Sja Chapter 8 notes: volume forms PhL 8.19.15

Section 8.1. Volume of an n-piped in R^N. [91]

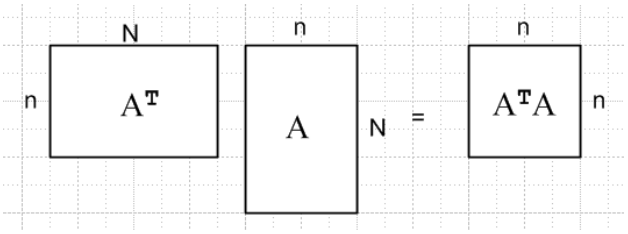
Only to me could this be fascinating. In tensor doc I show that the volume of an N-piped in R^N is det($\mathbf{a}_i$) where the vectors $\mathbf{a}_i$ span the N-piped. This result appears as Corollary 8.4 on page 93. But Sja is looking for an expression for an n-piped in R^N where n ≤ N. A tensor doc example of this would be the area of a face on the N-piped.

The answer says Sjamaar is the following. If an n-piped is spanned by vectors $\mathbf{a}_1$, $\mathbf{a}_2$....$\mathbf{a}_n$, then

volume of n-piped = $\sqrt{\det(A^T A)}$

where A = ($\mathbf{a}_1$, $\mathbf{a}_2$.....$\mathbf{a}_n$) = N rows and n columns, so Nxn = tall

I will review Sja's methods below, but first want to relate this to tensor doc. The conformation picture is this



so then $A^T A$ is indeed a square matrix n x n which then has a determinant.

Now let's track Ch 8 of tensor doc just a bit. This is my most confusing chapter. In my little world x'-space holds a cube, while x-space has the N-piped. Drawing is Fig 8.1. I am at this point only interested in areas of the cube, not sub-areas of lower dimension. In x'-space things are simple and I say

$d\mathcal{L}'^1 \equiv dx'^1$
$d\mathcal{A}'^3 = dx'^1 dx'^2$ $d\mathcal{A}'^1 = dx'^2 dx'^3$ $d\mathcal{A}'^2 = dx'^3 dx'^1$
$d\mathcal{V}' = dx'^1 dx'^2 dx'^3$ (8.2.4)

So 8.1 is overview, then 8.2 shows that drawing/ Then 8.3 summarizes Appendix B. Now in (8.3.7) I have a list of expressions for the vector face area $\mathbf{A}^n$. Not much else. Then into 8.4 for discussion of the differential N-piped and the drawing is repeated.

8.4(a) the setup with this picture repeated
8.4(b) edge transformation : $d\mathbf{x}'^{(n)} = R\, d\mathbf{x}^{(n)}$
8.4(c) area transformation: $d\mathbf{A}'^n = J\, R\, d\mathbf{A}^n$
8.4(d) volume transformation.

I then summarize things at this point:

edges	$d\mathbf{x}'^{(n)} = R\, d\mathbf{x}^{(n)}$	or	$[d\mathbf{x}'^{(n)}]^i = R^i{}_j\, [d\mathbf{x}^{(n)}]^j$	// ordinary vector
areas	$d\mathbf{A}'^n = J\, R\, d\mathbf{A}^n$	or	$(d\mathbf{A}'^n)^i = J\, R^i{}_j\, (d\mathbf{A}^n)^j$	// vector density W = -1
volume	$dV' = J\, dV$			// scalar density W = -1

(8.4.d.8)

I then get this final result set

$$d\mathbf{x}^{(n)} = \mathbf{e}_n\, d\mathcal{L}'^n \quad \Rightarrow |d\mathbf{x}^{(n)}| = h'_n\, d\mathcal{L}'^n \qquad d\mathcal{L}'^n \equiv dx'^n$$
$$d\mathbf{x}'^{(n)} = \mathbf{e}'_n\, d\mathcal{L}'^n \quad \Rightarrow |d\mathbf{x}'^{(n)}| = h'_n\, d\mathcal{L}'^n \quad `$$

$$d\mathbf{A}^n = J\, d\mathcal{A}'^n\, \mathbf{e}^n \quad \Rightarrow |d\mathbf{A}^n| = (|J|\sqrt{g'^{nn}})\, d\mathcal{A}'^n \qquad d\mathcal{A}'^n \equiv \Pi_{i\neq n} dx'^i$$
$$d\mathbf{A}'^n = J^2\, d\mathcal{A}'^n\, \mathbf{e}'^n \quad \Rightarrow |d\mathbf{A}'^n| = (J^2\sqrt{g'^{nn}})\, d\mathcal{A}'^n$$

$$dV = J\, d\mathcal{V}' \quad \Rightarrow |dV| = |J|\, d\mathcal{V}' \qquad d\mathcal{V}' \equiv \Pi_i dx'^i$$
$$dV' = J^2\, d\mathcal{V}' \quad \Rightarrow |dV'| = |J|^2\, d\mathcal{V}' \qquad (8.4.e.5)$$

