

Geometric Algebra Primer

Jaap Suter

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Abstract

Adopted with great enthusiasm in physics, geometric algebra slowly emerges in computational science. Its elegance and ease of use is unparalleled. By introducing two simple concepts, the multivector and its geometric product, we obtain an algebra that allows subspace arithmetic. It turns out that being able to ‘calculate’ with subspaces is extremely powerful, and solves many of the hacks required by traditional methods. This paper provides an introduction to geometric algebra. The intention is to give the reader an understanding of the basic concepts, so advanced material becomes more accessible.

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Chapter 1

Introduction

1.1 Rationale

Information about geometric algebra is widely available in the field of physics. Knowledge applicable to computer science, graphics in particular, is lacking. As Leo Dorst [1] puts it:

“... A computer scientist first pointed to geometric algebra as a promising way to ‘do geometry’ is likely to find a rather confusing collection of material, of which very little is experienced as immediately relevant to the kind of geometrical problems occurring in practice...”

... After perusing some of these, the computer scientist may well wonder what all the fuss is about, and decide to stick with the old way of doing things ...”

And indeed, disappointed by the mathematical obscurity many people discard geometric algebra as something for academics only. Unfortunately they miss out on the elegance and power that geometric algebra has to offer.

Not only does geometric algebra provide us with new ways to reason about computational geometry, it also embeds and explains all existing theories including complex numbers, quaternions, matrix-algebra, and Plückerspace. Geometric algebra gives us the necessary and unifying tools to express geometry and its relations without the need for tricks or special cases. Ultimately, it makes communicating ideas easier.

1.2 Overview

The layout of the paper is as follows; I start out by talking a bit about subspaces, what they are, what we can do with them and how traditional vectors or one-dimensional subspaces fit in the picture. After that I will define what a

geometric algebra is, and what the fundamental concepts are. This chapter is the most important as all other theory builds upon it. The following chapter will introduce some common and handy concepts which I call tools. They are not fundamental, but useful in many applications. Once we have mastered the fundamentals, and armed with our tools, we can tackle some applications of geometric algebra. It is this chapter that tries to demonstrate the elegance of geometric algebra, and how and where it replaces traditional methods. Finally, I wrap things up, and provide a few references and a roadmap on how to continue a study of geometric algebra..

1.3 Acknowledgements

I would like to thank David Hestenes for his books [7] [8] and papers [10] and Leo Dorst for the papers on his website [6]. Anything you learn from this introduction, you indirectly learned from them.

My gratitude to Per Vognsen for explaining many of the mathematical obscurities that I encountered, and providing me with some of the proofs in this paper. Thanks to Kurt Miller, Conor Stokes, Patrick Harty, Matt Newport, Willem De Boer, Frank A. Krueger and Robert Valkenburg for comments. Finally, I am greatly indebted to Dirk Gerrits. His excellent skills as an editor and his thorough proofreading allowed me to correct many errors.

1.4 Disclaimer

Of course, any mistakes in this text are entirely mine. I only hope to provide an easy-to-read introduction. Proofs will be omitted if the required mathematics are beyond the scope of this paper. Many times only an example or an intuitive outline will be given. I am certain that some of my reasoning won't hold in a thorough mathematical review, but at least you should get an impression. The enthusiastic reader should pick up some of the references to extend his knowledge, learn about some of the subtleties and find the actual proofs.

Chapter 2

Subspaces

It is often neglected that vectors represent 1-dimensional subspaces. This is mainly due to the fact that it seems the only concept at hand. Hence we abuse vectors to form higher-dimensional subspaces. We use them to represent planes by defining normals. We combine them in strange ways to create oriented subspaces. Some papers even mention quaternions as vectors on a 4-dimensional unit hypersphere.

For no apparent reason we have been denying the existence of 2-, 3- and higher-dimensional subspaces as simple concepts, similar to the vector. Geometric algebra introduces these and even defines the operators to perform arithmetic with them. Using geometric algebra we can finally represent planes as true 2-dimensional subspaces, define oriented subspaces, and reveal the true identity of quaternions. We can add and subtract subspaces of different dimensions, and even multiply and divide them, resulting in powerful expressions that can express any geometric relation or concept.

