Sjamaar Chapter 9 Notes PhL 8.20.15

9.1 Manifold with boundary

In the regular manifold definition 6.4 the last bullet item is this

$\psi(U) = V \cap M$

Here recall that V is a cushion in R^N around **x**, and $V \cap M$ is a patch on the surface of M (if 2D). We could only define the embedding function ψ on a set U which was *open* in R^n (the pullback flat space). In the new definition of "a manifold-with-boundary" we instead have

$\psi(U') = V \cap M$ $U' = U \cap H^n$

where H^n is the n^{th} half-plane in R^n whose boundary is an axis if R^2 or a half space if R^3 and so on. Since the set H^n includes its boundary at $x_n = 0$, it is not an open set. If U lies entirely in H^n then U' will be open, and if U lies in the other half plane, then U' = 0. But if U straddles the boundary of H^n, then the boundary of U is part open and part closed (the part on the boundary), so U' in that case is not open. Thus, we have a slight extension of the manifold idea to sets U' which might not be open. In the mapping ψ, since U' then has some boundary, the manifold M will also have some boundary. But this seems a very particular kind of boundary.

Note. H^i would be for an arbitrary axis i, whereas H^n is for the specific last axis and last variable x_n.

Suppose t in U' is a boundary point. Then I would think $\mathbf{x} = \boldsymbol{\psi}(\mathbf{t})$ on M would be a boundary point on the manifold because the manifold has to suddenly end at such a point. For example, consider the toroid picture on page 68. Suppose the red dot on the left is on the boundary of U'. That maps into a point like the red point **x** (not accurate) which is on the edge of the manifold M. Before Section 9.1, a manifold always had open edges because U was open. So now suddenly M has a boundary ∂M. Then

M - ∂M = M without any boundary = the **interior** of M = int(M) = "complement of M"

The word complement seems wrong here. I guess interior is complement of M-∂M within the world in which M lives and ignoring R^N surrounding space.

Suppose U = all of R^n which I guess one can regard as an open set. Then U' = H^n. This maps into some manifold with boundary under ψ, and that surface with boundary is a "manifold with boundary". If you were to use the map $\psi = 1$, then M = H^n and that is then a trivial example of "manifold with boundary".

Example 9.2. (a new meaning for symbol U') (this is a very long example for me)

U' = open subset of R^{n-1} , contains elements **t**

f = function such that f: U' → R which would be a functional f(**t**) for **t** in U'

U = U' x R = a space whose elements are generically $\begin{pmatrix} \mathbf{t} \\ u \end{pmatrix}$

If we were now to define $\psi_1(\mathbf{t}) = \begin{pmatrix} \mathbf{t} \\ f(\mathbf{t}) \end{pmatrix}$ as on page 68 D, the set of such points would be called the graph of f as discussed there. The region in the graph space is then the surface y = f(**t**).

But instead, we are supposed to think of all points where $y \leq f(\mathbf{t})$. Sja claims that this region is in fact a manifold M with boundary, and y = f(**t**) is the boundary ∂M. He says then that this M is the region "below" the function f(**t**) which the reader can interpret OK if n = 2 or 3 where then **x** as 1 or two components, so we can imagine a surface.

Now how do we prove this claim? We continue defining new things, first generically consider:

$\psi : U \to R^n$ so I think now of R^n as the target larger space.

This is going to be the embedding which defines our manifold within the R^n target space.

Define ψ in this strange manner:

$$\psi\begin{pmatrix} \mathbf{t} \\ u \end{pmatrix} = \begin{pmatrix} \mathbf{t} \\ f(\mathbf{t})-u \end{pmatrix} \qquad \mathbf{t} \text{ in } R^{n-1} \qquad \text{// this is page 104 A}$$

which you see has a left and right side different from the graph idea $\psi_1(\mathbf{t}) = \begin{pmatrix} \mathbf{t} \\ f(\mathbf{t}) \end{pmatrix}$. Notice that in the domain space if you set $\mathbf{v} = \begin{pmatrix} \mathbf{t} \\ u \end{pmatrix}$ in U, then you can say $v_n = u$, so u is then "the n^{th} coordinate". Then if we were to consider $\psi(U \cap H^n)$, then the domain would have the restriction $u \geq 0$.

