**Meta Notes for Chapter 7: k-forms on Manifolds** 8.18.15

## 7.1 First definition (of k-form on M)

The idea here is to think of a manifold as being covered by a set of embeddings:

$\psi_i: U_i \rightarrow V_i$

$f_i = \psi_i^* f$ // local representative $f_i$ is a pullback of f in the i-world

We then have the case of overlap between two regions which leads to

$f(\mathbf{x}) = f_i(\mathbf{t}) = f_j(\mathbf{u})$ t in $U_i$ u in $U_j$

This leads to a series of statements like

$f_j = (\psi_i^{-1} \circ \psi_j)^* f_i$ // how $f_i$ of two regions which overlap are related

When this is generalized from the 0-form $f_i$ to a k-form $\alpha_i$ we get

$\alpha_i = \psi_i^* \alpha$

$\alpha_j = (\psi_i^{-1} \circ \psi_j)^* \alpha_i$ which is p 82 (7.1)

In all these equations the * indicates a pullback operation.

Finally we note that if

$\alpha_i = \Sigma_I f_I dt_I$
$\alpha_j = \Sigma_J g_J dt_J$

then we can write

$g_J = \Sigma_I f_I(\varphi(\mathbf{t}))\, \det(D\varphi)_{I,J} = \Sigma_I \varphi^* f_I(\mathbf{t})\, \det(D\varphi)_{I,J}$

where $\varphi = \psi_i^{-1} \circ \psi_j$

No examples are given, this section just gives an *approach* of breaking things down into separate embeddings i (recall that a manifold M is patched together from such a set of embeddings $\psi_i$). By pondering overlap, one obtains strange pullback type relationships between the $f_i$ or the $\alpha_i$ of two regions. Nothing is too mysterious here, the reader just doesn't see the idea applied to anything and has to trust that it is useful.

## 7.2 Second definition (of k-form on M)

This section is immensely more complicated and perhaps is a presentation of the basics of the Grassman theory of differential forms. Two abstract mathematical tools are used. I was unfamiliar with both of them, so that slowed me down.

1. The concept of a **dual space** V* of a vector space being the space of linear functionals on the vector space.

Two examples are given.

Discrete. If $V = R^n$ (or some subset of $R^n$) with *basis vectors* $\mathbf{v}_i$, then $V^* = R^{n*}$ is the usual space whose vectors are "row vectors". For examples, suppose

$$\mathbf{v} = \Sigma_i c_i \mathbf{v}_i$$

Then we can regard $c_i(\mathbf{v})$ as a linear functional which maps $R^n$ to R and which gives the coefficients of the expansion of $\mathbf{v}$. $c_i(\mathbf{v}) = \mathbf{v}_i^T\mathbf{v}$ in matrix notation or $\mathbf{v}_i \bullet \mathbf{v}$ in dot notation. If you suppress the argument of $c_i(\mathbf{v})$ you then just have $c_i = \mathbf{v}_i^T$ . Note that $c_i$ is a linear functional, while $\mathbf{v}_i^T$ is a row vector.

$c_i = \mathbf{v}_i^T$ $\quad$ $c_i$ = a linear functional $\quad$ $\mathbf{v}_i^T$ = a row vector

This is at first very confusing! How can you have $c_i$ a scalar equal to the row vector $\mathbf{v}_i^T$? Well, in the equation $\Sigma_i c_i \mathbf{v}_i$ it is true that $c_i$ is a scalar and you could make a vector $\mathbf{c}$ out of these scalars, and also a row vector $\mathbf{c}^T$. But we have overloaded notation here. The other meaning of $c_i$ (and the main meaning) is that $c_i$ is a functional (not a scalar number) which acts on vectors in $R^n$. In this context, $c_i$ all by itself is a row vector in the dual space of V.