Later in 8.4 (g) I have "Cartesian view" results which are these

$$|\, d\mathbf{x}^{(n)}|/\, d\mathcal{L}'^n = h'_n = [g'_{nn}]^{1/2} = \text{the scale factor for edge } d\mathbf{x}^{(n)}$$

$$|\, d\mathbf{A}^{(n)}|/\, d\mathcal{A}'^n = \sqrt{g'^{nn}}\, |J| = \sqrt{g'^{nn}}\, g'^{1/2} = [\, g'^{nn}\, g']^{1/2} = [\text{cof}(g'_{nn})]^{1/2} \qquad \text{// Theorem 1 above}$$

$$|dV| / d\mathcal{V}' = |J| = g'^{1/2}\ . \qquad \text{// } g' \equiv \det(g'_{ij}) = J^2 \qquad (8.4.g.2)$$

The result of interest to me right now is this:

$$|\, d\mathbf{A}^{(n)}|/\, d\mathcal{A}'^n = [\text{cof}(g'_{nn})]^{1/2} \qquad \text{// Theorem 1 above}$$

Somehow this must agree with the Sjamaar result. The quantity $\text{cof}(g'_{nn})$ is the same as the "minor " associated with g'_{nn} since nn indicates a diagonal element. If G = 0, then I do have these results in devel notation.

$$g' = RR^T \qquad \text{// } g = 1$$
$$\bar{g}' = S^T S \qquad (5.7.9)$$

So maybe I can say

$$|\, d\mathbf{A}^{(n)}|/\, d\mathcal{A}'^n = [\text{cof}((S^T S)_{nn})]^{1/2}$$

so at least I am seeing an A^TA-like structure appearing. Here n denotes the area spanned by all vectors except n. These are the specific areas on my N-piped spanned by $\mathbf{e}_1, \mathbf{e}_2 .... \mathbf{e}_N$. So lets set n = N so we are then talking about that face N. Then

$$|\, d\mathbf{A}^{(N)}|/\, d\mathcal{A}'^N = [\text{cof}((S^T S)_{NN})]^{1/2} = [\text{minor}((S^T S)_{NN})]^{1/2}$$

Now minor($(S^TS)_{NN}$) = determinant of the matrix S^TS where labels run 1 to N-1. We already know that the matrix S = $\{\mathbf{e}_1, \mathbf{e}_2 ..... \mathbf{e}_N\}$ so let's then define A = $\{\mathbf{e}_1, \mathbf{e}_2 ..... \mathbf{e}_{N-1}\}$ Then I am claiming that

$$| d\mathbf{A}^{(N)} | / d\mathcal{A}'^N = [\det(A^TA)]^{1/2}$$

or

$$| d\mathbf{A}^{(N)} | = \sqrt{\det(A^TA)}\ d\mathcal{A}'^N = \sqrt{\det(A^TA)}\ dx'^1 dx'^2 .... dx'^{N-1}$$

If the N-piped is not differential, then we get

$$| \mathbf{A}^{(N)} | = \sqrt{\det(A^TA)}$$

and so this matches the Sjamaar result.

In 8.4 (h) I talk about "nested cofactor" things for subareas. I just mention one example in passing, but you get the idea of cof (cof(....)) and each iteration for the largest index just knocks another row&column out of the matrix g' = S^TS and then we get the Sjamaar formula! [Add this to tensor doc!! I added a note in the errata to do this.]

In (j) I refer to concatentation of transformations and maybe that should be called composition. Make a note.

Example: Suppose N = 4 so my picture is a hypercube. I knock out $\mathbf{e}_4$ and get A = ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) = the "area" of the face which is spanned by these three vectors = volume of 3-piped in R^4 = det(A^TA) . I could then knock out e_3 as well to get the area of one face of this 3-piped spanned by e_1 and e_2 and for that case I would have A = ($\mathbf{e}_1$, $\mathbf{e}_2$) and area = det(A^TA).