Now to compute $(D\psi)$ I have to write out details :

$\psi_i = t_i$	i = 1,2...n-1	a	upper rows
$\psi_i = f(t) - u$	i = n	b	bottom row

Let **x** be the variable name overall. Then

$x_j = t_j$	j = 1,2...n-1	c	left columns
$x_j = u$	j = n	d	right column

Consider then:

$(D\psi)_{ij} = \partial_j \psi_i$

I have to study this in each of four regions.

upper left = region ac: $\partial_j \psi_i = \partial_{tj}/t_i = \delta_{i,j}$

upper right = region ad: $\partial_j \psi_i = \partial_u / t_i = 0$

bottom left = region bc: $\partial_j \psi_i = \partial t_j / [\, f(t) - u \,] = \partial f(\mathbf{t}) / \partial t_j = [\nabla_t f(t)]_j = [Df(t)]_j$
bottom right = region bd: $\partial_j \psi_i = \partial_u [\, f(t) - u \,] = -1$

Thus with much effort we arrive at (Dψ) as the matrix shown in page 104 B,

$$(D\psi) = \begin{pmatrix} I_{n-1} & \mathbf{0} \\ [Df(\mathbf{t})] & -1 \end{pmatrix}$$ // this is page 104 B

Note that $Df = \partial f / \partial t_j$ = a row with n-1 partial derivatives in it.

Now Sja claims "by definition then $M = \psi(U \cap H^n)$ is a manifold". Why is that? I guess the main thing you need to show is that (Dψ) is 1-to-1 (hence invertible) so ψ is an embedding. The matrix (Dψ) here is n x n so is square. We have the same number of ψ functions as we have variables. If we evaluate det(Dψ) up the right column, we find that $\det(D\psi) = -\det(I_{n-1}) = -1$, and thus (Dψ) is 1-to-1.

But there is no mention of H^n in this example, at least so far. Now recall that $U = \begin{pmatrix} \mathbf{t} \\ u \end{pmatrix}$ elements, and so we could at least think about $U \cap H^n$ for the specific value of n. The n^{th} variable is u, so the half space H^n is then $u \geq 0$. So $U'' = U \cap H^m$ is the set of all $\mathbf{v} = \begin{pmatrix} \mathbf{t} \\ u \end{pmatrix}$ with $u \geq 0$. This is the function domain. Now look at

$$\psi\begin{pmatrix} \mathbf{t} \\ u \end{pmatrix} = \begin{pmatrix} \mathbf{t} \\ f(\mathbf{t})-u \end{pmatrix} \qquad \psi : U \rightarrow R^n$$

Since $u \geq 0$, on the right for $\psi(U \cap H^n)$, we must have that the lower coordinate is $\leq f(\mathbf{t})$. Think of points on the potential manifold as being **x** where

$$\mathbf{x} = \begin{pmatrix} \mathbf{t} \\ f(\mathbf{t})-u \end{pmatrix} = \psi\begin{pmatrix} \mathbf{t} \\ u \end{pmatrix}$$

The first coordinates are just t (think x,y) and the third is $f(\mathbf{t}) - u$ (think z) . So the manifold that you get is all points $\mathbf{x} = \begin{pmatrix} \mathbf{t} \\ f(\mathbf{t})-u \end{pmatrix}$ such that $x_n \leq f(\mathbf{t})$. This is of course the initially made claim and we now see why it is true after a lot of strange twists and turns. When u = 0 you are on the boundary of H_n and this maps into $x_n = f(\mathbf{t})$.

What is the dimensionality of this M? The ψ function has (n-1)+1 components ψ_i , so the target space is R^n. The arguments of ψ are also n in number! So I would say that M also has dimension n.

