Sjamaar Chapter 10 Notes: Topolopy PhL 8.21.15

I am just browsing this chapter for fun. These notes are only a few pages, so no meta notes were made.

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10.1 Brouwer's fixed point theorem.

Lots of strange new words are defined here:

1. Let A be a subset of M.
Consider the map $\varphi: M \to A$.
Suppose for $\mathbf{x}$ in A (thus also in M) this map reduces to $\varphi(\mathbf{x}) = \mathbf{x}$, the identity map.
So we have a map that is an identity map on A, and is some other map on the rest of M.
Such a mapping is called a "**retraction** of M onto subset A".

Example: let M = *punctured* solid unit sphere including boundary and for $\mathbf{x}$ in M we have $\varphi(x) = \mathbf{x}/\|\mathbf{x}\|$. On the boundary of this sphere ∂M we obviously have $\varphi(\mathbf{x}) = \mathbf{x}$ so this is our A. This mapping therefore is a retraction of the punctured unit sphere onto its surface. We need the puncture top remove x = 0 since for that point $\varphi(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$ is ill-defined, being 0/0.

Question: Is the punctured sphere a manifold? I suspect no because the puncture point is a singularity and derivatives mess up there I think. No! I think that may be OK. It is the compact requirement that says bounded and closed, and the punctured sphere is missing a limit point at the origin, so it is not closed!

Theorem 10.1: (**no retraction theorem**) If M is a compact orientable manifold with a non-empty boundary, there exists NO retraction of M to ∂M.

This is an exceedingly strange theorem. I would think that a solid sphere was a compact orientable manifold. So it must be that puncture that allows the result of the previous example!

The proof is pretty simple, you assume a retraction exists then get a contradiction.

Theorem 10.2: (**Brouwer's fixed point theorem**) .

If a mapping $\mathbf{f}(\mathbf{x})$ from a closed unit ball to itself is smooth, the mapping must have at least one fixed point.

The proof is quite simple and makes use of the retraction idea just defined. For each $\mathbf{x}$ in the sphere, draw the line segment shown through $\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$ and define the boundary point to be $\boldsymbol{\varphi}(\mathbf{x})$, some function. For $\mathbf{x}$ on the boundary, obviously $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{x}$. This is a retraction of the sphere onto the boundary. But Theorem 10.1 above says this is impossible.

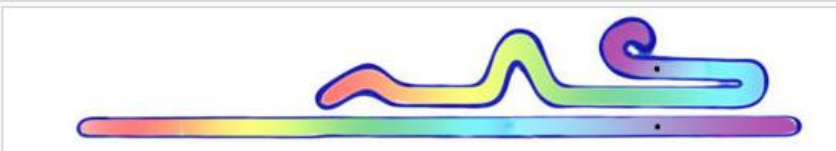
If there *were* a fixed point such that f(**x**) = **x**, then you could not uniquely draw that line segment for that value of x, so φ(**x**) would be undefined and the logic of the proof above falls apart at that **x**.

So the proof here depends on Theorem 10.l which Sja did not prove, fine.

The theorem really uses any compact convex set mapping into itself.

Example: in 1D suppose you map [0,1] to itself in some smooth manner. This is a curve inside the unit box. This curve must cross the diagonal where f(x) = x, so there will be a fixed point there.

Here is the "string" version of the 1D theorem



http://mathforum.org/mathimages/index.php/Brouwer_Fixed_Point_Theorem

Brouwer did this in 1886. Dutch math guy. (1881-1966, killed by car at age 85)
His pet theory: intuitionism. See wiki, things exist only in the mind, not in any assumed reality, sort of.

Notice that 1886 is regarded as ancient history in the realm of topology!

10.2 Homotopy.

Consider a smooth mapping φ : M x [0,1] → N where M and N are manifolds. The 2nd variable is often regarded as t, time, so you then have **n** = **φ**(**m**, t) . At t = 0 you get some N_0 from the map, while at t = 1 you get some different manifold N_1. As time runs from t = 0 to t = 1, you get a movie which shows that N_0 **morphs** in some smooth manner into N_1. The map φ is the **homotopy**.

Example 10.3: **φ**(**x**,t) = (1-t)**x**. Then at t = 0 we have **φ**(**x**,0) = **x** and then φ(x,1) = 0. So the movie shows for example the entire plane E^2 smoothly collapsing into a point at the origin, radial motion for all points.

Example 10.4. φ(**x**,t) = **x** / $\|\mathbf{x}\|^t$. At t = 0 you have an identity map. At t = 1, all points are on a spherical surface. So here Euclidian space is radially morphed into a unit spherical shell. But the Euclidean space has to be punctured at the origin so φ is well defined on domain.

Example 10.5 Here the plan is to have the movie (when run backwards) shrink your entire manifold M into some point $\mathbf{x}_0$ of M. You say that the two endpoints of the movie (two functions) are **homotopic** to each other. If you can do this for some M, that M is said to be **contractible**.

Exercise: Write a φ which contracts the plane to some point in the plane:

$\varphi(\mathbf{x},t) = \mathbf{x}t + \mathbf{x}_o(1-t)$ $\varphi(\mathbf{x},0) = x_0$ $\varphi(\mathbf{x},1) = x$

Homotopic of Curves. Here consider $\varphi : [a,b] \times [0,1] \to N$. For each t, the map takes the unit interval into some curve in N space. Write $\varphi([a,b],[0,1])$ and the domain is the rectangle shown on the left in A which has a continuum of horizontal segments for continuous t, The resulting continuum of curves in N space is shown on the right. Only selected equally spaced values of t are used. All is fine.

Homotopic of Loops. Now the left is a hollow cylinder where as before t goes up, and for every t we have a circle. So φ : (unit circle) x $[0,1] \to N$. Each circle is mapped into a closed curve in N space, and that is the homotopy of loops.

