Sja Chapter 8 meta notes: volume forms PhL 8.23.15

Section 8.1. Volume of an n-piped in R^N. [91]

Theorem 8.3: The volume of an n-piped spanned by $\{\mathbf{a}_i\}$ within N dimensional space is this $(n \le N)$

$$\text{volume of n-piped} = \sqrt{\det(A^T A)}$$

where $A = (\mathbf{a}_1, \mathbf{a}_2 ..... \mathbf{a}_n)$ = N rows and n columns, so A = Nxn = tall

I eventually will get this concept integrated into tensor doc. [did that task on 5.8.16 in Section 8.4 (h)]
I tried that today, but it is a larger subject than I at first thought, since I want to write something about faces of faces of faces in a vector sense. This could be a major overhaul of Appendix B. So don't do that right now! It will be an interesting tensor doc edit session when I get around to it.

Definition 8.1: defining the volume of an n-piped in R_N by a set of four axioms (typical Sjamaar):

So here Sja writes four axioms and uses them to define volume in R^N and that is what leads him to the above theorem.

Corollary 8.4. In the case n = N, A is square so $\det(A^T A) = [\det(A)]^2$ and we get volume = det(A) which is my main tensor doc result for an N-piped in R^N.

Section 8.2. Orientations [94]

In 2D this is the side of a square, in 3D this is the sign of det(A) where $A = [\mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_n]$. If you shuffle basis vectors, you can change the orientation as in $[\mathbf{e}_2, \mathbf{e}_1 .... \mathbf{e}_n] = - [\mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_n]$. He first talks about a generic oriented vector space in this manner, but then you can treat $T_x M$ at each point **x** on a manifold as such an oriented vector space. For this tangent space, he likes the idea of adding the normal first in the list so you would have $(\mathbf{n}, \mathbf{v}_1 .... \mathbf{v}_{n-1})$ where the last set of vectors spans the tangent space.

In more detail: You know there is some well-defined set of +1 orientation $[\mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_n]$ in R^n . Rotate things so that $[\mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_{n-1}]$ spans $T_x M$ for some particular **x**. Note that $\mathbf{e}_n$ is already determined in direction. Now the normal at **x** is orthogonal to the other $\mathbf{e}_i$ of $T_x M$, we know that much. If we were to add the normal to the *end* of the list to get $[\mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_n] = [\mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_{n-1}, \mathbf{n}]$, then you would select $\mathbf{n} = \mathbf{e}_n$ for the direction of your normal! However, I guess convention says to put **n** at the start of the list. Then you must make the choice that $\mathbf{n} = (-1)^{n+1}\mathbf{e}_n$, since in terms of orientation sign,

$$[\mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_n] = [(-1)^{n+1}\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_{n-1}]$$

So this example is showing how you should choose the direction of the normal at point x! This was a question I raised above.

I agree that you can map normals of a manifold to points on a sphere of the same dimension, which sphere points will have that normal. He calls this the **Gauss Map** of M. Idea of M being 2-sided or 1-sided but no mobius yet.

Claim 8.8. If a manifold M is defined by a single smooth equation $\mathbf{c} = \boldsymbol{\varphi}(\mathbf{x})$ where $\mathbf{c}$ is a regular point, then that manifold is orientable. Did not pursue this idea much. Mobius must violate this somehow.

Section 8.3. Volume Forms [96]

Here μ is the differential form for volume, but we don't have a general expression for it over all of M. In typical manifold fashion, we think of the manifold as a composite of mappings and then think about just one of those pieces or patches of M. Then for piece i we get [$\mu = \mu_M$]

$$\mu_i = \psi_i^* \mu = [\det\{(D\psi_i)^T(D\psi_i)\}]^{1/2}\, dt_1 dt_2 ....dt_n \qquad \text{// definitely a pullback statement!}$$

where ψ_i is the embedding function for patch i of the manifold. Notice the obvious A^TA volume thing going on here. Identify $A = (D\psi)$ whose column vectors are a basis of T_xM. This is the "volume form" or perhaps the "measure" at point x on some manifold. It might be the area measure on a 2D surface.