So how does Sjamaar come up with this result? He does one of his "axiom" things. He ponders an n-piped in R^N spanned by $\mathbf{a}_1 ..... \mathbf{a}_n$. Here are his axioms in my words:

Definition 8.1: defining the volume of an n-piped in R_N by a set of four axioms (typical Sjamaar):

(i) if you **scale** any one of the vectors by α, the volume should scale up by α

(ii) Now pause and think about "**shear**". Consider an aligned rectangle in R^2 which you shear to get a 2-piped. Since area = base x height, this "shear" operation does not change the area. Now in terms of the two basis vectors, how do you create a shear? The answer as you show on a piece of paper is that you replace one of the basis vectors like $\mathbf{a}_1$ with $\mathbf{a}_1 + \mathbf{c}$, where c is the constant shear vector. Let $\mathbf{a}_1' = \mathbf{a}_1 + \mathbf{c}$. Before the shear a point in the 2-piped (rectangle) is $\mathbf{r} = \alpha\mathbf{a}_1 + \beta\mathbf{a}_2$ where α,β are in (0.1). After the shear you instead have $\mathbf{r} = \alpha\mathbf{a}'_1 + \beta\mathbf{a}_2$. So Sja's axiom (ii) is that if you add a constant vector to any basis vector, the volume does not change.

(iii) **rotation** does not change a volume.

(iv) the unit cube has volume 1 (**normalization**)

Lemma 8.2. Using *just* his axioms, he shows the result we regard as obvious from physics that the volume of an n-cube is the product of the length of the sides.

Theorem 8.3. Volume on n-piped = $\sqrt{\det(A^T A)}$.

The reader is then tasked with the proof as Exercise 8.2. The proof should just show that this result satisfies the 4 axioms. Sjamaar then shows that the result is a unique result. To do this, he gets all involved in the GSO procedure for non-orthogonal a_i spanning vectors. This is of no interest to me right now so I skip his proof. He shows that if you GSO the $\mathbf{a}_i$ into some orthogonal $\mathbf{v}_i$, then he can use his shear axiom to show that $\det(A^T A) = \det(V^T V)$ = the known result.

Corollary 8.4. In the case n = N, A is square so $\det(A^T A) = [\det(A)]^2$ and we get volume = det(A) which is my main tensor doc result for an N-piped in R^N.

[94]

Section 8.2. Orientations

This is similar to the Buck discussion.

Handedness is determined from basis vectors by whether detA > 0 or < 0, as in tensor doc section 6.9.

Oriented vector space. The standard orientation. Whether a mapping $R^n \rightarrow R^n$ is orientation preserving or reversing.

Notation for "orientation": Use brackets to indicate "orientation" as a + or - quantity. For example

$[\mathbf{e}_1, \mathbf{e}_2 ....\mathbf{e}_n] = +1$ // right handed if you will, det A > 0

Then since det(A) = det $(\mathbf{e}_1, \mathbf{e}_2 ....\mathbf{e}_n)$, we get the alternating property, so for example

$[\mathbf{e}_2, \mathbf{e}_1 ....\mathbf{e}_n] = -[\mathbf{e}_1, \mathbf{e}_2 ....\mathbf{e}_n]$

[95]

Orientation of Manifolds of codimension 1. At any **x** on M we have $T_x M$ with n-1 basis vectors. To that we can add some form of a surface normal vector **n** to get a full basis of R^N. If we add this at the start of the list to get $(\mathbf{n}, \mathbf{v}_1 ....\mathbf{v}_{n-1})$ and if this is right-handed det > 0 then M is positively oriented. But Sja does not say which **n** to pick [but see below]. So you induce orientation by selecting one of these **n**'s.

Example 8.7. For hypersurface you have some orientation at **x** $[\mathbf{e}_1, \mathbf{e}_2 ....\mathbf{e}_{n-1}]$, while for R^n you have $[\mathbf{e}_1, \mathbf{e}_2 ....\mathbf{e}_n]$. The equation p 95A says this obvious fact (based on definitions above)

$[\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_n] = (-1)^{n-1}[\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}]$ but of course $(-1)^{n-1} = (-1)^{n+1}$

where we have slid $\mathbf{e}_n$ to the start position. So you can write

$[\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_n] = [(-1)^{n+1}\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}]$

The idea is this: If you imagine that your hyperspace T_xM has basis vectors named $[\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}]$ with this being +1 say, then if you want to add the local normal vector as "the extra basis vector" to get to R^n, and if you want this normal vector to be at the start of the list, then you should think of $\mathbf{n} = (-1)^{n+1}\mathbf{e}_n$.