$c_i = \mathbf{v}_i^T$ $\quad$ linear functional represented as a row vector

$c_i(\mathbf{v}) = \mathbf{v}_i^T\mathbf{v} = \mathbf{v}_i \bullet \mathbf{v}$ $\quad$ linear functional acts on a vector $\mathbf{v}$

Sja likes to refer to this $c_i(\mathbf{v})$ as $\lambda_i(\mathbf{v})$ perhaps to match historical notation, or perhaps to make the distinction between $c_i$ as a number and $\lambda_i$ as a functional. The $\lambda_i(\mathbf{v})$ functionals are called **coordinate functions** I guess since there is one for each coordinate i. We can repeat things above in this notation

$\mathbf{v} = \Sigma_i \lambda_i(\mathbf{v})\mathbf{v}_i$ $\quad$ // expansion

$\lambda_i = \mathbf{v}_i^T$ $\quad$ // linear functional number i = a row vector

Notice that $\lambda_i$ all by itself is a row vector , it is not $\boldsymbol{\lambda}$ which is a row vector.

It is easy to show that these functionals $\lambda_i$ form a basis in the dual space V*. This basis $\{\lambda_i\}$ is called the dual basis of basis $\{v_i\}$. The **dimensionality** of the dual space V* is the same as that of the original space V.

It is easy also to show that

$\lambda_i(\mathbf{v}_j) = \delta_{i,j}$ page 84 A $\mathbf{v}_i$ = basis vectors of V

Sjamaar confusion of notation:

(1) The notation $\mathbf{v}_i$ is used sometimes to indicate a basis vector, sometimes an arbitrary vector in $R^n$.

(2) Notation $\mathbf{v}_i$ indicates a vector, whereas $(\mathbf{v})_i$ is the scalar component of a vector $\mathbf{v}$. These two objects are easily confused. In physics one writes the latter as $v_i$ without bolding so I guess there is distinction.

Suppose you take the particular basis $\mathbf{v}_i = \mathbf{e}_i$ for $R^n$ which are the "unit vectors". Then equations above become

$\mathbf{v} = \Sigma_i \lambda_i(\mathbf{v})\mathbf{e}_i$ // expansion

$\lambda_i = \mathbf{e}_i^T$ // linear functional number i = row vector $\mathbf{e}_i^T$

Continuous. To get a continuum example we could write V = space of functions {f} and

$f = \Sigma_i c_i f_i$ f, $f_i$ = real functions on some interval

$c_i(f) = \int dx\, f(x)\, s_i(x)$ $s_i$ = some weight function, element of V

Here $c_i$ is then a linear functional of the vector f which is a function. I was happy to get to quote the Riesz Representation Theorem which says every linear functional can be associated with some function in the space V, and in the above $c_i(f)$ is associated with a weight function $s_i(x)$ in V, and in the discrete example, the linear functional $c_i(\mathbf{v})$ is associated with the vector $\mathbf{c}$ of $R^n$. Since there are an infinite number of $s_i(x)$ weight functions, the dimensionality of the dual space is $\infty$, the same as that of the functions space V.

I did some reading elsewhere to get this dual space stuff understood. In that scenario, a vector in V is called a **vector**, whereas a vector in the dual space V* is called a **covector**. This has nothing to do with covariant or contravariant vectors as far as I can tell. It is just the idea $\mathbf{v} = \Sigma_i c_i \mathbf{v}_i$ that a linear combination of vectors partnered with covectors gives you an arbitrary vector in V. It reminds me of how matrix elements and their "cofactors" get added to make a determinant.

2. The second abstract math idea is that of **multilinear functions**. When this maps to reals, you have a multilinear functional, and that is the obvious generalization of a regular linear functional from one vector argument to k vector arguments. A multilinear function has the usual two properties of a linear function but separately in each of its arguments. Very simple idea.

$\lambda(\mathbf{v}_1, \mathbf{v}_2, \alpha\mathbf{v}_3, \mathbf{v}_4) = \alpha\, \lambda(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ // scalar rule

$\lambda(\mathbf{v}_1, \mathbf{v}_2, \mathbf{u} + \mathbf{w}, \mathbf{v}_4) = \lambda(\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}, \mathbf{v}_4) + \lambda(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, \mathbf{v}_4)$ // addition rule

Sja likes to combine these into a single rule like this

$$\lambda(\mathbf{v_1},\mathbf{v_2}, \alpha\mathbf{u} + \beta\mathbf{w}, \mathbf{v_4}) = \alpha\lambda(\mathbf{v_1},\mathbf{v_2}, \mathbf{u}, \mathbf{v_4}) + \beta\lambda(\mathbf{v_1},\mathbf{v_2}, \mathbf{w}, \mathbf{v_4})$$ page 85A

Of particular interest are those multilinear functions which happen to be "alternating" in all arguments. An example is $f(\mathbf{v_1},\mathbf{v_2}...\mathbf{v_k}) = \det(\mathbf{v_1},\mathbf{v_2}...\mathbf{v_k})$.

