) = c(t_1,1,t_2)$ // an opposite side pair

And if we now define $\mathbf{t} = (t_1,t_2....t_{k-1})$ being on $[0,1]^{k-1}$ then [note the *notation* here!!]

$c_{3,0}(\mathbf{t}) = c(t_1,t_2,0)$
$c_{3,1}(\mathbf{t}) = c(t_1,t_2,1)$ // an opposite side pair
$c_{1,0}(\mathbf{t}) = c(0,t_1,t_2)$
$c_{1,1}(\mathbf{t}) = c(1,t_1,t_2)$ // an opposite side pair
$c_{2,0}(\mathbf{t}) = c(t_1,0,t_2)$
$c_{2,1}(\mathbf{t}) = c(t_1,1,t_2)$ // an opposite side pair

or reordering

$c_{1,\rho}(\mathbf{t}) = c(\rho,t_1,t_2)$ $\mathbf{t} = t_1,t_2$
$c_{2,\rho}(\mathbf{t}) = c(t_1,\rho,t_2)$
$c_{3,\rho}(\mathbf{t}) = c(t_1,t_2,\rho)$

where we have a pairwise labeling scheme. This is the meaning of page 60 C. The first index tells you the number of the slot where the 0 or 1 is located. Notice that these boundary faces are each defined on $[0,1]^2$ so whereas the 3-cube c is a map of $[0,1]^3$, the boundary ∂c is a 2-cube mapping $[0,1]^2$.

At this point the reader can look at a separate doc "detail work on k-cube faces.doc". It is 9 pages of fine detail, so I want to keep it out of this huge notes file. Here is a summary of these side notes:

1. A clarification the meaning of a k-cube ..... Error! Bookmark not defined.
2. Here is the Rule for creating faces: ..... Error! Bookmark not defined.
3. Face of a Face ..... Error! Bookmark not defined.
4. Sign of a Face ..... Error! Bookmark not defined.
5. Proof that $\partial(\partial c) = 0$ ..... Error! Bookmark not defined.

So this takes us through the end of Section 5.2, and I proved the fact that $\partial^2 = 0$ in full detail. I accept the formula p 60D for the boundary of a k-cube. I see how it works for a 2-cube, but I have no geometric interpretation for the signs of the 6 faces of a cube, although pairs always have opposite sign, and the

three + faces are adjacent, so in a huge pile of cubes you will get cancellation of all internal faces in some sense.

5.3 Cycles and Boundaries

Suppose you have a 2-cube where $c(t_1,t_2)$ normally describes a patch, but suppose this thing does not depend on t_2. Then $\mathbf{c}(t_1,t_2) = \mathbf{c}(t_1)$ = a 1-cube = a curve. This would be a *degenerate* 2-cube.

A degenerate 1-cube means $\mathbf{c}(t_1) = \mathbf{x}_0$, some constant, so your curve becomes just a point. You are losing a whole dimension in such "first order degeneracy". For a degenerate curve $\mathbf{c}(t_1) = \mathbf{k}$ if you integrate a form α over that curve, you really have no integration at all, so the integral is 0. Sja comments at this point that no work is done on a motionless particle, and that is what he means.

What happens if you do this integral

$$\int_{\mathbf{c}} \alpha$$

when α is a k-form, and where c is a degenerate k-cube? Mixing lots of concepts together here: k-forms, k-cubes (chains) and integration. To evaluate this integral, I would write it as a pullback to the unit cube in the "left" space to get

$$\int_{\mathbf{c}} \alpha = \int_{[0,1]^k} \mathbf{c}^*\alpha \qquad \alpha = \Sigma_I\, f_I(\mathbf{y})\, dy_I \qquad \mathbf{y} = \boldsymbol{\varphi}(\mathbf{x}) \text{ is } \mathbf{y} = \mathbf{c}(\mathbf{x})$$

$$\mathbf{c}^*\alpha = \Sigma_I\, f_I(\mathbf{c}(\mathbf{x}))\, dc_I = \Sigma_I\, f_I(\mathbf{c}(\mathbf{x}))\, \det(\mathbf{Dc})\, dx_I$$

So then

$$\int_{\mathbf{c}} \alpha = \int_{[0,1]^k} \mathbf{c}^*\alpha = \int_{[0,1]^k} \Sigma_I\, f_I(\mathbf{c}(\mathbf{x}))\, \det(\mathbf{Dc})\, dx_I \qquad (\mathbf{Dc})_{ij} = \partial_j c_i$$

But if c does not depend on one of the coordinates, then $\partial_j c_i$ for that j and all i, which is a whole column of the matrix (**D**c), so $\det(\mathbf{Dc}) = 0$ and then we have $\int_{\mathbf{c}} \alpha = 0$. The argument Sja gives for this proof requires the introduction of a new function g and we then have $c^* = f^*g^*$ and it just seems way complicated to me. The point to me is that if you have a degenerate c, then $\det(\mathbf{Dc}) = 0$ and you are done!