Example: See figure p 104. Suppose **t** = t, a scalar, so $R^{n-1} = R^1$ and then n = 2. The target space is R^2. The manifold is the region of R^2 below the curve as shown in gray. this region obviously is of dimension 2.

So this is something different for me: rather than being of dimensionality lower than that of the target space, this manifold is a *portion of* the target space, a subset of it.

Example: Suppose $\mathbf{t} = (t_1, t_2)$ so $R^{n-1} = R^2$ and n = 3. Target is R^3 and region is the 3D region lying below the 2D surface $x_n = x_3 = f(t_1,t_2)$. This region is a 3D portion of R^3.

Example 9.3. Now we try to enhance the previous example by having f: U' → R^m instead of R, so now f is a vector valued function with m components. Here is a comparison of the examples:

Example 9.2.	Example 9.3
U' = open subset of R^{n-1} with elements $\mathbf{t}$	same
f: U' → R f($\mathbf{t}$) for $\mathbf{t}$ in U'	$\mathbf{f}$: U' → R^m t($\mathbf{t}$) for $\mathbf{t}$ in U'
U = U' x R = a space whose elements are generically $\begin{pmatrix} \mathbf{t} \\ u \end{pmatrix}$	same
$\psi\begin{pmatrix} \mathbf{t} \\ u \end{pmatrix} = \begin{pmatrix} \mathbf{t} \\ f(\mathbf{t})-u \end{pmatrix}$ n equations	$\psi\begin{pmatrix} \mathbf{t} \\ u \end{pmatrix} = \begin{pmatrix} \mathbf{t} \\ \mathbf{f}(\mathbf{t})-u\mathbf{e}_m \end{pmatrix}$ n+m-1 equ
$(D\psi) = \begin{pmatrix} I_{n-1} & \mathbf{0} \\ [Df(\mathbf{t})] & -1 \end{pmatrix}$	$(D\psi) = \begin{pmatrix} I_{n-1} & \mathbf{0} \\ [D\mathbf{f}(\mathbf{t})] & -\mathbf{e}_m \end{pmatrix}$

In both cases the mapping has $\begin{pmatrix} \mathbf{t} \\ u \end{pmatrix}$ as the domain vector where $\mathbf{t}$ in U' in R^{n-1}.

But in the second case, ψ now has n-1 +m components ψ_i, so set N = n+m-1 as the target space dimension. Note that $\mathbf{e}_m$ is the last basis vector of R^m.

I won't do all the details but it seems reasonable that the last line above is valid. The $D\psi$ matrix now has m bottom rows instead of just 1 bottom row. Now the matrix $(D\psi)$ has n columns and n+m-1 rows, so cannot talk determinant. How do I know it is full rank? The max possible rank is n. If I look at the minor formed from I_{n-1} and the lower right element which is the 1 from $\mathbf{e}_m$, there is an nxn non vanishing minor so full rank n. Or you can see that the first n-1 rows and the last row must be linearly independent, so again rank = n. So this thing is full rank and that is what you need for 1-to-1 in a non-square matrix scenario, as I learned a while ago in the Sja notes. So this means we really have an embedding, and that is going to become a manifold.

The lower coordinates shown in F are these

$\mathbf{f}(\mathbf{t}) - u\mathbf{e}_m$ with $u \geq 0$

So I guess that means you start at point $\mathbf{f(t)}$ and you go along a ray $\mathbf{f(t)} - u\mathbf{e_m}$ described by $u \geq 0$. This ray is parallel to the m axis. The manifold is all points x of this form,

$$\mathbf{x} = \begin{pmatrix} \mathbf{t} \\ \mathbf{f(t)}\text{-}u\mathbf{e_m} \end{pmatrix}$$ n-1 coords in top, m on the bottom, total n+m-1

which means

$x_i = t_i$	i = 1,2...n-1		
$x_i = f_i(\mathbf{t})$	i = n, n+1..... n+m-2		
$x_i = f_i(\mathbf{t}) - u$	i = n+m-1	=>	$x_{n+m-1} \leq f_i(t)$