This chapter will demonstrate how vectors represent 1-dimensional subspaces and uses this knowledge to express subspaces of arbitrary dimensions. However, before we get to that, let us consider the very basics by using a familiar example.

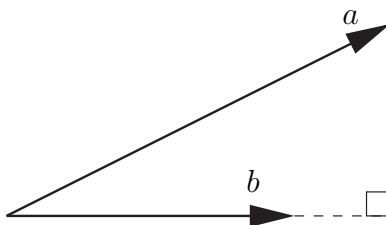


Figure 2.1: The dot product

What if we project a 1-dimensional subspace onto another? The answer is well known; For vectors a and b , the dot product $a \cdot b$ projects a onto b resulting in the scalar magnitude of the projection relative to b 's magnitude. This is

depicted in figure 2.1 for the case where b is a unit vector.

Scalars can be treated as 0-dimensional subspaces. Thus, the projection of a 1-dimensional subspace onto another results in a 0-dimensional subspace.

2.1 Bivectors

Geometric algebra introduces an operator that is in some ways the opposite of the dot product. It is called the outer product and instead of projecting a vector onto another, it extends a vector along another. The $\wedge$ (wedge) symbol is used to denote this operator. Given two vectors a and b , the outer product $a \wedge b$ is depicted in figure 2.2.

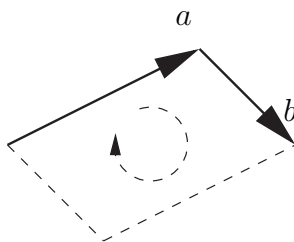


Figure 2.2: Vector a extended along vector b

The resulting entity is a 2-dimensional subspace, and we call it a bivector. It has an area equal to the size of the parallelogram spanned by a and b and an orientation depicted by the clockwise arc. Note that a bivector has no shape. Using a parallelogram to visualize the area provides an intuitive way of understanding, but a bivector is just an oriented area, in the same way a vector is just an oriented length.

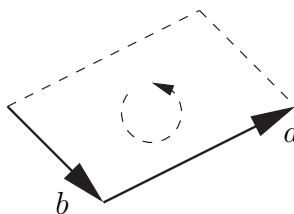


Figure 2.3: Vector b extended along vector a

If b were extended along a the result would be a bivector with the same area but an opposite (ie. counter-clockwise) orientation, as shown in figure 2.3. In mathematical terms; the outer product is anticommutative, which means that:

$$a \wedge b = -b \wedge a \quad (2.1)$$

With the consequence that:

$$a \wedge a = 0 \quad (2.2)$$

which makes sense if you consider that $a \wedge a = -a \wedge a$ and only 0 equals its own negation ($0 = -0$). The geometrical interpretation is a vector extended along itself. Obviously the resulting bivector will have no area.

Some other interesting properties of the outer product are:

$$(\lambda a) \wedge b = \lambda(a \wedge b) \quad \text{associative scalar multiplication} \quad (2.3)$$

$$\lambda(a \wedge b) = (a \wedge b)\lambda \quad \text{commutative scalar multiplication} \quad (2.4)$$

$$a \wedge (b + c) = (a \wedge b) + (a \wedge c) \quad \text{distributive over vector addition} \quad (2.5)$$

For vectors a , b and c and scalar λ . Drawing a few simple sketches should convince you, otherwise most of the references provide proofs.

2.1.1 The Euclidian Plane

Given an n -dimensional vector a there is no way to visualize it until we see a decomposition onto a basis $(e_1, e_2, \dots, e_n)$. In other words, we express a as a linear combination of the basis vectors e_i . This allows us to write a as an n -tuple of real numbers, e.g. (x, y) in two dimensions, (x, y, z) in three, etcetera. Bivectors are much alike; they can be expressed as linear combinations of basis bivectors.

To illustrate, consider two vectors a and b in the Euclidian Plane $\mathbb{R}^2$. Figure 2.4 depicts the real number decomposition $a = (\alpha_1, \alpha_2)$ and $b = (\beta_1, \beta_2)$ onto the basis vectors e_1 and e_2 .