Just above I have proved

[61] **Lemma 5.5: " The integral of a k-form α over a *degenerate* k-cube is zero. "**

This is then of course true for a degenerate k-chain which is a linear combination of degenerate k-cubes.

Suppose you had a k-cube c such that ∂c is a *degenerate* (k-1)-cube. In this situation, one says that the original k-cube c is **closed**. One also says it is "**a cycle**". The reasons for these words are not clear yet.

Definition: A k-cube c is **closed or a cycle** if the (k-1)-cube ∂c is degenerate.

Now suppose a k-cube c can be written as $c = \partial b + c'$ *where* c' is degenerate. If this is the case, then c is said to be a **boundary** for the (k+1)-cube b. [A little like a form being exact, but an extra term].

Definition. If a k-cube c can be written as $c = \partial b + c'$ where c' is a degenerate k-cube, then c is the **boundary** of the (k+1)-cube b.

Example 5.6. Let c_1 and c_2 be two curves which form a closed loop. These curves are 1-cubes. Perhaps if $c = c_1+c_2$ then ∂c is degenerate since there is then no boundary. I would say $\partial c = 0$. I guess such a c is regarded as a degenerate 1-cube and therefore you would say that $c = c_1+c_2$ was closed or was a cycle using the above definitions. Now the words closed and cycle make sense.

2.13.16: If a 1-cube is *non*-degenerate, then it's boundary is a set of points each of which has dimension 0. For example, a semicircle curve has a boundary consisting of the two endpoints. If a 1-cube is degenerate, then the boundary is lower than the official dimension, and in this case instead of having a set of points, you have no points at all in the boundary! You might call that dimension = -1, so we dropped a dimension. If you could have a circle be a 1-cube, we know it has no boundary so $\partial c = 0$ and $\dim(\partial c) = -1$, so to speak. In any event, this would be an example of a degenerate 1-cube because the boundary has lower than the expected dimension. A half spherical shell is a 2-cube and it has a circular 1D boundary, so it would be a normal non-degenerate 2-cube. But if a full spherical shell could be a 2-cube, then since it has no 1D boundary at all, it is degenerate. I think that a circle can only be a 1-chain and a spherical shell a 2-chain, but I will hedge on that for now.

[62] Lemma 5.7: The boundary of a degenerate k-cube is a degenerate (k-1)-cube.

Proof: Let c be a degenerate k-cube. Focus just on argument t_i and assume c no depend on t_i. This means that $c_{i,0} = c_{i,1}$. Now what is ∂c for our c? The full formula is this

$$\partial c = \Sigma_{j=1}^{k} (-1)^j (c_{j,0} - c_{j,1}).$$

The term in this sum with j = i then vanishes so we then have

$$\partial c = \Sigma_{j\neq i} (-1)^j (c_{j,0} - c_{j,1}).$$ // this is page 62 D (*)

Since c does not depend on t_i, it seems that $c_{j,\rho}$ also does not depend on i. Remember

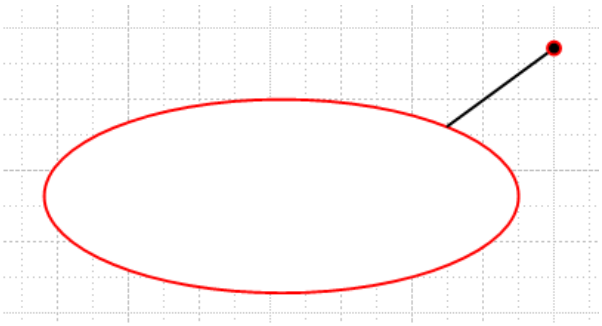
$$[c(t_1,t_2,t_3..\ t_{j-1},t_j,t_{j+1}..t_k)]_{j,\rho} \rightarrow c(t_1,t_2,t_3..\ t_{j-1},\rho,t_{j+1}..t_k) \rightarrow c(t_1,t_2,t_3..\ t_{j-1},\rho,t_j..t_{k-1})$$

If t_i is to the left of j, we could just put it in and find that if c does not depend on t_i, neither does $c_{j,\rho}$. But suppose t_i is to the right of j (like the t_k is). Then this t_i gets renumbered to be t_{i-1}. In either case, we find that $c_{j,\rho}$ is degenerate because its dependence on at least one variable is missing. Thus from (*) above we conclude that ∂c is degenerate. QED.

Corollary: Every boundary is a cycle

Proof: If ∂c is a "boundary" of c, then we can write $c = \partial b + c'$ by the boundary definition above, where c' is degenerate. Now consider $\partial c = \partial^2 b + \partial c' = \partial c'$. Since c' is degenerate, so is $\partial c'$ by the above lemma 5.7. Therefore ∂c is degenerate. By the definition above, c is then a cycle (it is closed). QED

Comment: In the formula $c = \partial b + c'$ in 2D. This issue has come up somewhere before. Suppose the ellipse including interior plus whisker sticking out (line segment) is patch b. Then $c = \partial b$ is the red perimeter of the ellipse plus the boundary points of the whisker (I show one in red). But the whisker boundary points are just points, so c' = those points = a degenerate curve. The points are loci of one less dimension and would contribute nothing to a line integral of something around the boundary.



