Ways to write a determinant PhL 7.9.15

Lagrange doc Appendix B has the key facts, and here I just do a little extension of those facts.

1. I show in (B.2.5) and following that you have these two ways to write det(M):

$$\det(M) = \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \Pi(\mathbf{z}_0,\mathbf{a};M) = \Sigma_{\mathbf{a}}\ \varepsilon_{\mathbf{a}}\ M_{\mathbf{z}_0\mathbf{a}} = \Sigma_{\mathbf{a}}\ \varepsilon_{a_1a_2\cdots a_n}\ M_{1a_1}M_{2a_2}\ \ldots\ldots\ M_{na_n}$$

$$\det(M) = \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \Pi(\mathbf{a},\mathbf{z}_0;M) = \Sigma_{\mathbf{a}}\ \varepsilon_{\mathbf{a}}\ M_{\mathbf{a}\mathbf{z}_0} = \Sigma_{\mathbf{a}}\ \varepsilon_{a_1a_2\cdots a_n}\ M_{a_11}M_{a_22}\ \ldots\ldots\ M_{a_nn} \qquad (1)$$

2. Now here is a slight extension of these two results: start with the first line:

$$\det(M) = \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \Pi(\mathbf{z}_0,\mathbf{a};M)$$

$$= \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \Pi(C\mathbf{z}_0,C\mathbf{a};M)$$

$$= \Sigma_{\mathbf{a}}\ \text{Parity}(C\mathbf{a})\ \text{Parity}(\mathbf{c})\ \Pi(C\mathbf{z}_0,C\mathbf{a};M) \qquad \text{// Parity}(C\mathbf{a})\ \text{Parity}(\mathbf{c}) = \text{Parity}(\mathbf{a})$$

$$= \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \text{Parity}(\mathbf{c})\ \Pi(C\mathbf{z}_0,\mathbf{a};M) \qquad \text{// rearrangement}$$

$$= \text{Parity}(\mathbf{c})\ \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \Pi(\mathbf{c},\mathbf{a};M)$$

$$= \text{Parity}(\mathbf{c})\ \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \ M_{c_1a_1}M_{c_2a_2}\ldots.\ M_{c_na_n} \qquad \mathbf{c} = C\mathbf{z}_0$$

Had we started with the second form in 1, the a and c labels are reversed. So we get

$$\text{Parity}(\mathbf{c})\ \det(M) = \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \ M_{c_1a_1}M_{c_2a_2}\ldots.\ M_{c_na_n}$$

$$\text{Parity}(\mathbf{c})\ \det(M) = \Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \ M_{a_1c_1}M_{a_2c_2}\ldots.\ M_{a_nc_n} \qquad (2a)$$

In compact notation we have

$$\varepsilon_{\mathbf{c}}\ \det(M) = \Sigma_{\mathbf{a}}\ \varepsilon_{\mathbf{a}}\ M_{\mathbf{c}\mathbf{a}}$$
$$\varepsilon_{\mathbf{c}}\ \det(M) = \Sigma_{\mathbf{a}}\ \varepsilon_{\mathbf{a}}\ M_{\mathbf{a}\mathbf{c}} \qquad (2b)$$

In expanded notation

$$\varepsilon_{c_1c_2\cdots c_n}\ \det(M) = \Sigma_{a_1a_2\cdots a_n}\ \varepsilon_{a_1a_2\cdots a_n}\ M_{c_1a_1}M_{c_2a_2}\ldots.\ M_{c_na_n}$$

$$\varepsilon_{c_1c_2\cdots c_n}\ \det(M) = \Sigma_{a_1a_2\cdots a_n}\ \varepsilon_{a_1a_2\cdots a_n}\ M_{a_1c_1}M_{a_2c_2}\ldots.\ M_{a_nc_n} \qquad (2c)$$

If you select $\mathbf{c} = \mathbf{z}_0$ then (2c) replicates (1).