Example 10.6. Here the homotopy movie is of a circle that slides to the right over time. A problem with the move is this: if the loops are all in the punctured plane, then as you slide the loop, it snags on the origin and cannot slide though. It is stuck there. One loop will be forced to hit this point and this that loop (that circle) will have a point which does not live in the punctured plane. As we see later, the real problem is that the initial loop has winding number 1, while the final one has winding number 0, and you cannot have a homotopy that links these two situations.

The cylinder with base M. Think of M x [1,0] where M is the base of some object, and then you make copies of M at various t. All these copies have the same shape, since they are all M, This is different from our loop thing above.

At this point Sja gets into writing differential forms on this "cylinder" thing.

He defines a certain cylinder operator κ which maps a form defined on the cylinder into a form defined only on the base, as shown in p 116 A.

At this point I think I will stop tracking the details.

He eventually gets to Corollary 10.11 which says that homotopic loops must have the same winding number about the origin. This relates to Example 10.6 above where two extremal loops had different winding numbers. I have seen the subject of homotopy talked about for the first time, and it has a connection to manifolds and forms defined on manifolds.

10.3 Closed and exact forms: revisited.

Earlier in the PDF we had much talk of forms being closed and/or exact, and here the subject is resumed.

New definition: a **null-homotopic loop** is one which can be morphed into a point (I think). So a loop around a puncture is not such a loop. Also, if you have a manifold with a hole in it (not simply connected), then a loop around that hole cannot be morphed to a point

Claim 10.14. If c is a null-homotopic loop in M, then $\int_c \alpha = 0$ for all forms α on M. This fact is true for an exact 1-form, by the way.

Fact: A manifold is **simply connected** if every loop is null-homotopic, ie, can be shrunk to a point.

So now we have mixed in exact and closed forms with winding numbers and with connectivity of a manifold. This must have some connection to complex variables for n = 2.

Theorem 10.17. (The Poincare Lemma). If a manifold M is contractible, then all closed k-forms are exact ($k \geq 1$).

Theorem 10.19. Punctured n-space is not contractible.

Theorem 10.20. On punctured R^n (and on S^{n-1}) every closed form of degree $k \neq 1$ and $k \neq n-1$ is exact, The problem is only with 1-forms and n-1 forms.

Prop 10.21. A contractible manifold is simply connected.

Definitions:

2-sphere = 2D unit spherical shell of a 3D sphere. It is a 2-manifold.	S^2
3-sphere = 3D unit spherical shell of a 4D sphere.. It is a 3-manifold.	S^3
n-sphere = nD unit spherical shell of a (n+1)D sphere. It is an n-manifold.	S^n

Poincare Conjecture for n = 2: Is any compact simply-connected 2-manifold M homeomorphic to the surface of a sphere (that surface is called a 2-sphere or S^2). Restated: if every loop on M can be shrunk to a point, then the answer is yes. I think this was proven early on.

The Poincaré Conjecture. The question is this: in n=3 dimensions, is any compact simply-connected 3-manifold M homeomorphic (meaning smooth 1-1 map etc) to a 3-sphere? The "generalized" Poincare conjecture asks for general dimension n, not just dimension n = 3.

If your n-manifold is homotopic to S^n, then maybe it is also homeomorphic to S^n. The problem is a hard one, and as people tried to solve it, the field of topology was "developed". This problem was a driver.

Solution:

n = 2	solved by Poincare I suppose around 1910
$n \geq 5$	solved by Smale in 1960
n = 4	solved by Freedman in 1982
n = 3	solved by Perelman in 2002/2003.

"I once heard an expert "explain" the difficulty of the n=3 case to a general audience by saying something like this: when $n \leq 2$, there isn't enough room for anything to go wrong, while for $n \geq 4$, there's enough room to fix anything that goes wrong; for n=3, there's enough room for something to go wrong, and (this was 15 years ago) it's not clear whether there's enough room to fix things when they go wrong."

OK, so what about fractional dimensions? :)

Perelman used the method of Ricci flow used by Hamilton. He used "Ricci flow with surgery" because he had to show you could fix up all possible problems. Then he refused the Fields Medal and later the $1M Millennium prize feeling he had not earned it, Hamilton did at least in part. He is a Russian born 1966, now age 49. Jewish of course. He can name his game from now on, but seems to have some issues.

Comments on the Sjmaar Book

1. Nowhere is there a metric tensor. Consider this from wiki:

Riemannian manifold

From Wikipedia, the free encyclopedia

> *This article is about a concept from differential geometry. For the algebraic concept, see Zariski–Riemann space.*
>
> *Not to be confused with Riemann surface.*

In differential geometry, a **(smooth) Riemannian manifold** or **(smooth) Riemannian space** (*M*,*g*) is a real smooth manifold *M* equipped with an inner product g_p on the tangent space T_pM at each point p that varies smoothly from point to point in the sense that if *X* and *Y* are vector fields on *M*, then $p \mapsto g_p(X(p), Y(p))$ is a smooth function. The family g_p of inner products is called a Riemannian metric (tensor). These terms are named after the German mathematician Bernhard Riemann. The study of Riemannian manifolds constitutes the subject called Riemannian geometry.

Perhaps in order to have an "inner product" on T_pM at some point p, you have to have a metric tensor on that tangent space, since then

$$\mathbf{a} \bullet \mathbf{b} = \Sigma_{ij}\, g_{ij} a_i b_j \qquad\qquad g_{ij} = \mathbf{e}_i \bullet \mathbf{e}_j$$

So a Riemannian manifold starts out as a regular issue smooth manifold to which we attach a metric tensor at every point. This seems very much like general relativity to me. The tangent space might be the local space at some spacetime point. We shall see eventually.