So the point is that in this case, the ellipse c is regarded as the boundary of the entire k-chain and the fact that you have $c = \partial b$ + something else, does not matter.

[62] 5.9 **Unit Circle Example**. This took me a while to decipher. Consider this 2-cube b

$$\mathbf{b}(t_1,t_2) = (1-t_2)\mathbf{c}(t_1)$$

where $c(t_1)$ is a unit circle parametrized by t_1, meaning $\mathbf{c}(t_1) = (\sin(2\pi t),\cos(2\pi t)) = (x,y)$. In this example, we shall show that

$\partial\mathbf{b} = \mathbf{c} - \mathbf{c}'$ where $\mathbf{c}' = (0,0)$ = a degenerate 1-cube

and this provides an example of the idea that $\mathbf{c} = \partial\mathbf{b} + \mathbf{c}'$

Details: looking at **b** above, you can see that as you vary the two parameters, you sort of fill out the area of a disk of radius 1, so this is certainly a good candidate for a 2-cube with boundary being a circle. So let's use our formula of page 60 D to compute $\partial\mathbf{b}$:

$\partial\mathbf{b} = \Sigma_{i=1}^{2} (-1)^i [\, \mathbf{b}_{i,0} - \mathbf{b}_{i,1}]$ // a 2-cube is a sum of 1-cubes = a sum of curves

$= - [\, \mathbf{b}_{1,0} - \mathbf{b}_{1,1}] + [\, \mathbf{b}_{2,0} - \mathbf{b}_{2,1}] = - \mathbf{b}_{1,0} + \mathbf{b}_{1,1} + \mathbf{b}_{2,0} - \mathbf{b}_{2,1}$ = four terms

We then go off and compute these four terms: Be careful with the argument renaming!

$$\mathbf{b}(t_1,t_2) = (1-t_2)\mathbf{c}(t_1)$$

$\mathbf{b}_{1,0} = \mathbf{b}(0,t_1) = (1-t_1)\mathbf{c}(0) = (1-t_1)(1,0) = (1-t_1, 0)$ this curve is a hor line segment.
$\mathbf{b}_{1,1} = \mathbf{b}(1,t_1) = (1-t_1)\mathbf{c}(1) = (1-t_1)(1,0) = (1-t_1, 0)$

$\mathbf{b}_{2,0} = \mathbf{b}(t_1,0) = (1-0)\mathbf{c}(t_1) = \mathbf{c}(t_1)$
$\mathbf{b}_{2,1} = \mathbf{b}(t_1,1) = (1-1)\mathbf{c}(t_1) = (0,0)$

Adding up the four terms we then get

$$\partial\mathbf{b} = -\mathbf{b}_{1,0} + \mathbf{b}_{1,1} + \mathbf{b}_{2,0} - \mathbf{b}_{2,1}$$

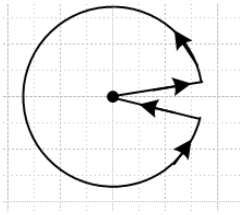
$$= -(1-t_1, 0) + (1-t_1, 0) + \mathbf{c}(t_1) - (0,0)$$

$$= \mathbf{c}(t_1) - (0,0)$$

and therefore

$$\mathbf{c}(t_1) = \partial\mathbf{b} + (0,0)$$

In this case, the extra piece c' = (0,0) is a degenerate 1-curve. Here is a picture that might explain what is happening here (this picture is apparently implied by the explicit form of **b** stated above)



The four terms of ∂b are the four pieces of this boundary. The two line segments $(1-t_1, 0)$ cancel, and you end up with the circle plus the point at the origin which is the (0,0).

5.4 Stokes's Theorem

Recall Theorem 4.4 from page 49, where α is an exact 1-form and c is a curve and g is a 0-form = function:

$$\int_{\mathbf{c}} \alpha = g(\mathbf{c}(b)) - g(\mathbf{c}(a)) \qquad \text{where } \alpha = dg \qquad (*)$$

which I derived in this manner, noting that $c^*\alpha = c^*(dg) = d(c^*g) = dh$ where $h(t) = c^*g = g(c(t))$,

$$\int_{\mathbf{c}} \alpha \equiv \int_{[a,b]} c^*\alpha = \int_{[a,b]} dh = h(b) - h(a) = g(\mathbf{c}(b)) - g(\mathbf{c}(a)) \qquad \text{QED}$$

Notice that the proof here builds on the notions of:

- the pullback to be the definition of the integral over a form and the definition of φ* in this context
- the theorem that d(φ*α) = φ*(dα)
- the general theory of differential forms

This (*) is the "fundamental theorem of calculus in R^n (FTOC)" written in the language of 1-forms. Rewrite then as

$$\int_{\mathbf{c}} dg = g(\mathbf{c}(b)) - g(\mathbf{c}(a))$$

where **c** is a curve with endpoints **c**(a) and **c**(b). Note that $c:R^1 \rightarrow R^n$ so in general the curve lies in R^n. A special case is for n = 1 and then the curve is like y = f(x) in regular 1D calculus, here y = c(x). Here we are integrating an exact 1-form α = dg.