The space of such *alternating* multilinear functions with k arguments is called $AV^{\mathbf{k}}$ . It is shown that this space is in fact a vector space due to the obvious scalar and addition rules. The space $AV^{\mathbf{k}}$ is very much like a dual space, but it is not clear what the corresponding underlying space would be. If you think of the starting space as being the direct product space $V^{\mathbf{k}}$, then maybe the dual space would be the space of multilinear functionals with k arguments, but the alternating property would not be there.

So much then for these two math ideas: dual spaces and alternating multilinear functions.

Sja then goes on to define a function whose name is not f or g but "$\lambda_1\lambda_2....\lambda_k$" . Here is that definition:

$$"\lambda_1\lambda_2....\lambda_k" = \det[\lambda_i(\mathbf{v_j})]$$

where the matrix is

$$\begin{matrix}
\lambda_1(\mathbf{v_1}) & \lambda_1(\mathbf{v_2}) & \lambda_1(\mathbf{v_3}) & ..... & \lambda_1(\mathbf{v_k}) \\
\lambda_2(\mathbf{v_1}) & \lambda_2(\mathbf{v_2}) & \lambda_2(\mathbf{v_3}) & ..... & \lambda_2(\mathbf{v_k}) \\
... & & & & \\
\lambda_k(\mathbf{v_1}) & \lambda_k(\mathbf{v_2}) & \lambda_k(\mathbf{v_3}) & ..... & \lambda_k(\mathbf{v_k})
\end{matrix}$$

Here these $\lambda_i(\mathbf{v})$ are the coordinate functions described just above in the dual space notes. In this definition, we happen to have 1,2,3... in order, but you could think of them in any order, and in fact you could say

$$"\lambda_{i_1}\lambda_{i_2}....\lambda_{i_k}" = \det[\lambda_{i_k}(\mathbf{v_j})]$$

where the matrix is

$$\begin{matrix}
\lambda_{i_1}(\mathbf{v_1}) & \lambda_{i_1}(\mathbf{v_2}) & \lambda_{i_1}(\mathbf{v_3}) & ..... & \lambda_{i_1}(\mathbf{v_k}) \\
\lambda_{i_2}(\mathbf{v_1}) & \lambda_{i_2}(\mathbf{v_2}) & \lambda_{i_2}(\mathbf{v_3}) & ..... & \lambda_{i_2}(\mathbf{v_k}) \\
... & & & & \\
\lambda_{i_k}(\mathbf{v_1}) & \lambda_{i_k}(\mathbf{v_2}) & \lambda_{i_k}(\mathbf{v_3}) & ..... & \lambda_{i_k}(\mathbf{v_k})
\end{matrix}$$

Notice that the det[...] notation is not very clear until you actually write out the matrix.

We think of $k \leq n$ where n is from $R^{\mathbf{n}}$.

Notice that "$\lambda_{i_1}\lambda_{i_2}....\lambda_{i_k}$" is an element of the space $AV^k$ since it is in fact an alternating multilinear function of order k. As a shorthand, you could write the above as

$$"\lambda_I" = "\lambda_{i_1}\lambda_{i_2}....\lambda_{i_k}" = \det[\lambda_{i_k}(\mathbf{v}_j)] \qquad I = \text{multi-index } i_1 i_2....i_k$$

Or, you could state that I was an *ordered* multi-index, same notation for both. Sja refers this as an *increasing* multi-index which is I guess clearer.

We know from earlier that $\lambda_i(\mathbf{v}_j) = \delta_{i,j}$ . Sja shows then on page 86 this interesting generalization:

$$\lambda_I(\mathbf{e}_J) = \delta_{I,J} \qquad \text{where I and J are increasing multi-indices of order k}$$

This is a very dense notation! If you write it out, it really says

$$"\lambda_{i_1}\lambda_{i_2}....\lambda_{i_k}"(\mathbf{e}_{j_1},\mathbf{e}_{j_2}....\mathbf{e}_{j_k}) = \delta_{i_1,j_1}\,\delta_{i_2,j_2}\,.....\,\delta_{i_k,j_k}$$

Here I and J are both *increasing* multi-indices of order k. I don't think this claim is true if you make them arbitrary multi-indices, but I don't know and that is not our interest here.