Written down, this decomposition looks as follows:

$$a = \alpha_1 e_1 + \alpha_2 e_2$$

$$b = \beta_1 e_1 + \beta_2 e_2$$

The outer product of a and b becomes:

$$a \wedge b = (\alpha_1 e_1 + \alpha_2 e_2) \wedge (\beta_1 e_1 + \beta_2 e_2)$$

Using (2.5) we may rewrite the above to:

$$\begin{aligned} a \wedge b = & (\alpha_1 e_1 \wedge \beta_1 e_1) + \\ & (\alpha_1 e_1 \wedge \beta_2 e_2) + \\ & (\alpha_2 e_2 \wedge \beta_1 e_1) + \\ & (\alpha_2 e_2 \wedge \beta_2 e_2) \end{aligned}$$

Equation (2.3) and (2.4) tell us we may reorder the scalar multiplications to obtain:

$$\begin{aligned} a \wedge b = & (\alpha_1 \beta_1 e_1 \wedge e_1) + \\ & (\alpha_1 \beta_2 e_1 \wedge e_2) + \\ & (\alpha_2 \beta_1 e_2 \wedge e_1) + \\ & (\alpha_2 \beta_2 e_2 \wedge e_2) \end{aligned}$$

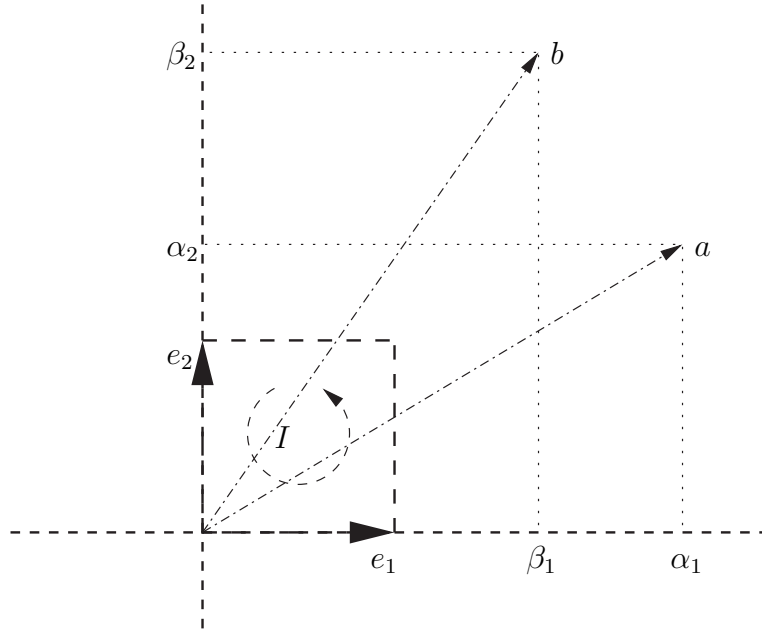


Figure 2.4: A two dimensional basis

Now, recall equation (2.2) which says that the outer product of a vector with itself equals zero. Thus we are left with:

$$a \wedge b = (\alpha_1 \beta_2 e_1 \wedge e_2) + (\alpha_2 \beta_1 e_2 \wedge e_1)$$

Now take another look at figure 2.4. There, I represents the outer product $e_1 \wedge e_2$. This will be our choice for the basis bivector. Because of (2.1) this means that $e_2 \wedge e_1 = -I$. Using this information in the previous equation, we obtain:

$$a \wedge b = (\alpha_1 \beta_2 - \alpha_2 \beta_1) I \quad (2.6)$$

Which is how to calculate the outer product of two vectors $a = (\alpha_1, \alpha_2)$ and $b = (\beta_1, \beta_2)$. Thus, in two dimensions, we express bivectors in terms of a basis bivector called I . In the Euclidian plane we use I to represent $e_{12} = e_1 \wedge e_2$.

2.1.2 Three Dimensions

In 3-dimensional space $\mathbb{R}^3$, things become more complicated. Now, the orthogonal basis consists of three vectors: e_1 , e_2 , and e_3 . As a result, there are three basis bivectors. These are $e_1 \wedge e_2 = e_{12}$, $e_1 \wedge e_3 = e_{13}$, and $e_2 \wedge e_3 = e_{23}$, as depicted in figure 2.5.