Now consider our earlier result from page 57 where we integrate a 0-form over a 0-cube **c** where **c** = **x** = just a point.

$$\int_{\mathbf{c}} f = f(\mathbf{x})$$

Using this notion, we can write $g(\mathbf{c}(b)) = \int_{\mathbf{c}} g = \int_{\{\mathbf{c}(b)\}} g$. Then our FTOC reads

$$\int_{\mathbf{c}} dg = \int_{\{\mathbf{c}(b)\}} g - \int_{\{\mathbf{c}(a)\}} g = \int_{\{\mathbf{c}(b)\}-\{\mathbf{c}(a)\}} g$$

Now we know from the bottom of page 59 or from the general formula p 60 D that

$$\{c(b)\}-\{c(a)\} = \partial c$$

Therefore our FTOC appears as (replace g with symbol α)

$$\int_{\mathbf{c}} d\alpha = \int_{\partial \mathbf{c}} \alpha$$ where α is a 0-form and c is a 1-cube, ∂c = 0-cube

It turns out that this theorem stated exactly this way is also valid for α = k-form and c = (k+1)-cube and then ∂c = k-cube. This more general theorem is the famous Stokes's Theorem which then will encompass all my "integral theorems". For the proof we will instead say

α = (k-1)-form	c = k-cube	(c-chain, etc etc)
dα = k-form	∂c = (k-1)-cube	

Side Question: Why isn't the above FTOC on my list of "integral theorems"? I think it only applies to the case that you have α = **F**•d**x** where **F** is a conservative force and only in that case is α a perfect differential.

[I had trouble with this proof because you need to show the wedge product symbols in some places and not in others. I will add those symbols where I think they go:]

Proof: We know that

$$\int_c d\alpha = \int_{[0,1]^k} c^*(d\alpha) = \int_{[0,1]^k} d(c^*\alpha) \qquad \text{// pullback and theorem on d and *}$$

We know that α is a (k-1) form, so the pulled back $c^*\alpha$ is also a (k-1)-form since the degree of forms does not change when you do the pullback. Therefore we can write

$c^*\alpha$ = (k-1)-form $= \Sigma_i\, g_i(\mathbf{t})\, dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k$ dt_i (red) is missing **page 63 B**

This is the most general form you can have for a (k-1)-form and g_i are the coefficient functions.

More detail on this:

$$c^*\alpha = c^*[\Sigma_I\, f_I\, dx_I] = \Sigma_I\, c^*(f_I)\, c^*(dx_I) = \Sigma_I\, f_I(c(\mathbf{x}))dc_I = \Sigma_I\, f_I(c(\mathbf{x}))(Dc)dt_I = \Sigma_I\, g_I\, dt_I \quad \text{p 42A,B}$$

Note that the t_j are all in proper order. Now apply d:

$$d(c^*\alpha) = \Sigma_i\, (dg_i)\, dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k = \Sigma_i\, \Sigma_j\, [(\partial_j g_i) dt_j]\, dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k$$

In the Σ_j only j = i will contribute since otherwise you get $dt_j dt_j$ appearing which is 0. So,

$$d(c^*\alpha) = \Sigma_i\, (\partial_i g_i) dt_i \wedge dt_1 \wedge dt_2 \ldots dt_i \ldots \wedge dt_k = \Sigma_i\, (-1)^{i+1}\, (\partial_i g_i)\, dt_1 \wedge dt_2 \ldots \wedge dt_k$$

where we pick up the sign putting dt_i into is correct location.

Aside: Proof that correct sign is $(-1)^{i+1}$:

Consider: $dt_1 dt_2 \ldots dt_i \ldots dt_k = (-1)^s\, dt_i\, dt_1 \ldots dt_i \ldots dt_k$

Suppose i = 1. this then reads

$dt_1 dt_2 \ldots dt_i \ldots dt_k = (-1)^s\, dt_1 dt_2 \ldots dt_i \ldots dt_k \quad \Rightarrow$ s = even so $s = (-1)^{i+1}$ is correct

Suppose i = 2. Then

$dt_1 dt_2 \ldots dt_i \ldots dt_k = (-1)^s\, dt_2\, dt_1 dt_3 \ldots dt_k$ and here need s = odd

Then we have

$$\int_c d\alpha = \int_{[0,1]^k} d(c^*\alpha) = \int_{[0,1]^k} \{\Sigma_i\, (-1)^{i+1}\, (\partial_i g_i)\, dt_1 \wedge dt_2 \wedge \ldots \wedge dt_k\}$$

$= \Sigma_i\ (-1)^{i+1} \int_{[0,1]^k}\ (\partial_i g_i)\ dt_1 \wedge dt_2 \wedge \ldots\ldots \wedge dt_k$ **page 63 C**