We then arrive at this claim:

**Theorem 7.14:** The set of $\lambda_I$ multilinear alternating functionals for all increasing multi-indices I forms a complete basis for the space $AV^k$ of general alternating multilinear functionals.

The proof is easy once you have shown that $\lambda_I(\mathbf{e}_J) = \delta_{I,J}$ .

Now I have intentionally *deferred* the next four very important claims of Sjamaar which serve to conclude the Chapter:

Claim 1. (**$dx_i$ connection**) If we take $\{\mathbf{e}_i\}$ as our basis of interest in $V = R^n$, we showed above that

$$\lambda_i = \mathbf{e}_i^T \qquad = \text{linear functional} = \text{row vector in the dual space}$$

Now we suddenly associate this functional or row vector with the symbol "$dx_i$" which somehow is associated with our differential forms differential object $dx_i$. So we now suddenly write

$$dx_i = \lambda_i = \mathbf{e}_i^T \qquad = \text{linear functional} = \text{row vector in the dual space}$$

and so we now have three notations for the same thing. Then we can write for example

$$"dx_I" = "dx_{i_1}dx_{i_2}....dx_{i_k}" = \det[dx_{i_k}(\mathbf{v}_j)] \qquad I = \text{multi-index } i_1 i_2....i_k$$

and this function name in quotes is reminiscent of the products we had in our differential forms in Ch 2. So many equations above can be restated using this new $dx_i$ symbol! For example,

$dx_I(\mathbf{e}_J) = \delta_{I,J}$ I.J both increasing multi-indices

So this claim is what starts to make a connection between the math of this chapter and differential forms.

Claim 2. The **wedge product** notation

Recall that we defined the function name

$$"\lambda_{i_1}\lambda_{i_2}....\lambda_{i_k}" = \det[\lambda_{i_k}(\mathbf{v}_j)]$$

which we can now write as

$$"dx_{i_1}dx_{i_2}....dx_{i_k}" = \det[dx_{i_k}(\mathbf{v}_j)]$$

Instead of writing out this long name which looks like a product of factors but really is not, we can write things in this way

$$(\lambda_{i_1} \wedge \lambda_{i_2}.... \wedge \lambda_{i_k})(\mathbf{v}_1,\mathbf{v}_2, ...\mathbf{v}_k) = \det[\lambda_{i_k}(\mathbf{v}_j)]$$

$$(dx_{i_1} \wedge dx_{i_2}.... \wedge dx_{i_k})(\mathbf{v}_1,\mathbf{v}_2, ...\mathbf{v}_k) = \det[dx_{i_k}(\mathbf{v}_j)]$$

This then is the definition of the "wedge product" of k linear functionals to form a k-multilinear alternating functional. Various properties of the wedge product are then immediate:

(a) if two factors in the wedge product are the same, the wedge product vanishes
(because in this case the matrix has two rows which are the same)

(b) the wedge product of a $dx_i$ with itself must vanish (for same reason)

Examples for k = 2

For the case k = 2 we find that

$$(\lambda_1 \wedge \lambda_2)(\mathbf{v}_1, \mathbf{v}_2) = \det\left[\begin{pmatrix} \lambda_1(\mathbf{v}_1) & \lambda_1(\mathbf{v}_2) \\ \lambda_2(\mathbf{v}_1) & \lambda_2(\mathbf{v}_2) \end{pmatrix}\right] = \det(\mathbf{a}, \mathbf{b})$$

where $\mathbf{a} = \begin{pmatrix} \lambda_1(\mathbf{v}_1) \\ \lambda_2(\mathbf{v}_1) \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \lambda_1(\mathbf{v}_2) \\ \lambda_2(\mathbf{v}_2) \end{pmatrix}$ which are the two column vectors. I know from tensor doc that this quantity det(**a**,**b**) is in fact the area of a 2-piped spanned by **a** and **b**. Now if we assume the usual $\mathbf{e}_i$ basis we could write

$$\mathbf{a} = \begin{pmatrix} \lambda_1(\mathbf{v}_1) \\ \lambda_2(\mathbf{v}_1) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1^T\mathbf{v}_1 \\ \mathbf{e}_2^T\mathbf{v}_1 \end{pmatrix} = \begin{pmatrix} (\mathbf{v}_1)_1 \\ (\mathbf{v}_1)_2 \end{pmatrix} = \mathbf{v}_1$$