Sja shows in Theorem 8.10 that the above construct satisfies the consistency condition

$$\mu_j = (\psi_i^{-1} \circ \psi_j)^* \mu_i \equiv \varphi^* \mu_i . \qquad \text{where} \quad \psi_i^{-1} \circ \psi_j \equiv \varphi$$

Example 8.11. For the identity map $\psi(\mathbf{t}) = \mathbf{t}$ for U→U We know that $(D\psi) = 1$ and then $(D\psi)^T(D\psi) = 1$ and then the volume form page 96 E is just $\mu_i = 1\ dt_1....dt_n$, the obvious result.

Example 8.12. Here f(t) is a scalar function and we take $\psi(t) = (t,f(t))$ which is that graph thing. We find that

$$(D\psi)(t) = (1, f'(t)) \qquad (D\psi)^T(D\psi) = (1, f'(t))^T (1, f'(t)) = 1 + f'(t)^2$$
$$\mu = \psi^*\mu_M = [\det((D\psi)^T(D\psi))]^{1/2} dt = \sqrt{1 + f'(t)^2}\ dt$$

So this is what you integrate to get the arc length of a curve. I show how this relates to wiki and Buck. Note that here t is really the x of (x,y) for a function y = f(x). This is different from Buck's s which is a parameter along the curve.

Sja uses the word **hypersurface** to refer to a surface of dimension n-1 which lies in R^n. Wiki agrees with this usage, I guess I never really used that word strictly. For me, n-2 surface would also be a hypersurface if n-2 = 7, say.

Comment

1. $*d\mathbf{x}$ as a product of differentials represents a volume element of M which has dim n-1
   For example, if M has dim = 2 within R^3, then $*d\mathbf{x}$ = an area (vector).
   The area elements would be $(*d\mathbf{x})_i$ such as $(*d\mathbf{x})_3 = dx_1dx_2$

So (*d**x**) is an n-1 dimension volume element on M

Theorem 8.14. In n dimensions we know what vector d**x** looks like, page 98 A. Each entry is a trivial 1-form. The Hodge * acts on these to give "all the others" and so the rest of A makes fine sense. The theorem makes this claim:

$$\mathbf{F} \bullet (*d\mathbf{x}) = (\mathbf{F} \bullet \mathbf{n})\, \mu_M$$

For a regular 2D surface in E^3 this is basically the $\mathbf{F} \bullet d\mathbf{A}$ thing where we want to integrate over area of the surface. I usually write this as

$$\mathbf{F} \bullet d\mathbf{A} = (\mathbf{F} \bullet \hat{\mathbf{n}})\, dA$$

and so this theorem is just saying that $dA = \mu_M$ which is the "area form" in my example here.

He gives two proofs of theorem 8.14, I follow through only on the first. Set F = n to get

$$\mathbf{n} \bullet *d\mathbf{x} = \mu_M \qquad \text{like} \qquad n \bullet d\mathbf{A} = dA$$ // this is page 99 A

Example 8.16. Consider hypersurface $\varphi(\mathbf{x}) = c$ where c = scalar = regular value of φ. Then

$$\mathbf{n} = \frac{\nabla\varphi}{\|\nabla\varphi\|} \qquad \frac{\nabla\varphi}{\|\nabla\varphi\|} \bullet *d\mathbf{x} = \mu_M \qquad d\mathbf{x} = \mu_M \frac{\nabla\varphi}{\|\nabla\varphi\|}$$

For a sphere, $\varphi(\mathbf{r}) = |\mathbf{r}| = r$, $\nabla\varphi = \hat{\mathbf{r}}$ and the above says

$$\hat{\mathbf{r}} \bullet *d\mathbf{x} = \mu_M \qquad \text{or} \qquad d\mathbf{x} = \mu_M\, \hat{\mathbf{r}} \qquad \text{and } \mathbf{n} = \hat{\mathbf{r}}$$