This is the point where we have things in standard order and we want to actually do the integration. If we do that right here, we then have

$\int_c d\alpha\ =\ \Sigma_i\ (-1)^{i+1} \int_{[0,1]^k}\ (\partial_i g_i)\ dt_1 dt_2 \ldots\ldots dt_k$ // no ^

We would then continue this way

$\int_{[0,1]^k}\ (\partial_i g_i)\ dt_1 dt_2 \ldots\ldots dt_k\ =\ \int_{[0,1]^k}\ (\partial_i g_i)\ dt_i\ dt_1 dt_2 \ldots dt_i \ldots dt_k$ // no ^ **page 63 D1**

So this is the only way I can obtain page 63 D1. He slips dt_i to the left without a sign because we have already collapsed out the ^ symbols and we are just working on the multivariable integration. At this point we then have

$\int_c d\alpha\ = \Sigma_i\ (-1)^{i+1} \int_{[0,1]^k}\ (\partial_i g_i)\ dt_i\ dt_1 dt_2 \ldots dt_i \ldots dt_k$

$= \Sigma_i\ (-1)^{i+1} \int_{[0,1]^{k-1}}\ [\ \int_{[0,1]}\ (\partial_i g_i)\ dt_i\ ]\ dt_1 dt_2 \ldots dt_i \ldots dt_k\ .$

Ignoring other coordinates we find

$\int_{[0,1]}\ (\partial_i g_i)\ dt_i\ =\ \int_0^1\ (d_i g_i(t_i))dt_i\ =\ \int_0^1 \frac{dg_i(t)}{dt}\ dt\ = g_i(t_i{=}1) - g_i(t_i{=}\ 0)$

$= g_i(t_1,t_2,\ldots t_{i-1},1,t_{i+1}\ldots\ t_k)\ -\ g_i(t_1,t_2,\ldots t_{i-1},0,t_{i+1}\ldots\ t_k)$

We then have

$\int_c d\alpha\ =$

$\Sigma_i(-1)^{i+1} \int_{[0,1]^{k-1}}[g_i(t_1,t_2,\ldots t_{i-1},1,t_{i+1}\ldots\ t_k)\ -\ g_i(t_1,t_2,\ldots t_{i-1},0,t_{i+1}\ldots\ t_k)]\ dt_1 dt_2 \ldots dt_i \ldots dt_k$

Now at this point we can reinsert the ^ symbols, without changing any ordering

$\int_c d\alpha\ =\ \Sigma_i(-1)^{i+1} \int_{[0,1]^{k-1}}[g_i(t_1,t_2,\ldots t_{i-1},1,t_{i+1}\ldots\ t_k)\ -\ g_i(t_1,t_2,\ldots t_{i-1},0,t_{i+1}\ldots\ t_k)]$

$dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k$ // dt_i is missing

Now a notation detail we have to work on. Let $\mathbf{x} = \mathbf{c}(\mathbf{t})$. If we fix t_i at value ρ, we can define a new mapping

$\mathbf{x} = \mathbf{d}(t_1,t_2....t_i....t_k;\ t_i = \rho)$ mapping of only k-1 variables

I could then define the mapping $\mathbf{c}_{i,\rho}$ as

$\mathbf{c}_{i,\rho} \equiv \mathbf{d}$

Now go back to

$\mathbf{c}^*\alpha$ = (k-1)-form $= \Sigma_i\ g_i(t_1,t_2,t_3\ ..t_i\ ...t_k)\ dt_1\wedge dt_2\wedge ...dt_i....\wedge dt_k$

Then how about

$\mathbf{d}^*\alpha$ = (k-1)-form $= \Sigma_i\ g_i(t_1,t_2,t_3\ ..\rho\ ...t_k)\ dt_1\wedge dt_2\wedge ...dt_i....\wedge dt_k$

So nothing illegal here. But then with the above definition we get

$$[\mathbf{c}_{i,\rho}]^*\ \alpha = \Sigma_i\ g_i(t_1,t_2,t_3\ ..\rho\ ...t_k)\ dt_1\wedge dt_2\wedge ...dt_i....\wedge dt_k = \Sigma_i\ [g_i]_{i,\rho}\ dt_1\wedge dt_2\wedge ...dt_i....\wedge dt_k \quad (*)$$

where I have NOT made any variable name adjustments in $[g_i]_{i,\rho}$. So OK, then ignoring that little adjustment issue, we have

$$\int_{\mathbf{c}} d\alpha = \Sigma_i\ (-1)^{i+1} \int_{[0,1]^{k-1}} [g_i(t_1,t_2,....t_{i-1},1,t_{i+1}... t_k) - g_i(t_1,t_2,....t_{i-1},0,t_{i+1}... t_k)]$$
$$dt_1\wedge dt_2\wedge ...dt_i....\wedge dt_k$$

$$= \Sigma_i(-1)^{i+1} \int_{[0,1]^{k-1}} [\ [g_i]_{i,1} - [g_i]_{i,0}]\ dt_1\wedge dt_2\wedge ...dt_i....\wedge dt_k$$