In more detail: You know there is some well-defined set of +1 orientation $[\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_n]$. Rotate things so that $[\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}]$ spans T_xM for some particular **x**. Note that $\mathbf{e}_n$ is already determined in direction. Now the normal at **x** is orthogonal to the other $\mathbf{e}_i$ of T_xM, we know that much. If we were to add the normal to the end of the list to get $[\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_n] = [\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}, \mathbf{n}]$, then you would select $\mathbf{n} = \mathbf{e}_n$ for the direction of your normal! However, I guess convention says to put **n** at the start of the list. Then you must make the choice that $\mathbf{n} = (-1)^{n+1}\mathbf{e}_n$. So this example is showing how you should choose the direction of the normal at point x! This was a question I raised above.

I agree that you can map normals of a manifold to points on a sphere of the same dimension, which sphere points will have that normal. He calls this the **Gauss Map** of M. Idea of M being 2-sided or 1-sided but no mobius yet.

Claim 8.8. If a manifold M is defined by a single smooth equation $\mathbf{c} = \boldsymbol{\varphi}(\mathbf{x})$ where **c** is a regular point, then that manifold is orientable. You have then n = gradφ. But I am missing the point as to why Mobius would be excluded from these arguments. Like the examples here, Mobius would have two normals at any point. Something has to be said about going on a global voyage on the surface and finding that your normal has flipped when you get back home. Mobius not mentioned even once in this section.

Section 8.3. Volume Forms [96]

Note the buzzword "forms" (as in differential forms) in the title. Take it slow.

The first idea is that we have some M with its usual $\{\psi_i(U_i)\}$ of embeddings onto chunks of M. In fact, for the first time Sja writes this out as $M = \bigcup_i \psi_i(U_i)$, think of in the sense of $S = \Sigma_i a_i$. This I think is the first time I have every used this "union sum" symbol (used 14 pt bold on usual $\cup$), and it is exactly the right way to state this since there is in general overlap.

Recall from Chapter 6 notes that we can write (today I cleared this notation up!) [see Ch 6 meta notes for my most recent explanation of this notation, search for word "challenge"]

$(D\psi_i) : R^n \rightarrow T_xM$

and in this case, if **t**' exhausts R^n in t-space, we would exhaust T_xM, but here we have in mind that **t**' only runs over a small region U_i in t-space (where of course U_i is in R^n), and so we cover only a portion of T_xM, which is accounted for by the → mapping symbol. [correct]

Now recall from page 89 C or page 82 B that we write $\alpha_i = \psi_i^*\alpha$ which relates α_i which is the local representative of the form α to the form α. [Actually, I think α_i is the local pullback of α] In Section 7.1 this was analogous to the idea for a function f (1 0-form) that we got $f_i = \psi_i^* f$ as the local representative of the function f. The idea is that f and α are defined over all of M, but f_i and α_i are what it looks like in some local region i.

Now consider this definition of a local representative μ_i of a form μ where

$$\mu_i = \psi_i^* \mu = [\det\{(D\psi_i)^T(D\psi_i)\}]^{1/2}\, dt_1 dt_2 .... dt_n \qquad \text{// definitely a pullback statement!}$$

Earlier we showed that if $A = (\mathbf{c}_1, \mathbf{c}_2 .... \mathbf{c}_n)$ and if these vectors $\mathbf{c}_i$ span an n-piped, then the volume of that n-piped is given by $\det(A^TA)$. Now in our tangent space context, we know that $(D\psi) = (\mathbf{c}_1, \mathbf{c}_2 .... \mathbf{c}_n)$ which are the tangent space basis vectors. and so $[\det\{(D\psi)^T(D\psi)\}]^{1/2}$ is exactly the volume of the n-piped formed by these tangent basis vectors! We have to think of (**x**,**t**) here as some unstated reference point for the representative i.