If we instead describe our manifold point this way

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{t} \\ \mathbf{f(t)}\text{-}u\mathbf{e_m} \end{pmatrix}$$ n-1 coords in top, m on the bottom, total n+m-1

Then we would say

$x_i = t_i$	i = 1,2...n-1		
$y_i = f_i(\mathbf{t})$	i = 1,2..m-1	=>	$y_1 = f_1(\mathbf{x}), y_2 = f_2(\mathbf{x}). \ldots y_{m-1} = f_{m-1}(\mathbf{x})$
$y_i = f_i(\mathbf{t}) - u$	i = m	=>	$y_m \leq f_m(\mathbf{x})$

So the set of m-1 *equations* and one *inequality* on the right somehow describe manifold M. It must have dimension n since the domain of ψ has n components. For each $\mathbf{x} = t$ in R^{n-1} we get a specific value of y_1, y_2... through y_{m-1} and we get a range for y_m. So the boundary ∂M has a specific value also for y_m and then ∂M must be a surface of dimension n-1since x spans a portion of R^{n-1}. Then M has dimension n and I guess this is a portion of R^n. So I guess you can think about the y_m axis and our manifold M then includes only that portion of R^n for which $y_m \leq f_m(\mathbf{x})$. As $\mathbf{x}$ varies, this endpoint $f_m(\mathbf{x})$ varies, so the boundary ∂M is not a plane, but some curved sheet within R^n which is of dimension n-1. Enough!

Well one more thing. That boundary ∂M is defined by $\mathbf{y} = \mathbf{f(x)}$. Now

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{f(x)} \end{pmatrix}$$ = the graph of $\mathbf{f}$ = the boundary ∂M.

Theorem 9.4 . Regular Value Theorem for a Manifold with Boundary

The following items are the same as for the original theorem:

$U = R^m$ $\mathbf{x}$ in R^N

If we add $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{c}$ where c is a regular value of c, we learned that $\varphi^{-1}(\mathbf{c})$ = manifold of dimension N-m, and thus codimension m.

Now *suppose* we require $\varphi_m \leq c_m$ instead of $\varphi_m = c_m$. Then our previous theorem applies here to the *boundary*, and we would now say $\partial M = \varphi^{-1}(\mathbf{c})$ = a manifold of dimension N-m and codimension m. This is the boundary of some M which has one higher dimension, so

Properties of ∂M
dim(∂M) = N-m codimension(∂M) = N - [N-m] = m

Properties of M
dim(M) = N-m+1 codimension(∂M) = N - [N-m+1] = m-1

So the theorem makes complete sense. Changing to an inequality raises the dimension of M by 1, and thus lowers the codimension by 1. Thankfully Sjamaar does not prove this theorem.

Example 9.5 In this example m = 1 so there is only one function which we take as $\varphi(\mathbf{x}) = c$ where $\mathbf{x}$ is in R^n. Specifically we take $\varphi(\mathbf{x}) = ||\mathbf{x}||$ so then $\varphi(\mathbf{x}) = c$ is a spherical surface of radius c and M = this surface = S^{n-1} if c = 1 (shell embedded in R^n). But now we go to $\boldsymbol{\varphi}(x) \leq c$. Now it is the *boundary* ∂M which is this spherical surface, and M = the solid sphere including the surface ∂M. All points with $||\mathbf{x}|| \leq c$. As before, as long as c > 0 c is a regular value.

Fact: This shows that a closed sphere of positive radius is a manifold with boundary, and the boundary in this case is the surface.

Comments: Suppose you have more than one inequality? If they are in different coordinate components, such as $x \geq 0$ and $y \geq 0$, you have a domain say which is a quadrant, but this has a sharp corner, so we are not led to any manifolds. That quadrant itself for example is not a manifold with boundary because it has a sharp corner. The boundary we know is not a manifold because it has two tangent vectors at that corner, so the Dφ matrix is not defined there. A square is another example of not a manifold for both the closed M and for the boundary ∂M.