$$= \Sigma_i(-1)^{i+1} \int_{[0,1]^{k-1}} [\ (\mathbf{c}_{i,1})^*\ \alpha\ -\ (\mathbf{c}_{i,0})^*\ \alpha\ ] \qquad \text{// using (*) above}$$

$$= \Sigma_i(-1)^{i} \int_{[0,1]^{k-1}} [\ (\mathbf{c}_{i,0})^*\ \alpha\ -\ (\mathbf{c}_{i,1})^*\ \alpha\ ]$$

$$= \Sigma_i\ \Sigma_\rho\ (-1)^{i+\rho} \int_{[0,1]^{k-1}} [\ (\mathbf{c}_{i,\rho})^*\ \alpha\ ]$$

Now use the pullback definition to write this as

$$= \Sigma_i\ \Sigma_\rho\ (-1)^{i+\rho} \int_{\mathbf{c}(i,\rho)}\ \alpha$$

Now do the sum acting on the integration domain and use formula p 60 D and you get

$= \int_{\partial c} \alpha$

and this then concludes the proof.

Summary of this proof of the general Stokes's Theorem

Step 1: $\int_c d\alpha = \int_{[0,1]^k} c^*(d\alpha) = \int_{[0,1]^k} d(c^*\alpha)$

pullback form $\varphi^* d = d\varphi^*$ p 39

Step 2: $c^*\alpha$ = (k-1)-form $= \Sigma_i\, g_i(\mathbf{t})\, dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k$

$d(c^*\alpha) = \Sigma_i\, (dg_i)\, dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k = \Sigma_i \Sigma_j\, [(\partial_j g_i) dt_j]\, dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k$

$= \Sigma_i\, (-1)^{i+1}\, (\partial_i g_i)\, dt_1 \wedge dt_2 \ldots \wedge dt_k$ // slide dt_i into place

Step 3: $\int_c d\alpha = \int_{[0,1]^k} d(c^*\alpha) = \Sigma_i\, (-1)^{i+1} \int_{[0,1]^k} (\partial_i g_i)\, dt_1 \wedge dt_2 \wedge \ldots \wedge dt_k$

Step 4: $\int_c d\alpha = \int_{[0,1]^k} d(c^*\alpha) = \Sigma_i\, (-1)^{i+1} \int_{[0,1]^k} (\partial_i g_i)\, dt_1 dt_2 \ldots dt_k$ // no ^

Step 5: $\int_c d\alpha = \Sigma_i\, (-1)^{i+1} \int_{[0,1]^{k-1}} \left[\int_{[0,1]} (\partial_i g_i)\, dt_i \right] dt_1 dt_2 \ldots dt_i \ldots dt_k$

Step 6: $\int_c d\alpha = \Sigma_i\, (-1)^{i+1} \int_{[0,1]^{k-1}} [\, [g_i]_{i,1} - [g_i]_{i,0}]\, dt_1 dt_2 \ldots dt_i \ldots dt_k$

$= \Sigma_i\, \Sigma_\rho\, (-1)^{i+\rho}\, \Sigma_\rho \int_{[0,1]^{k-1}} [g_i]_{i,\rho}\, dt_1 dt_2 \ldots dt_i \ldots dt_k$

$= \Sigma_i\, \Sigma_\rho\, (-1)^{i+\rho}\, \Sigma_\rho \int_{[0,1]^{k-1}} [g_i]_{i,\rho}\, dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k$

Step 7: $[\mathbf{c}_{i,\rho}]^*\, \alpha = \Sigma_i\, [g_i]_{i,\rho}\, dt_1 \wedge dt_2 \wedge \ldots dt_i \ldots \wedge dt_k$

Step 8: $\int_c d\alpha = \Sigma_i\, \Sigma_\rho\, (-1)^{i+\rho}\, \Sigma_\rho \int_{[0,1]^{k-1}} [\mathbf{c}_{i,\rho}]^*\, \alpha$

$= \Sigma_i\, \Sigma_\rho\, (-1)^{i+\rho}\, \Sigma_\rho \int_{c(i,\rho)}\, \alpha$

$= \int_{\partial c} \alpha$ // using page 60 D for ∂c expression

I had to add the wedge stuff to make it work right and to agree with Sja's waypoints. Spivak has a similar proof on page 102 that I have not studied. It is not clear, but Spivak also seems to play with removing the ^ operators in a judicious fashion.

[Fla89] H. Flanders, *Differential forms with applications to the physical sciences*, second ed., Dover Publications, New York, 1989.

After a long search I finally landed this Flanders book. But it is like all the rest, there is no definition of the wedge product, he just starts using it. [downloaded again on about 2.11.16].