Think of that torus case where the manifold is 2 dimensional, and parameters in U would be t_1 and t_2. There are then only two tangent vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. The above "form" represents the area of a piece of this manifold!!! If the manifold were 3-dimensional, it would be the volume. If M were 1-dimensional (a curve in R^N) this would be the arc-length.

2.15.16. I think this is the first time Sjamaar has ever mentioned computing *area* on a 2D manifold. Bucks deal with this in their book. Here we use the $\det(A^TA)$ formula (in a non-square application) to describe a piece of area on a manifold, but pulled back to t-space where we always deal with actually doing something.

But we are now in the manifold M context, so we can only think of μ_i for a given portion of M.

μ is also called μ_M .

Theorem 8.10. To show that our newly defined "volume form" is "well-defined", Sjamaar shows that it obeys the consistency rule 7.1 page 82, which in the current context requires that one show,

$$\mu_j = (\psi_i^{-1} \circ \psi_j)^* \mu_i \equiv \varphi^* \mu_i . \qquad \text{where} \quad \psi_i^{-1} \circ \psi_j \equiv \varphi$$

In verifying that this works, he starts with p 96 E for μ_i. To this he applies the pullback φ*. This as usual affects the function part and the differentials part of the form, For the function part, we replace **t** by φ(**u**) as the argument. The differential part is the Jacobian rule. Now how does he get j to suddenly appear in the processing? He has a chain rule deal going on:

$D\psi_i(\varphi(\mathbf{u}))\ \ D\varphi(\mathbf{u}) \ = D\psi_j(\mathbf{u})$ is his claim

Where does this come from? Go back to the definition of φ,

$\psi_i^{-1} \circ \psi_j = \varphi$

or

$\psi_j = \psi_i \circ \varphi$

Then this is where you apply the chain rule which reads

$D\psi_j = D\psi_i(\varphi)\, D\varphi$

which he has formalized in App B as p 131 A. So that is how j appears!

[97]

Example 8.11. For the identity map $\psi(\mathbf{t}) = \mathbf{t}$ for U→U We know that $(D\psi) = 1$ and then $(D\psi)^T(D\psi) = 1$ and then the volume form page 96 E is just $\mu_i = 1\ dt_1....dt_n$, the obvious result.

Example 8.12. Now f : I→R (I = an interval) is a simple real function, and (x,f(x)) is the graph thing which we know makes a manifold M of dim = 2. It is of course just the "graph" of the function f and as such is a 1-manifold (a curve in R^2). Think of (t,f(t)) = ψ(t) as the embedding function. What is the arc length?

$(D\psi)(t) = (1, f'(t))$ $\qquad$ $(D\psi)^T(D\psi) = (1, f'(t))^T\,(1, f'(t)) = 1 + f'(t)^2$

where we have reproduced a classic result (see below).

Question: In a situation where one ψ covers all of M, how do we express p 96 E? I guess I need to reinterpret

$\mu = \psi^*\mu_M = [\det((D\psi)^T(D\psi))]^{1/2}dt = \sqrt{1 + f'(t)^2}\ dt$

Here μ is really the pullback of the form μ_M which is abstract because it gets back to the raw pullback dt_i differentials. Confirmed in text! $\psi^*\mu_M$ is the pullback. Note that here t is really the x of (x,y) for a function y = f(x). This is different from Buck's s which is a parameter along the curve.

This is a classical arc length formula, and here is the usual derivation as in wiki

$$(ds)^2 = (dx)^2 + (dy)^2$$
$$\frac{(ds)^2}{(dx)^2} = 1 + \frac{(dy)^2}{(dx)^2}$$
$$\sqrt{\frac{(ds)^2}{(dx)^2}} = \sqrt{1 + \frac{(dy)^2}{(dx)^2}}$$
$$\frac{ds}{dx} = \sqrt{1 + \frac{(dy)^2}{(dx)^2}}$$
$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

So dt in Sja plays the role of x above. Variable s is the arc length parameter, and so we basically have

$\mu = ds = \sqrt{1 + f'(t)^2}\ dt$ = a differential form (but in pulled back form)

$L = \int ds = \int \sqrt{1 + f'(t)^2}\ dt$

2.15.16. In earlier notes (Ch 4?) I pointed out that the 1-form $\alpha = \mathbf{f} \bullet d\mathbf{x}$ was related to a *line integral* and was those separate from the arc length type integral. I felt that an arc length integral did not fit into the 1-form framework. But here we have a differential form μ_M which in fact gives arc length when you look at the pulled-back form of μ_M .