3. Now go back to this idea that

$\mathbf{c} = C\mathbf{z_0}$ = a permutation of (1,2,3...n).

$c_i = \Sigma_j C_{ij}\, j$ since $(\mathbf{z_0})_j = j$

One can think of a permutation as a **mapping**

$c: Z_n \rightarrow Z_n$ $Z_n = \{1,2,....n\}$

In this mapping we have

$1 \rightarrow c_1 = c(1)$ $c_i = c(i) = \Sigma_j C_{ij}\, j$
$2 \rightarrow c_2 = c(2)$
...
$i \rightarrow c_i = c(i)$
....
$n \rightarrow c_n = c(n)$

4. **Sjamaar** writes on page 33

$\det(M) = \Sigma_{\sigma \in Sn} \text{sign}(\sigma)\, M_{1,\sigma(1)} M_{2,\sigma(2)} \ldots\ldots M_{n,\sigma(n)}$ S_n = set of all permutations of Z_n

Well this is really the same as my result above where for fun I replace **a** by **c**

$\det(M) = \Sigma_{\mathbf{c}}\, \text{Parity}(\mathbf{c})\; M_{1c_1} M_{2c_2} \ldots\ldots M_{nc_n}$ $c = \sigma$

So I already have Sjamaar's bases covered!

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5. Update on 10/18/15. Here is yet another form that I need in tensor and wedge

Consider the above result,

$\det(M) = \Sigma_{\sigma \in Sn} \text{sign}(\sigma)\, M_{1,\sigma(1)} M_{2,\sigma(2)} \ldots\ldots M_{n,\sigma(n)}$ S_n = set of all permutations of Z_n

Write this as

$\det(M) = \Sigma_P\, (-1)^{S(P)}\, M_{1,P(1)} M_{2,P(2)} \ldots\ldots M_{n,P(n)}$

$= \Sigma_{a_1 a_2 \cdots a_n}\, \varepsilon_{a_1 a_2 \cdots a_n}\; M_{1a_1} M_{2a_2} \ldots\ldots M_{na_n}$ $P\{1,2....n\} = \{a_1, a_2, ...a_n\}$

We know the following from the rearrangement theorem: (Q can be on either side, a new fact!_

$$\Sigma_P\, f(QP) = \Sigma_P f(P) = \Sigma_P\, f(PQ)$$

so can write the above as

$$\det(M) = \Sigma_P\, (-1)^{S(PQ)}\, M_{1,PQ(1)} M_{2,PQ(2)} \ldots\ldots M_{n,PQ(n)}$$

Now select a particular permutation Q = I which does this:

$$I\{1,2....n\} = \{i_1, i_2, ...i_n\}\ .$$

Then we have

$$\det(M) = \Sigma_P\, (-1)^{S(PI)}\, M_{1,P(i_1)} M_{2,P(i_2)} \ldots\ldots M_{n,P(i_n)}$$

$$= (-1)^{S(I)}\, \Sigma_P\, (-1)^{S(P)}\, M_{1,P(i_1)} M_{2,P(i_2)} \ldots\ldots M_{n,P(i_n)}$$

$$= \varepsilon_{i_1 i_2 \cdots i_n}\, \Sigma_P\, (-1)^{S(P)}\, M_{1,P(i_1)} M_{2,P(i_2)} \ldots\ldots M_{n,P(i_n)}$$

which can be written

$$\Sigma_P\, (-1)^{S(P)}\, M_{1,P(i_1)} M_{2,P(i_2)} \ldots\ldots M_{n,P(i_n)} = \varepsilon_{i_1 i_2 \cdots i_n} \det(M)$$

Notice that this claim is valid only if $\{i_1, i_2, ...i_n\}$ is a permutation of $\{1,2....n\}$.

Noted added 10.3.16. None of this "ways to write" doc relates to the cofactor method of appendix B.3. In fact only Section B.2 is involved, so maybe I can just edit that section to work in the main stuff above. Let's try that now in scratch. // Done, and added to Lagrange Appendix D.