The last Sja comment concerns the angle form and the circle once again. Given the puncture, we know we cannot write $c = \partial b + c'$ because when we did that, we had to have $c' = \{0\}$ which is not present in the punctured domain U. We know that $\int_c \alpha = 2\pi$. We know that the angle form is closed but not exact, and we showed earlier that $d\alpha = 0$ (closed). But then if $c = \partial b + c'$ *were* true and if you *could* apply Stokes, it would say $2\pi = \int_c \alpha = \int_{\partial b} \alpha = \int_b d\alpha = 0 = \int_b 0 = 0$ and we would get a contradiction.

Stokes

Exercise 5.6. This is the big payoff for the general theorem $\int_c d\alpha = \int_{\partial c} \alpha$.

Recall this table from earlier:

$\alpha = f$	0-form in R^n	$\alpha \leftrightarrow f$	PL1
$d\alpha = df$	1-form in R^n	$d\alpha \leftrightarrow \nabla f$	// namely, $d\alpha = \nabla f \bullet d\mathbf{x}$
$\alpha = f$	0-form in R^n	$\alpha \leftrightarrow f$	
$*d[*(d\alpha)] = \partial_i{}^2 f = \nabla^2 f$	0-form on R^n	$*d[*(d\alpha)] \leftrightarrow \nabla^2 f$	PL2
$\alpha = \mathbf{F} \bullet d\mathbf{x}$	1-form on R^n	$\alpha \leftrightarrow \mathbf{F}$	
$*[d(*\alpha)] = (\text{div } \mathbf{F})$	0-form on R^n	$*[d(*\alpha)] = \text{div } \mathbf{F}$	PL3
$\alpha = \mathbf{F} \bullet d\mathbf{x}$	1-form in R^3	$\alpha \leftrightarrow \mathbf{F}$	one-to-one
$*(d\alpha) = [\text{curl } \mathbf{F}] \bullet d\mathbf{x}$	1-form in R^3	$*(d\alpha) \leftrightarrow \text{curl } \mathbf{F}$	PL4

(i) Let $d\alpha = \nabla f \bullet d\mathbf{x}$ = a 1-form. Integrate this over a curve. Then

$$\int_c d\alpha = \int_c \nabla f \bullet d\mathbf{x} = \int_c \text{grad } f \bullet d\mathbf{x} = \int_{\partial c} \alpha = \int_{\partial c} f = f(\text{final end}) = f(\text{initial end}) = f(\mathbf{c}(1)) - f(\mathbf{c}(0))$$

This is for any curve parametrized over [0,1]. This appears as my "**other theorem**" in integral theorems doc:

$$\int_{\mathbf{A}}^{\mathbf{B}} \nabla\varphi \bullet d\boldsymbol{\ell} \;=\; \varphi(B) - \varphi(A) \qquad (9) \qquad \text{MM 4-36 p 156}$$

(ii) **Green's Formula**: Let $\alpha = f dx+g dy$. Then $d\alpha = df\,dx + dg\,dy = \partial_{\mathbf{y}} f\, dy dx + \partial_{\mathbf{x}} g dx dy = (\partial_{\mathbf{x}} g - \partial_{\mathbf{y}} f) dx dy$ So

$\alpha = f dx+g dy$	1-form
$d\alpha = (\partial_{\mathbf{x}} g - \partial_{\mathbf{y}} f) dx dy$	2-form

Then let c = 2-cube or 2-chain.

$$\int_{\mathbf{c}} d\alpha \;=\; \int_{\mathbf{c}} (\partial_{\mathbf{x}} g - \partial_{\mathbf{y}} f)\, dx dy \;=\; \int_{\partial \mathbf{c}} \alpha \;=\; \int_{\partial \mathbf{c}} (f dx + g dy) \;\text{ so } \partial c = \text{a curve}$$

This is usually called "Green's Theorem" and appears as a one-liner in the Stokes part of my doc. This can be interpreted as Stokes for $\mathbf{A} = f\hat{\mathbf{x}} + g\hat{\mathbf{y}}$ as I show there.

(iii) The **divergence theorem** (Gauss's theorem, or Gauβ' formula says Sja)

$$\alpha = \mathbf{F} \bullet [*d\mathbf{x}] \qquad d\alpha = d\mathbf{F} \bullet [*d\mathbf{x}] = dF_{\mathbf{i}}\, [*d\mathbf{x}]_{\mathbf{i}} = (\partial_{\mathbf{j}} F_{\mathbf{i}})\, dx_{\mathbf{j}}\, [*d\mathbf{x}]_{\mathbf{i}} = (\partial_{\mathbf{i}} F_{\mathbf{i}})\, dV$$

So Stokes says:

$$\int_{\mathbf{c}} d\alpha \;=\; \int_{\mathbf{c}} [\text{div } \mathbf{F}]\, dV \;=\; \int_{\partial \mathbf{c}} \alpha \;=\; \int_{\partial \mathbf{c}} \mathbf{F} \bullet [*d\mathbf{x}]$$

If x is in $R^{\mathbf{n}}$, then c is a region of $R^{\mathbf{n}}$ and and ∂c is a boundary which is a surface of degree n-1. The object $[*d\mathbf{x}]$ for n = 3 would be $*dx_{\mathbf{1}} = dx_{\mathbf{2}} dx_{\mathbf{3}}$ so that $dx_{\mathbf{1}} * dx = dx_{\mathbf{1}}\, dx_{\mathbf{2}} dx_{\mathbf{3}} = +dV$. Write $[*d\mathbf{x}] = d\mathbf{A}$ if you want to think of this as a patch of area of dimension n-1. But you see that the version here is clearer that my version $\int_{\mathbf{V}} dV\, \nabla \bullet \mathbf{A} \;=\; \int_{\mathbf{S}} d\mathbf{S} \bullet \mathbf{A}$ which is hazy about the meaning of dA except when n = 3.