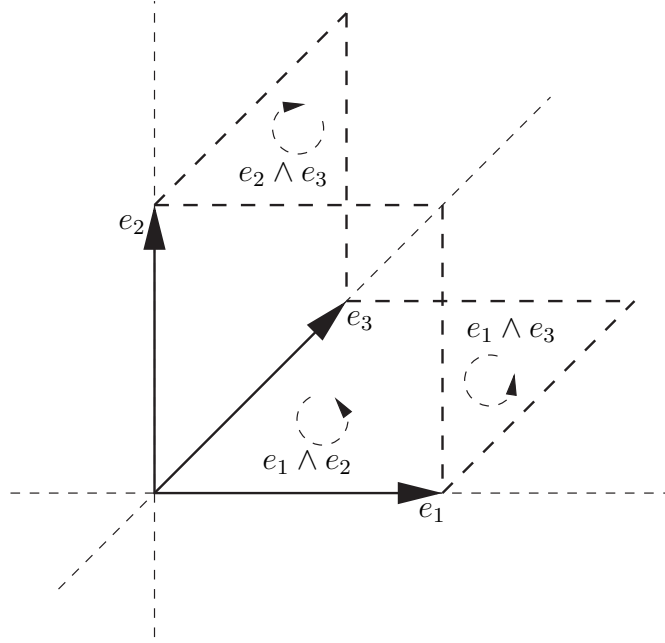


Figure 2.5: A 3-dimensional bivector basis

It is worth noticing that the choice between using either e_{ij} or e_{ji} as a basis bivector is completely arbitrary. Some people prefer to use $\{e_{12}, e_{23}, e_{31}\}$ because it is cyclic, but this argument breaks down in four dimensions or higher; e.g. try making $\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$ cyclic. I use $\{e_{12}, e_{13}, e_{23}\}$ because it solves some issues [12] in computational geometric algebra implementations.

The outer product of two vectors will result in a linear combination of the three basis bivectors. I will demonstrate this by using two vectors a and b :

$$a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

$$b = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3$$

The outer product $a \wedge b$ becomes:

$$a \wedge b = (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) \wedge (\beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3)$$

Using the same rewrite rules as in the previous section, we may rewrite this to:

$$\begin{aligned} a \wedge b = & \alpha_1 e_1 \wedge \beta_1 e_1 + \alpha_1 e_1 \wedge \beta_2 e_2 + \alpha_1 e_1 \wedge \beta_3 e_3 + \\ & \alpha_2 e_2 \wedge \beta_1 e_1 + \alpha_2 e_2 \wedge \beta_2 e_2 + \alpha_2 e_2 \wedge \beta_3 e_3 + \\ & \alpha_3 e_3 \wedge \beta_1 e_1 + \alpha_3 e_3 \wedge \beta_2 e_2 + \alpha_3 e_3 \wedge \beta_3 e_3 \end{aligned}$$

And reordering scalar multiplication:

$$\begin{aligned} a \wedge b = & \alpha_1 \beta_1 e_1 \wedge e_1 + \alpha_1 \beta_2 e_1 \wedge e_2 + \alpha_1 \beta_3 e_1 \wedge e_3 + \\ & \alpha_2 \beta_1 e_2 \wedge e_1 + \alpha_2 \beta_2 e_2 \wedge e_2 + \alpha_2 \beta_3 e_2 \wedge e_3 + \\ & \alpha_3 \beta_1 e_3 \wedge e_1 + \alpha_3 \beta_2 e_3 \wedge e_2 + \alpha_3 \beta_3 e_3 \wedge e_3 \end{aligned}$$

Recalling equations (2.1) and (2.2), we have the following rules for $i \neq j$:

$$\begin{array}{ll} e_i \wedge e_i = 0 & \text{outer product with self is zero} \\ e_i \wedge e_j = e_{ij} & \text{outer product of basis vectors equals basis bivector} \\ e_j \wedge e_i = -e_{ij} & \text{anticommutative} \end{array}$$

Using this, we can rewrite the above to the following:

$$a \wedge b = (\alpha_1 \beta_2 - \alpha_2 \beta_1) e_{12} + (\alpha_1 \beta_3 - \alpha_3 \beta_1) e_{13} + (\alpha_2 \beta_3 - \alpha_3 \beta_2) e_{23} \quad (2.7)$$

which is the outer product of two vectors in 3-dimensional Euclidian space.