Buck page 320 has a different integral for arc length of a 3D curve.

$$L(\gamma) = \int_a^b \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right\}^{1/2} dt$$

where you integrate the speed. For 1D curve in R^2 this would say

$L = \int dt \sqrt{(dx/dt)^2 + (dy/dt)^2} = \int dt\ (dx/dt) \sqrt{1 + (dy/dx)^2} = \int dx \sqrt{1 + (dy/dx)^2}$

Proposition 8.13. Says that the volume form is the volume of an n-piped of the tangent space. I thought we already knew this? The only issue maybe is the sign?

Proof:

$\omega_{\mathbf{x}}(\mathbf{v}_1....)$ is a function which seems to be exactly the same as $\text{vol}_n(\mathbf{v}_1...)$ at that point.

ω = some n-form, perhaps $\omega_{\mathbf{x}}(\mathbf{v}_1....)\ dx_1....dx_n$? Not clear

$\psi^*\omega = g\ dt_1....dt_n$ = this is the usual general form of a pullback, so I agree there exists some g

$g = \psi^*\omega_{\mathbf{x}}(\mathbf{e}_1....)$ // this is the step I don't follow.

He refers to Lemma 7.12 but this seems unrelated to g. I think he means to quote p 88 D which says

$g_I = \varphi^*\alpha(e_J)$

which in our context becomes

$g = \psi^*\omega(\mathbf{e}_1, \mathbf{e}_2....)$

where e_J = this simple case which is the only thing you can have for a full volume form. This thing is a definition on page 88 which has no reference number!

I agree that he has shown that $g = \psi^*\omega(\mathbf{e}_1, \mathbf{e}_2....)$ = ω_X(of the columns of $D\psi$) = the tangent space volume. We can use 8.3 then to write ω_X(of the columns of $D\psi$) = $sqrt((D\psi)^T(D\psi))$ = vol_n. But I guess his point is that he ends up with p 96 E. This all seems rather circular, as if we are just verifying something, not proving anything.

[98]

Sja uses the word **hypersurface** to refer to a surface of dimension n-1 which lies in R^n. Wiki agrees with this usage, I guess I never really used that word strictly. For me, n-2 surface would also be a hypersurface if n-2 = 7, say.

Theorem 8.14. In n dimensions we know what vector d**x** looks like, page 98 A. Each entry is a trivial 1-form. The Hodge * acts on these to give "all the others" and so the rest of A makes fine sense. The theorem makes this claim:

$\mathbf{F} \bullet (*d\mathbf{x}) = (\mathbf{F} \bullet \mathbf{n})\, \mu_M$

For a regular 2D surface in E^3 this is basically the $\mathbf{F} \bullet d\mathbf{A}$ thing where we want to integrate over area of the surface. I usually write this as

$\mathbf{F} \bullet d\mathbf{A} = (\mathbf{F} \bullet \hat{\mathbf{n}})\, dA$

and so this theorem is just saying that $dA = \mu_M$ which is the "area form" in my example here. For a surface of dimension n-1 within R^n the same idea applies, but that area becomes the n-1 volume element. We will use page 96 E but in n-1 dimensions because we want the n-1 dimension volume element on our hypersurface.

It is 6:45 PM and I am tired for today. Only 2 more pages in Ch 8, things are on the move.

Next day 8/20/15 continue: There are two proofs of Theorem 8.14:

First Proof. I follow it down to equation C. Here are some steps

$\mu_M(\mathbf{e}_1 \ldots) = +1$ because $A^T A = 1$ or just because we know the volume element in p 97C is 1

$\mathbf{n} = (-1)^{n+1}\, \mathbf{e}_n$ we know this from Ex 8.7 above, the choice of **n** direction.

$*dx_i = (-1)^{i+1} dx_1 dx_2 \ldots dx_i \ldots dx_n$ where dx_i is missing, sign due to $dx_i(*dx_i) = +\,dV$

$\mathbf{F} \bullet *d\mathbf{x} = \Sigma_i F_i (*dx)_i = \Sigma_i F_i (-1)^{i+1} dx_1 dx_2 \ldots dx_i \ldots dx_n$ // so C is derived!