However, for the sphere case if you had $r_1 \leq \varphi(\mathbf{x}) = ||\mathbf{x}|| \leq r_2$ where now both inequalities are on $\varphi(\mathbf{x})$ and not on coordinates, then you get a thick spherical shell and it *is* a manifold, and ∂M is the union of the two spherical surfaces. No sharp corners, things are happy.

Pair of pants is a manifold M, three closed curves comprise ∂M. No sharp corners. Mobius is a manifold, but not orientable because only has one side, a topic Sja did not really explore earlier or here.

What is the tangent space of a ∂M? Nice if you can arrange to have one of the tangent vectors point along the boundary as shown twice in page 105 drawing B for the pants. In that example the other tangent vector at the top "points out" while the other one at the bottom "points in".

Seems to me that : If you arrange to have one of the tangent vectors at a boundary point $\mathbf{x}$ be normal to the boundary, then the tangent space for the boundary would be all the "other" tangent vectors of M. I think this is then the basis for saying

$T_{\mathbf{x}}M = [(D\psi)(\mathbf{t})](R^n)$ tangent space is a full R^n space at point $\mathbf{x}$

$T_{\mathbf{x}}\partial M = [(D\psi)(\mathbf{t})](R^{n-1})$ tangent space is a full R^{n-1} space at point **x**

In the second line you just lose that one perp tangent base vector from the set for M (all at **x**).

In the last paragraph Sja does my idea: select the outward pointing normal for **x** at a boundary. Then if it happens that for this **n** you have $[\mathbf{n}, \mathbf{v}_1, \mathbf{v}_2.....\mathbf{v}_{n-1}]$ = for $T_{\mathbf{x}}M$ = +, then $[\mathbf{v}_1, \mathbf{v}_2.....\mathbf{v}_{n-1}]$ is + for ∂M. So the orientation on M **induces** an orientation on ∂M.

Example with $M = H^n$ which is a manifold with boundary. I would say that $\mathbf{n} = -\mathbf{e}_n$ points out. Then we have

$$[\mathbf{n}, \mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_{n-1}] = [-\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_{n-1}] = -(-1)^{n-1}[\mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_n] = (-1)^n[\mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_n] = (-1)^n$$

So $[\mathbf{n}, \mathbf{e}_1, \mathbf{e}_2.....\mathbf{e}_{n-1}]$ could be + or - depending on n.

9.2 Integration over orientable manifolds

This is THE major-major topic of this entire Sjamaar book. We want to know about doing such integrations in general. We have three pages here of text to hack through.

Earlier for example we talked about integrating 1-forms over a curve in R^n, Chapter 4. We did not discuss then whether or not such a curve was a manifold (because manifolds were not really introduced until Ch 6). In Chapter 3 page 41 we had the general pullback formula for a k-form which involved the (Dφ) determinant object, and this means things have to be smooth! So probably these things *will* be smooth if your integration domains are manifolds, just my guess. The manifold requirement makes (Dψ) exist and have full rank so the $T_{\mathbf{x}}M$ space is fully spanned.

I suspect many pieces of the puzzle are going to come together in this final section of the book (apart from Chap 10 which is "advanced topics". I will resume tomorrow.

Footnote: Sjamaar has never mentioned any "**cotangent space**". I see that in other sources. Wiki says that the cotangent space at any point **x** is the dual space to the tangent space which I guess means if would be the space of linear combinations of the tangent vectors, which is to say, the space of all linear functionals of the tangent space vectors. Recall that the dual space contains co-vectors, so co-tangent space parallels that notation. In a linear combination, a covector would be the set of coefficients, a row vector, etc etc.

Now to prep for this section 9.2, go review Chapter 5 notes please. // Done, meta notes only.