For some, this looks remarkably like the definition of the cross product. But they are not the same. The outer product works in all dimensions, whereas the cross product is only defined in three dimensions.¹ Furthermore, the cross product calculates a perpendicular subspace instead of a parallel one. Later we will see why this causes problems in certain situations² and how the outer product solves these.

¹Some cross product definitions are valid in all spaces with uneven dimension.

²If you ever tried transforming a plane, you will remember that you had to use an inverse of a transposed matrix to transform the normal of the plane.

2.2 Trivectors

Until now, we have been using the outer product as an operator of two vectors. The outer product extended a 1-dimensional subspace along another to create a 2-dimensional subspace. What if we extend a 2-dimensional subspace along a 1-dimensional one?

If a , b and c are vectors, then what is the result of $(a \wedge b) \wedge c$? Intuition tells us this should result in a 3-dimensional subspace, which is correct and illustrated in figure 2.6.

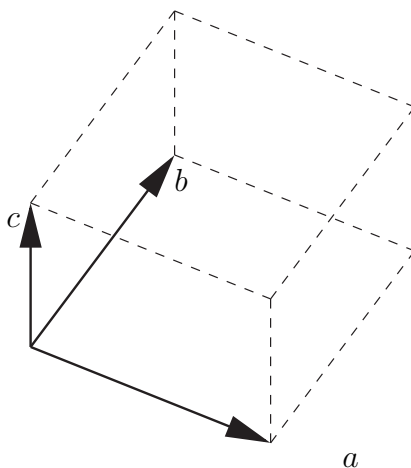


Figure 2.6: A Trivector

A bivector extended by a third vector results in a directed volume element. We call this a trivector. Note that, like bivectors, a trivector has no shape; only volume and sign. Even though a box helps to understand the nature of trivectors intuitively, it could have been any shape.

In 3-dimensional Euclidian space $\mathbb{R}^3$, there is one basis trivector equal to $e_1 \wedge e_2 \wedge e_3 = e_{123}$. Sometimes, in Euclidian space, this trivector is called I . We already saw this symbol being used for e_{12} in the Euclidian plane, and we'll return to it when we discuss pseudo-scalars.

The result of the outer product of three arbitrary vectors results in a scalar multiple of this basis trivector. In 4-dimensional space $\mathbb{R}^4$, there are four basis trivectors e_{123} , e_{124} , e_{134} , and e_{234} , and consequently an arbitrary trivector will be a linear combination of these four basis trivectors. But what about the Euclidian Plane? Obviously, there can be no 3-dimensional subspaces in a 2-dimensional space $\mathbb{R}^2$. The following informal proof demonstrates why trivectors do not exist in two dimensions.

We need to show that for arbitrary vectors a , b , and $c \in \mathbb{R}^2$ the following holds:

$$(a \wedge b) \wedge c = 0$$

Again, we will decompose the vectors onto the basis vectors, using real numbers (α_1, α_2) , (β_1, β_2) , and (γ_1, γ_2) :

$$\begin{aligned}a &= \alpha_1 e_1 + \alpha_2 e_2 \\b &= \beta_1 e_1 + \beta_2 e_2 \\c &= \gamma_1 e_1 + \gamma_2 e_2\end{aligned}$$

Using equation (2.6), we may write:

$$(a \wedge b) \wedge c = ((\alpha_1 \beta_2 - \alpha_2 \beta_1) e_1 \wedge e_2) \wedge (\gamma_1 e_1 + \gamma_2 e_2)$$

We can rewrite this to:

$$((\alpha_1 \beta_2 - \alpha_2 \beta_1) e_1 \wedge e_2) \wedge (\gamma_1 e_1) + ((\alpha_1 \beta_2 - \alpha_2 \beta_1) e_1 \wedge e_2) \wedge (\gamma_2 e_2)$$

Which becomes:

$$(\gamma_1 (\alpha_1 \beta_2 - \alpha_2 \beta_1) e_1 \wedge e_2 \wedge e_1) + (\gamma_2 (\alpha_1 \beta_2 - \alpha_2 \beta_1) e_1 \wedge e_2 \wedge e_2)$$

The scalar parts are not really important. Take a good look at the outer product of the basis vectors. We have:

$$e_1 \wedge e_2 \wedge e_1, \text{ and } e_1 \wedge e_2 \wedge e_2$$

Because the outer product is anticommutative (equation (2.1)), we may rewrite the first one:

$$-e_1 \wedge e_1 \wedge e_2, \text{ and } e_1 \wedge e_2 \wedge e_2$$

And using equation (2.2) which says that the outer product of a vector with itself equals zero, we are left with:

$$-0 \wedge e_2, \text{ and } e_1 \wedge 0$$

From here, it does not take much to realize that the outer product of a vector and the null vector results in zero. I'll come back to a more formal treatment of null vectors later, but for now it should be enough to understand that if we extend a vector by a vector that has no length, we are left with zero area. Thus we conclude that $a \wedge b \wedge c = 0$ in $\mathbb{R}^2$.