$\mathbf{F} \bullet \mathbf{n} = (-1)^{n+1} F_n$ // since $\mathbf{F} = \Sigma_i F_i \mathbf{e}_i$ and $\mathbf{n} = (-1)^{n+1} \mathbf{e}_n$, picks out last term

Now how do we then arrive at equation B? I don't see it. Go back to the line above

$\mathbf{F} \bullet *d\mathbf{x} = \Sigma_i F_i (*dx)_i = \Sigma_i F_i (-1)^{i+1} dx_1 dx_2 \ldots dx_i \ldots dx_n$

Where are "the arguments" here? In Section 7.2 a differential form has all those arguments. OK, we know that the object "$dx_1 dx_2 \ldots dx_i \ldots dx_n$" is an alternating n-1 multilinear function (since one missing), so the above can be written

$(\mathbf{F} \bullet *d\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_{n-1}) = \Sigma_i F_i (*dx)_i = \Sigma_i F_i (-1)^{i+1}\, "dx_1 dx_2 \ldots dx_i \ldots dx_n"(\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_{n-1})$

Now take this particular set of vectors

$(\mathbf{F} \bullet *d\mathbf{x})(\mathbf{e}_1, \mathbf{e}_2 \ldots \mathbf{e}_{n-1}) = \Sigma_i F_i (*dx)_i = \Sigma_i F_i (-1)^{i+1}\, "dx_1 dx_2 \ldots dx_i \ldots dx_n"(\mathbf{e}_1, \mathbf{e}_2 \ldots \mathbf{e}_{n-1})$

Now want to study this object,

$"dx_1 dx_2 \ldots dx_i \ldots dx_n"(\mathbf{e}_1, \mathbf{e}_2 \ldots \mathbf{e}_{n-1})$

and we can then use page 85C to write this as $= \det(dx_i(\mathbf{e}_j))$.

Let's first do a specific example. Let n = 4 and i = 3 so then we have the following matrix,

$dx_1(\mathbf{e}_1)$	$dx_2(\mathbf{e}_1)$	$dx_4(\mathbf{e}_1)$
$dx_1(\mathbf{e}_2)$	$dx_2(\mathbf{e}_2)$	$dx_4(\mathbf{e}_2)$
$dx_1(\mathbf{e}_3)$	$dx_2(\mathbf{e}_3)$	$dx_4(\mathbf{e}_3)$

or

1	0	0
0	1	0
0	0	0

This says that

$"dx_1 dx_2 dx_4"(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 0$

On the other hand, suppose n = 4 and i = 4. Then we get

$$\begin{matrix} dx_1(\mathbf{e}_1) & dx_2(\mathbf{e}_1) & dx_3(\mathbf{e}_1) \\ dx_1(\mathbf{e}_2) & dx_2(\mathbf{e}_2) & dx_3(\mathbf{e}_2) \\ dx_1(\mathbf{e}_3) & dx_2(\mathbf{e}_3) & dx_3(\mathbf{e}_3) \end{matrix}$$

or

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

This says that

$$"dx_1dx_2dx_3"(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$$

So I have shown I think this fact:

$$"dx_1dx_2....dx_i\ ....dx_n"(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}) = \delta_{i,n}$$

In our earlier shorthand notation you could write this as

$$dx_I(\mathbf{e}_J) = \delta_{I,J}$$

and the only want to get I= J is to have i = n. So then we have

$$(\mathbf{F} \bullet *d\mathbf{x})(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}) = \Sigma_i F_i (*dx)_i = \Sigma_i F_i (-1)^{i+1} "dx_1dx_2....dx_i\ ....dx_n"(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1})$$

$$= \Sigma_i F_i (-1)^{i+1} \delta_{i,n}$$

$$= F_n(-1)^{n+1}$$

which says

$$(\mathbf{F} \bullet *d\mathbf{x})(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}) = F_n(-1)^{n+1}$$ // which finally is page 98 B

I wonder how many readers would really follow all this detail? Here then are facts so far:

$$(\mathbf{F} \bullet *d\mathbf{x})(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}) = F_n(-1)^{n+1} \quad (1)$$

$$\mathbf{F} \bullet \mathbf{n} = (-1)^{n+1} F_n \quad (2)$$

$$\mu_M(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}) = +1 \quad (3)$$

From these we then have

$$(\mathbf{F} \bullet *d\mathbf{x})(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1}) = \mathbf{F} \bullet \mathbf{n} = (\mathbf{F} \bullet \mathbf{n})\ \mu_M(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_{n-1})$$ // which is page p98 E.