Here we go. Assume that a manifold M of dimension n is the image of a map **c** on the unit cube $[0,1]^n$. So we write

$\mathbf{c}([0,1]^n) = M$ $\quad\quad$ $\mathbf{c}: R^n \to R^n$ $\quad\quad$ // n was called k in Chapter 5

Clearly **c** would have to have some properties such that the image M really is a manifold.

Next Sja talks about "the restriction of $\mathbf{c}$ to $(0,1)^n$ is an orientation-preserving embedding" . I guess the idea is that $[0,1]^n$ is a closed set and $(0,1)^n$ is an open set, so we are "restricting" $\mathbf{c}$ to not operate on the boundary of the n-cube ?? Or is there some other meaning of "restriction"? No, I think that is it.

Meanwhile, "orientation" was discussed page 94. There he talked about whether or not a mapping between two vector spaces of the same dimension either preserves or reverses the "orientation" say of the basis vector set $(\mathbf{e}_1, \mathbf{e}_2....\mathbf{e}_n)$. If you think of the function $\mathbf{c}$ above as an embedding $\boldsymbol{\psi}$, then if the mapping $\mathbf{c}$ preserves orientation, it is then an orientation-preserving embedding. Recall that embeddings must have *open* domains, see Def 6.1 on page 67. So I guess that is why we need the "restriction". Good.

So then we have this embedding $\mathbf{c}([0,1]^n) = M$ doing $\mathbf{c}$: $R^n \rightarrow R^n$ and it preserves orientation.

As examples, Sja refers in passing to various Exercises. Ex 3.18 defines "polar" coordinates for spherical coordinates in n dimensions. Ex 5.5 talks about mapping $[0,1]^n$ into a sphere, and I think if you define basis vectors in the usual manner they will be "orientation preserving" under this map. Ex 9.4 seems to do all this for a torus, though the exercise itself makes student figure out this is a torus.

Question: In $\mathbf{c}([0,1]^n) = M$ we have c being an embedding. But you see it is here defined on a non-open set. Sja says nothing about this. The non-open set is *not* just $\mathbf{c}(U \cap H^n)$. I can only assume that the M generated here is a manifold *with boundary* in a more general way than was given earlier. We know from Stokes that there is a boundary ∂c, but back in Chap 5 we had not yet discussed embeddings and manifolds, so it was no big deal that $[0,1]^n$ was a non-open set. So OK, there is a boundary ∂M.

Page 106 A then says that if $\alpha = \Sigma_I \, f_I(\mathbf{x}) \, dx_I$

$$\int_M \alpha = \int_{[0,1]^n} c^*\alpha = \int \Sigma_I \, f_I(\mathbf{c}(\mathbf{x})) \, \det(D\mathbf{c}) \, dt_1 dt_2 .....dt_n$$

using our Chapter 5 notation.

In **Lemma 9.6** Sja pauses to show that if two mappings $\mathbf{c}$ and $\mathbf{c}'$ have the same image M and have the same orientation preserving property as well, then integrals $\int_{[0,1]^n} c^*\alpha$ and $\int_{[0,1]^n} c'^*\alpha$ are the same. I think this would just be a reparametrization situation (with same orientation however).

Now here is a **major claim** without proof: It is possible to represent *any* n-manifold M as the *union* of a finite set of mappings $\mathbf{c}_i$(each of an n-cube) such that these separate mappings have no overlap on M! Thus we write

$M = \cup_{i=1}^{k} \mathbf{c}_i([0,1]^n)$ where we assume k mappings are required

Each of these $\mathbf{c}_i$ mappings is orientation-preserving.