(iv) **Stokes's theorem** for n = 3

$\alpha = \mathbf{F} \bullet d\mathbf{x}$ = 1-form

$d\alpha = \partial_{\mathbf{j}} F_{\mathbf{i}} dx_{\mathbf{j}} dx_{\mathbf{i}} = \Sigma_{\mathbf{ij,j}\neq\mathbf{i}}\, \partial_{\mathbf{j}} F_{\mathbf{i}} dx_{\mathbf{j}} dx_{\mathbf{i}} = \Sigma_{\mathbf{i<j}} (\partial_{\mathbf{j}} F_{\mathbf{i}} - \partial_{\mathbf{i}} F_{\mathbf{j}}) dx_{\mathbf{j}} dx_{\mathbf{i}}$

$= \Sigma_{\mathbf{i<j}} (\partial_{\mathbf{i}} F_{\mathbf{j}} - \partial_{\mathbf{j}} F_{\mathbf{i}}) dx_{\mathbf{i}} dx_{\mathbf{j}} = [\text{curl F}]_{\mathbf{k}}\, (*dx_{\mathbf{k}}) = \text{curl } \mathbf{F} \bullet (*d\mathbf{x})$ = 2-form

Then if c is a 2-form meaning a surface (open) and ∂c is the boundary of that surface.

$$\int_{\mathbf{c}} d\alpha \;=\; \int_{\mathbf{c}} \text{curl } \mathbf{F} \bullet (*d\mathbf{x}) \;=\; \int_{\partial \mathbf{c}} \mathbf{F} \bullet d\mathbf{x}$$

Compare to integral theorems doc, where again (*d**x**) = d**A** = d**S** = area element.

$$\int_S d\mathbf{S} \bullet (\nabla \times \mathbf{A}) = \oint d\boldsymbol{\ell} \bullet \mathbf{A} \qquad \textbf{(7)} \qquad \text{Schaum 22.60}$$

This concludes (finally) my voyage through Sjamaar Chapter 5. No meta notes yet on Ch 5.

Comment on [*dx] in an area integral.

$$\mathbf{F} \bullet d\mathbf{A} = F_i\, dA_i = F_1 dA_1 + F_2 dA_2 + F_3 dA_3 = F_1 dx_2 dx_3 + F_2 dx_3 dx_1 + F_3 dx_1 dx_2$$

$$= F_1[*dx_1] + F_2[*dx_2] + F_3[*dx_3]$$

$$= \mathbf{F} \bullet [*d\mathbf{x}]$$

So this is not so strange after all.

Exercise 5.5. Consider the famous equations

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Now consider a block such that r = [0,a], θ = [0,π], φ = [0,2π]. This is close enough to a cube. Call this block B. Then the above equations can be written as a set S = **f**(B) . As r,θ,φ scan over the block, we get a certain set S in R^3. This set is "the sphere". So roughly speaking, this is a mapping from the unit cube to a sphere. We could redo it this way

$$x = u_1\sin(\pi u_2)\cos(2\pi u_3)$$
$$y = u_1\sin(\pi u_2)\sin(2\pi u_3)$$
$$z = u_1\cos(\pi u_2)$$

Then this map **x** = **F**(**u**) is a mapping from $[0,1]^3$ a 3D sphere. See Fig 1.7 of tensor doc for the picture of the block. I suppose I could have add a picture of a sphere on the RHS.

So, I know how to map from a unit cube to a sphere. I new it all along really.

Exercise 5.6 to show now the general Stokes' theorem generates all the famous integral theorems:

$$\int_c \nabla f \bullet d\mathbf{x} = f(\mathbf{c}(1)) - f(\mathbf{c}(0)) \qquad \text{"other theorem"}$$

$$\int_c (\partial_x g - \partial_y f)\, dxdy = \int_{\partial c} (f dx + g dy) \qquad \partial c = \text{a curve} \qquad \text{"Green's theorem"}$$

$\int_{\mathbf{c}} [\text{div } \mathbf{F}]\, dV = \int_{\partial \mathbf{c}} \mathbf{F} \bullet [*d\mathbf{x}]$ divergence theorem (Gauss's theorem)

$\int_{\mathbf{c}} \text{curl } \mathbf{F} \bullet (*d\mathbf{x}) = \int_{\partial \mathbf{c}} \mathbf{F} \bullet d\mathbf{x}$ Stokes's theorem