Then we have

$$(\lambda_1 \wedge \lambda_2)(\mathbf{v}_1, \mathbf{v}_2) = \det(\mathbf{v}_1, \mathbf{v}_2)$$

Note: So far, the wedge product is between linear functionals in V*. One could regard two row vectors $\mathbf{a}^T$ and $\mathbf{b}^T$ as linear functionals. Then you could say

$$(\mathbf{a}^T \wedge \mathbf{b}^T)(\mathbf{v}_1, \mathbf{v}_2) = \det \left[ \begin{pmatrix} \mathbf{a}^T\mathbf{v}_1 \\ \mathbf{b}^T\mathbf{v}_1 \end{pmatrix} \begin{pmatrix} \mathbf{a}^T\mathbf{v}_2 \\ \mathbf{b}^T\mathbf{v}_2 \end{pmatrix} \right]$$

which does not seem very exciting. As s special case

$$(\mathbf{a}^T \wedge \mathbf{b}^T)(\mathbf{e}_1, \mathbf{e}_2) = \det \left[ \begin{pmatrix} \mathbf{a}^T\mathbf{e}_1 \\ \mathbf{b}^T\mathbf{e}_1 \end{pmatrix} \begin{pmatrix} \mathbf{a}^T\mathbf{e}_2 \\ \mathbf{b}^T\mathbf{e}_2 \end{pmatrix} \right] = \det \left[ \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right]$$

$$= \det \left[ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right] = \det(\mathbf{a},\mathbf{b}) = \text{number} = \text{area of 2-piped.}$$

Comments.

- So far, a 2-wedge product is only defined between two linear functionals. I can regard these as being the two row vectors $\mathbf{a}^T$ and $\mathbf{b}^T$.

- The *value* of a 2-wedge product is unspecified unless you display the 2-wedge product *arguments*!

Claim 3. **Connection to k-forms** :

Since the $dx_I$ form a *basis* in $AV^k$, the *most general* element of $AV^k$ can be written

$$\alpha = \Sigma_I \; f_I \, dx_I \qquad \text{sum over increasing multi-indices I}$$

or

$$\alpha(\mathbf{v}_1,\mathbf{v}_2, \ldots\mathbf{v}_k) = \Sigma_I \; f_I \, dx_I(\mathbf{v}_1,\mathbf{v}_2, \ldots\mathbf{v}_k)$$

Due to the earlier noted fact that $dx_I(\mathbf{e}_J) = \delta_{I,J}$ we can show that

$$f_I = \alpha(\mathbf{e}_I)$$

Then we can rewrite the above as

$$\alpha = \Sigma_I \; \alpha(\mathbf{e}_I) \, dx_I \qquad \text{sum over increasing multi-indices I}$$

or

$$\alpha(\mathbf{v}_1,\mathbf{v}_2, \ldots\mathbf{v}_k) = \Sigma_I \; \alpha(\mathbf{e}_I) \, dx_I(\mathbf{v}_1,\mathbf{v}_2, \ldots\mathbf{v}_k)$$

In the first notation, Sjamaar "tacks on" **x** in order to make a connection to Chapter 2:

$\alpha_{\mathbf{x}} = \Sigma_{\mathbf{I}}\ f_{\mathbf{I}}(\mathbf{x})\ dx_{\mathbf{I}}$ sum over increasing multi-indices I

or

$\alpha_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}, ...\mathbf{v_k}) = \Sigma_{\mathbf{I}}\ f_{\mathbf{I}}(\mathbf{x})\ dx_{\mathbf{I}}(\mathbf{v_1},\mathbf{v_2}, ...\mathbf{v_k})$

$f_{\mathbf{I}}(\mathbf{x}) = \alpha_{\mathbf{x}}(\mathbf{e_I})$

So finally

$\alpha_{\mathbf{x}} = \Sigma_{\mathbf{I}}\ \alpha_{\mathbf{x}}(\mathbf{e_I})\ dx_{\mathbf{I}}$