Given this theorem, we can then say

$$\int_M \alpha = \Sigma_{i=1}^{k} \int_{[0,1]n} c_i^*\alpha = \Sigma_{i=1}^{k} \int \Sigma_I f_I(\mathbf{c_i(x)}) \det(D\mathbf{c_i})\, dt_1 dt_2 .....dt_n \qquad \text{// p 107 B}$$

Except for the right expression, this is page 107 B. This then is *a prescription* for doing an integration of any n-form α over any n-manifold M!! So we are combining Chapter 3 on pullbacks with Chapter 5 on Stokes and general pullbacks and "boundaries" ∂c and with Chap 6 on manifolds, and Chap 7 on differential forms on manifolds. Sja points out that the no-intersection requirement removes double counting in overlap regions when doing an integral.

Note: I suppose I could state this form k-forms on an n-manifold, but Sja is anticipating the Stokes' theorem which deals only with n-forms on an n-manifold.

OK, now there are many types of integration that might be of interest to the reader, and each appears in a block of definitions. Recall that μ is the "volume form" of page 96 E, and in our current context which in our current context the pullback of this form is given by

$$\mu_{pullback} = c^*\mu = \sqrt{\det(\mathbf{Dc})^T \det(\mathbf{Dc})}\, dt_1 dt_2 .....dt_n.$$

but in the integrals here we will just show μ and save the pullback for calculation time. So here are those definitions where M is a manifold

volume of M = $\int_M \mu_M$ n=3 **volume**, n=2 **surface area**, n=1 **arc length**

integral of f on M = $\int_M \mu_M f$

mean of f on M = $\int_M \mu_M f / \int_M \mu_M$

mean of x_i over M = $\int_M \mu_M x_i / \int_M \mu_M$ centroid, barycenter

Not surprisingly, Stokes theorem is 5.10 just restated with c → M

$$\int_M d\alpha = \int_{\partial M} \alpha \qquad \text{// page 108 A}$$

Comment. In Buck p 320, we first deal with an arc length integral of this form:

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt \qquad \gamma = \text{a curve}$$

In Sja language, I think you would write this as

$L(\gamma) = \int_a^b ds = \int_a^b \mu_M$ $\qquad\qquad$ $\mu_M = |\gamma'(t)|\, dt = |D\gamma|\, dt = |D\psi|\, dt$

In this Buck section we also have an area integral which has this form

$A(\Sigma) = \int\int_D |\mathbf{n}(u,v)| du dv$ = see page 338 for the long version

How does this appear in Sjamaar language? He would say,

area of $M = \int_M \mu_M = \int\int \sqrt{\det(\mathbf{D\psi})^T \det(\mathbf{D\psi})}\, dt_1 dt_2$ $\qquad\qquad$ $\psi: U \to M$

I will have to ponder this question later. Maybe $\boldsymbol{\psi}(\mathbf{t}) = (\mathbf{t}, \mathbf{f}(t))$?

Now much later on page 367 Bucks talk about integrating a *function* over a curve or surface. EG,

$$\int_\gamma f = \int_a^b f(\gamma(t))|\gamma'(t)|\, dt$$

Sja would write this as $\int_\gamma f\, \mu_M$ where he shows the measure (see just above). Notice that Bucks' result above is in fact in "pullback form", though Bucks do not mention the word pullback. Before doing areas, the Bucks dive into the topic of differential forms! They then show an area integral page 380. I am going to have to review Buck in light of Sjamaar, but not right now! There is certainly a direct connection.

9.3 Gauss' and Stokes' classical theorems

Here is the list of theorems from Chapter 5 and

$\alpha = f$	0-form in R^n	$\alpha \leftrightarrow f$	PL1
$d\alpha = df$	1-form in R^n	$d\alpha \leftrightarrow \nabla f$	// namely, $d\alpha = \nabla f \bullet d\mathbf{x}$
$\alpha = f$	0-form in R^n	$\alpha \leftrightarrow f$	
$*d[*(d\alpha)] = \partial_i^2 f = \nabla^2 f$	0-form on R^n	$*d[*(d\alpha)] \leftrightarrow \nabla^2 f$	PL2
$\alpha = \mathbf{F} \bullet d\mathbf{x}$	1-form on R^n	$\alpha \leftrightarrow \mathbf{F}$	
$*[d(*\alpha)] = (\text{div } \mathbf{F})$	0-form on R^n	$*[d(*\alpha)] = \text{div } \mathbf{F}$	PL3
$\alpha = \mathbf{F} \bullet d\mathbf{x}$	1-form in R^3	$\alpha \leftrightarrow \mathbf{F}$	one-to-one
$*(d\alpha) = [\text{curl } \mathbf{F}] \bullet d\mathbf{x}$	1-form in R^3	$*(d\alpha) \leftrightarrow \text{curl } \mathbf{F}$	PL4

