Sjamaar Chapter 9 Meta Notes PhL 8.23.15

Chapter is 9 pages total include Ex, this meta is 3 pages long.

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9.1 Manifold with boundary

Up to this point, a manifold was always described with embedding $\psi(U) = V \cap M$ where the domain U had to be an *open* set. Here that requirement is softened in a particular way. A special $U' = U \cap H^n$ allows your set U' to have a boundary along $x_n = 0$ where n is the highest coordinate. If this creates a boundary in U', then the boundary M will also have a boundary. I suspect the idea really applies to any U which has any kind of reasonable boundary. The resulting M is called "a **manifold with boundary**" and the boundary is called ∂M of course. Then $M\text{-}\partial M \equiv \text{int}(M)$ is the interior.

Example 9.2 is quite long in my raw notes. It is a particular ψ which results in M being a region in R^2 which is bounded by a curve, as shown page 104. I think this is the first real example of a manifold which has the same dimensionality as the target space (the large space).

Example 9.3 is an enhanced version of the previous example where you end up with a manifold which is a region of R^n which is bounded by a curved hypersurface.

Theorem 9.4 extends the regular value theorem RVT $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{c}$ idea to a case where $\varphi_m \le c_m$ for I guess one of the φ components (instead of =). Obviously now $\varphi_m = c_m$ will produce ∂M instead of M, so

Properties of ∂M
dim(∂M) = N-m codimension(∂M) = N - [N-m] = m // original RVT

Properties of M
dim(M) = N-m+1 codimension(∂M) = N - [N-m+1] = m-1

2.15.16. In Ch 6 we had $\mathbf{c} = \boldsymbol{\varphi}(\mathbf{x})$ being m equations with $\mathbf{x}$ in R^N and this defined a surface of dimension N - m. In the above, we still have m equations but for the *boundary*, so *it* has dim = N-m. The inequality applies to M.

Example 9.5 With $\varphi(\mathbf{x}) = \|\mathbf{x}\|$ and $\varphi = c$ we get a spherical surface as M (radius c). But then $\varphi \le c$ causes that surface to be ∂M and then M is the solid sphere interior.

Fact: This shows that a closed sphere of positive radius is a manifold with boundary, and the boundary in this case is the surface.

For the sphere case if you had $r_1 \le \varphi(\mathbf{x}) = \|\mathbf{x}\| \le r_2$ where now both inequalities are on $\varphi(\mathbf{x})$ and not on coordinates, then you get a **thick spherical shell** and it *is* a manifold, and ∂M is the union of the two spherical surfaces. No sharp corners, things are happy.

Pair of pants is a manifold M, three closed curves comprise ∂M. No sharp corners. Mobius is a manifold, but not orientable because only has one side, a topic Sja did not really explore earlier or here.

Tangent space for ∂M ? If you arrange to have $[\mathbf{n}, \mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_{n-1}]$ be a $T_{\mathbf{x}}M$ basis where **n** is normal to the assumed hypersurface, then $[\mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_{n-1}]$ is a basis for ∂M. If this IS the ∂M basis, then the M basis will be $[\mathbf{n}, \mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_{n-1}] = (-1)^n [\mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_n]$ and so the "orientation" sign (+ or -) will depend on the integer n. You are inducing an orientation on ∂M from M is how he says it.

9.2 Integration over orientable manifolds

I comment on cotangent space.

In **Lemma 9.6** Sja pauses to show that if two mappings **c** and **c'** have the same image M and have the same orientation preserving property as well, then integrals $\int_{[0,1]^n} c^*\alpha$ and $\int_{[0,1]^n} c'^*\alpha$ are the same. I think this would just be a reparametrization situation (with same orientation however).

Now here is a **major claim** without proof: It is possible to represent *any* n-manifold M as the *union* of a finite set of mappings $\mathbf{c}_i$ (each of an n-cube) such that these separate mappings have no overlap on M! Thus we write

$M = \cup_{i=1}^{k} \mathbf{c}_i([0,1]^n)$ where we assume k mappings are required

Each of these $\mathbf{c}_i$ mappings is orientation-preserving.

Given this theorem, we can then say

$$\int_M \alpha = \Sigma_{i=1}^{k} \int_{[0,1]^n} c_i^*\alpha = \Sigma_{i=1}^{k} \int \Sigma_I f_I(\mathbf{c}_i(\mathbf{x})) \det(D\mathbf{c}_i)\, dt_1 dt_2 .....dt_n \qquad \text{// p 107 B}$$

Except for the right expression, this is page 107 B. This then is *a prescription* for doing an integration of any n-form α over any n-manifold M!! So we are combining Chapter 3 on pullbacks with Chapter 5 on Stokes and general pullbacks and "boundaries" ∂c and with Chap 6 on manifolds, and Chap 7 on differential forms on manifolds. Sja points out that the no-intersection requirement removes double counting in overlap regions when doing an integral.

Here are a few sample integral types using the volume measure described above: (the Buck integrals!)

volume of M = $\int_M \mu_M$ n=3 **volume**, n=2 **surface area**, n=1 **arc length**

integral of f on M = $\int_M \mu_M f$

I comment a bit in the raw notes about the connection to Buck, but that topic is really not yet resolved fully. I will do it eventually.

9.3 Gauss' and Stokes' classical theorems

One *major* result now is that Stokes' Theorem can be applied to an "orientable manifold with boundary":

$\int_{\mathbf{M}} d\alpha = \int_{\partial \mathbf{M}} \alpha$ // page 108 A

This section ends with a statement of special cases of this theorem:

M = a 1-manifold:

$\int_{\mathbf{M}} \nabla g \bullet d\mathbf{x} = g(\mathbf{c}(1)) - g(\mathbf{c}(0)) = g(\mathbf{b}) - g(\mathbf{a})$ "other theorem" $\mathbf{F} = \nabla g$

$\int_{\mathbf{M}} \mathbf{F} \bullet d\mathbf{x} = g(\mathbf{c}(1)) - g(\mathbf{c}(0)) = g(\mathbf{b}) - g(\mathbf{a})$ "other theorem" $\mathbf{F} = \nabla f$ p 108 B

This one is not mentioned in Chapter 9

$\int_{\mathbf{M}} (\partial_{\mathbf{x}} g - \partial_{\mathbf{y}} f)\, dxdy = \int_{\partial \mathbf{M}} (f dx + g dy)$ ∂c = a curve "Green's theorem"

M = an n-manifold:

$\int_{\mathbf{M}} [\text{div } \mathbf{F}]\, dx_{\mathbf{1}} dx_{\mathbf{2}} ..... dx_{\mathbf{n}} = \int_{\partial \mathbf{M}} (\mathbf{F} \bullet \mathbf{n})\, \mu_{\partial \mathbf{M}}$ divergence theorem (Gauss's theorem) p 108 D

$\int_{\mathbf{M}} [\text{curl } (\mathbf{F} \bullet \mathbf{n})]\, \mu_{\mathbf{M}} = \int_{\partial \mathbf{M}} \mathbf{F} \bullet d\mathbf{x}$ Stokes's theorem p 109 A

Notice in the second last line that we have $\mu_{\delta \mathbf{M}}$ which is the volume form on the boundary.

For an n→n mapping, you get $\sqrt{\det(A^{\mathbf{T}}A)} = A$ = Jacobian, so then $\varphi^* d\mu$ = Jacobian * $dt_{\mathbf{1}} dt_{\mathbf{2}} ..... dt_{\mathbf{n}}$