$\alpha_{\mathbf{x}}(\mathbf{v_1},\mathbf{v_2}, ...\mathbf{v_k}) = \Sigma_{\mathbf{I}}\ \alpha_{\mathbf{x}}(\mathbf{e_I})\ dx_{\mathbf{I}}(\mathbf{v_1},\mathbf{v_2}, ...\mathbf{v_k})$

Claim 4. In the parallel Grassmann World view of differential forms, we have these **pullback facts**:

$\varphi^*[\alpha_{\mathbf{y}}(\mathbf{e_I})] = \alpha_{\boldsymbol{\varphi}(\mathbf{x})}(\mathbf{e_I})$ // action on the coefficient

$[\varphi^* dy_{\mathbf{I}}(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k})] = dy_{\mathbf{I}}(\ (D\varphi)\mathbf{v_1},\ (D\varphi)\mathbf{v_2}, .,.,.,\ (D\varphi)\mathbf{v_k})$ // action on the differential

We then find that

$\varphi^*\alpha_{\mathbf{y}}(\mathbf{v_1},\mathbf{v_2}.....\mathbf{v_k}) = \alpha_{\boldsymbol{\varphi}(\mathbf{x})}((D\varphi)\mathbf{v_1},(D\varphi)\mathbf{v_2}.....(D\varphi)\mathbf{v_k})$ // page 88A

He and I show that *if* you make this definition of $g_J$ in the Grassmann World,

$g_J = \varphi^*\alpha_{\mathbf{y}}(\mathbf{e_J})$ // which Sja writes as $g_J = (\varphi^*\alpha\ )_{\mathbf{x}}(\mathbf{e_J})$

then you find that $g_J = \Sigma_{\mathbf{I}}\ f_{\mathbf{I}}(\boldsymbol{\varphi}(\mathbf{x}))\ \det(D\varphi)_{\mathbf{I},J}$ which is the same as the Chapter 2 $g_J$.

**Forms on Manifolds.**

Finally we come to the main topic of this chapter! Back in Chap 2 we learned about forms defined on $R^{\mathbf{N}}$, a whole space, or perhaps an open subset of the space. In the current Chap 7 the goal is to show how you deal with forms which are defined not on $R^{\mathbf{N}}$, but on some surface within $R^{\mathbf{N}}$. Rather than try to do this for an arbitrary surface in $R^{\mathbf{N}}$, we do this for a surface which has the properties of being a manifold within $R^{\mathbf{N}}$. Perhaps that manifold is an n-dimensional surface. We know there is a set of embeddings $\psi_{\mathbf{i}}$ which together characterize this manifold and its "cushion" at every point x which allows differentiation in all directions. We are able on this manifold to define differential forms $\alpha_{\mathbf{x}}$ where the k in k-form is ≤ n. The Grassmann theory produces k-forms which are alternating multilinear functions of k vectors. In the final step, when **x** is on a manifold, we take these vectors to lie in the tangent space $T_{\mathbf{x}}M$ which we know has a local $R^{\mathbf{n}}$ of n dimensions. This tangent space is of course different at every **x**, and so we cannot have just a single $R^{\mathbf{n}}$ to deal with, but a continuum of local $R_{\mathbf{n}}$ spaces $T_{\mathbf{x}}M$. The arguments of $\alpha_{\mathbf{x}}$ must be vectors in this local $R_{\mathbf{n}}$ tangent space.

Example 7.17. In this case M = a curve, $\alpha_{\mathbf{x}}(\mathbf{v})$ has only one vector since k = n = 1 of interest here. An example of a 1-form is $\alpha_{\mathbf{x}}(\mathbf{v}) = \pm \| \mathbf{v}(x) \|$. The tangent space $T_{\mathbf{x}}M$ has only one vector v. As we saw in

Buck, this form is related to arc length. Remember:  you always evaluate a k-form on M at a set of vectors which lie in the tangent space.

This whole idea sounds like general relativity and I think I am going to be happy I learned about manifolds now and not at some future time.

The final section on page 89 shows that this Grassmann theory of forms on manifolds produces exactly the same relationship between local representative $\alpha_i$ that we conjectured in Section 7.1. Here they are derived from the theory, not conjectured.

So ends the chapter. There is much still to digest!!

I am now ready to start into Chapter 8 which is already printed.