Then recall this from the Ch 5 meta notes:

Exercise 5.6 to show now the general Stokes' theorem generates all the famous integral theorems:

$\int_{\mathbf{c}} \nabla f \bullet d\mathbf{x} \;=\; f(\mathbf{c}(1))-f(\mathbf{c}(0))$ "other theorem"

$\int_{\mathbf{c}} (\partial_{\mathbf{x}} g-\partial_{\mathbf{y}} f)\, dxdy = \int_{\partial \mathbf{c}} (fdx+gdy)$ ∂c = a curve "Green's theorem"

$\int_{\mathbf{c}} [\text{div } \mathbf{F}]\, dV = \int_{\partial \mathbf{c}} \mathbf{F} \bullet [*d\mathbf{x}]$ divergence theorem (Gauss's theorem)

$\int_{\mathbf{c}} \text{curl } \mathbf{F} \bullet (*d\mathbf{x}) = \int_{\partial \mathbf{c}} \mathbf{F} \bullet d\mathbf{x}$ Stokes's theorem

Now recall from Chapter 8 page 98 B that

$(\mathbf{F} \bullet *d\mathbf{x}) = (\mathbf{F} \bullet \mathbf{n})\ \mu_{\mathbf{M}}$

$dV = dx_1 dx_2 .....dx_n$

So if I insert this in two places in the above list, and replace c → M and ∂c → ∂M we get

<u>M = a 1-manifold:</u>

$\int_{\mathbf{M}} \nabla g \bullet d\mathbf{x} \;=\; g(\mathbf{c}(1))-g(\mathbf{c}(0)) = g(\mathbf{b}) - g(\mathbf{a})$ "other theorem" $\mathbf{F} = \nabla g$

$\int_{\mathbf{M}} \mathbf{F} \bullet d\mathbf{x} \;=\; g(\mathbf{c}(1))-g(\mathbf{c}(0)) \;=\; g(\mathbf{b}) - g(\mathbf{a})$ "other theorem" $\mathbf{F} = \nabla f$ p 108 B

<u>This one is not mentioned in Chapter 9</u>

$\int_{\mathbf{M}} (\partial_{\mathbf{x}} g-\partial_{\mathbf{y}} f)\, dxdy = \int_{\partial \mathbf{M}} (fdx+gdy)$ ∂c = a curve "Green's theorem"

<u>M = an n-manifold:</u>

$\int_{\mathbf{M}} [\text{div } \mathbf{F}]\, dx_1 dx_2 .....dx_n = \int_{\partial \mathbf{M}} (\mathbf{F} \bullet \mathbf{n})\, \mu_{\partial \mathbf{M}}$ divergence theorem (Gauss's theorem) p 108 D

$\int_{\mathbf{M}} [\text{curl } (\mathbf{F} \bullet \mathbf{n})]\, \mu_{\mathbf{M}} \;=\; \int_{\partial \mathbf{M}} \mathbf{F} \bullet d\mathbf{x}$ Stokes's theorem p 109 A

Notice in the second last line that we have $\mu_{\delta \mathbf{M}}$ which is the volume form on the boundary.

For an n→n mapping, you get $\sqrt{\det(A^{\mathbf{T}}A)} = A$ = Jacobian, so then $\varphi^* d\mu$ = Jacobian * $dt_1 dt_2 .....dt_n$