Comparing both sides says without arguments says

$(\mathbf{F} \bullet *d\mathbf{x}) = (\mathbf{F} \bullet \mathbf{n})\ \mu_M$ // this *is* Theorem 8.14

which is the claim of our theorem!

Second Proof. This one is embedding-based and starts out just fine. But then it devolves into a long complicated thing with about 30 steps each of which I would have to check. The first proof is for people who invested in reading Section 7.2, and I did that, so I will just ignore this second proof. I then skip to below the pencil line on page 99 and continue reading.

Comments on the above theorem.

1. $*d\mathbf{x}$ as a product of differentials represents a volume element of M which has dim n-1
   For example, if M has dim =2 within R^3, then $*d\mathbf{x}$ = an area (vector).
   The area elements would be $(*d\mathbf{x})_i$ such as $(*d\mathbf{x})_3 = dx_1 dx_2$
   So $(*d\mathbf{x})$ is an n-1 dimension volume element on M

2. Write the theorem in this manner

   $(\mathbf{F} \bullet *d\mathbf{x}) = \mathbf{F} \bullet (\mu_M \mathbf{n})$

since this is true for any vector F, it seems clear that in an operational or distributional sense,

$*d\mathbf{x} = \mu_M \mathbf{n}$ like $d\mathbf{A} = dA\ \mathbf{n}$

Here μ_M is the scalar volume of a tiny chunk of M (a volume of dimension n-1)

This shows that $*d\mathbf{x}$ is just a vector version of μ_m. Dot with **n** to get

$\mathbf{n} \bullet *d\mathbf{x} = \mu_M$ like $\mathbf{n} \bullet d\mathbf{A} = dA$ // this is page 99 A

This last result is obtained also from the theorem $(\mathbf{F} \bullet *d\mathbf{x}) = (\mathbf{F} \bullet \mathbf{n})\ \mu_M$ by setting $\mathbf{F} = \mathbf{n}$.

If **F** were a fluid velocity flux $\mathbf{F} = \rho\mathbf{v}$ = (kg/m^3) (m/sec) = mass per area per second, then yes, you would interpret $\mathbf{F} \bullet \mathbf{n}$ as the flux in direction n, and then $(\mathbf{F} \bullet \mathbf{n})\ \mu_M$ = (kg/m^2-sec) m^2 = kg/sec and this is the total mass flow per second per unit area across the surface. So I guess you just imagine this happening for some hypersurface when n > 3.

Example 8.16. Consider hypersurface $\varphi(\mathbf{x}) = c$ where c = scalar = regular value of φ (this is just one equation, hence you get a hypersurface). Such values were discussed on page 73. A regular value for c means that the matrix $(D\varphi)$ has full rank, and also means that the surface is in fact a manifold, and finally that the tangent space is the nullspace of $D\varphi$. What does this last mean?

$(D\varphi)\,\mathbf{v} = 0$ for v in the nullspace of $(D\varphi)$

$(\partial_i \varphi) v_i = 0$

$(\nabla\varphi) \bullet \mathbf{v} = 0$

So the nullspace is any vector perp to the vector $\nabla\varphi$. This is consistent with what I know that $\nabla\varphi$ is the normal to the surface. So one writes that $T_x M = (\nabla\varphi)^{\perp}$ = the space that is perp to vector $\nabla\varphi$.

So for this kind of manifold, we have

$$\mathbf{n} = \frac{\nabla\varphi}{\|\nabla\varphi\|}$$

and then our little results above become

$$\frac{\nabla\varphi}{\|\nabla\varphi\|} \bullet *d\mathbf{x} = \mu_M \qquad \text{or} \qquad d\mathbf{x} = \mu_M \frac{\nabla\varphi}{\|\nabla\varphi\|}$$

Consider the case $\varphi(\mathbf{r}) = |\mathbf{r}| = r$ which is a sphere of radius r. Then $\nabla\varphi = \hat{\mathbf{r}}$ and we get

$$\hat{\mathbf{r}} \bullet *d\mathbf{x} = \mu_M \qquad \text{or} \qquad d\mathbf{x} = \mu_M\, \hat{\mathbf{r}} \qquad \text{and } \mathbf{n} = \hat{\mathbf{r}}$$

And so finally ends Chapter 8!