2.3 Blades

So far we have seen scalars, vectors, bivectors and trivectors representing 0-, 1-, 2- and 3-dimensional subspaces respectively. Nothing stops us from generalizing all of the above to allow subspaces with arbitrary dimension.

Therefore, we introduce the term k -blades, where k refers to the dimension of the subspace the blade spans. The number k is called the grade of a blade. Scalars are 0-blades, vectors are 1-blades, bivectors are 2-blades, and trivectors are 3-blades. In other words, the grade of a vector is one, and the grade of a trivector is three. In higher dimensional spaces there can be 4-blades, 5-blades, or even higher. As we have shown for $n = 2$ in the previous section, in an n -dimensional space the n -blade is the blade with the highest grade.

Recall how we expressed vectors as a linear combination of basis vectors and bivectors as a linear combination of basis bivectors. It turns out that every k -blade can be decomposed onto a set of basis k -blades. The following tables contain all the basis blades for subspaces of dimensions 2, 3 and 4.

k	basis k -blades	total
0-blades (scalars)	$\{1\}$	1
1-blades (vectors)	$\{e_1, e_2\}$	2
2-blades (bivectors)	$\{e_{12}\}$	1

Figure 2.7: Basis blades in 2 dimensions

k	basis k -blades	total
0-blades (scalars)	$\{1\}$	1
1-blades (vectors)	$\{e_1, e_2, e_3\}$	3
2-blades (bivectors)	$\{e_{12}, e_{13}, e_{23}\}$	3
3-blades (trivectors)	$\{e_{123}\}$	1

Figure 2.8: Basis blades in 3 dimensions

k	basis k -blades	total
0-blades (scalars)	$\{1\}$	1
1-blades (vectors)	$\{e_1, e_2, e_3, e_4\}$	4
2-blades (bivectors)	$\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$	6
3-blades (trivectors)	$\{e_{123}, e_{124}, e_{134}, e_{234}\}$	4
4-blades	$\{e_{1234}\}$	1

Figure 2.9: Basis blades in 4 dimensions

Generalizing this; how many basis k -blades are needed in an n -dimensional space to represent arbitrary k -blades? It turns out that the answer lies in the

binomial coefficient:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

This is because a basis k -blade is uniquely determined by the k basis vectors from which it is constructed. There are n different basis vectors in total. $\binom{n}{k}$ is the number of ways to choose k elements from a set of n elements and thus it is easily seen that the number of basis k -blades is equal to $\binom{n}{k}$.

Here are a few examples which you can compare to the tables above. The number of basis bivectors or 2-blades in 3-dimensional space is:

$$\binom{3}{2} = \frac{3!}{(3-2)!2!} = 3$$

The number of basis trivectors or 3-blades in 3-dimensional space equals:

$$\binom{3}{3} = \frac{3!}{(3-3)!3!} = 1$$

The number of basis bivectors or 2-blades in 4-dimensional space is:

$$\binom{4}{2} = \frac{4!}{(4-2)!2!} = 6$$

Chapter 3

Geometric Algebra

All spaces $\mathbb{R}^n$ generate a set of basis blades that make up a *Geometric Algebra* of subspaces, denoted by $\mathcal{C}\ell_n$.¹ For example, a possible basis for $\mathcal{C}\ell_2$ is:

$$\left\{ \underbrace{1}_{\text{basis scalar}}, \underbrace{e_1, e_2}_{\text{basis vectors}}, \underbrace{I}_{\text{basis bivector}} \right\}$$

Here, 1 is used to denote the basis 0-blade or scalar-basis. Every element of the geometric algebra $\mathcal{C}\ell_2$ can be expressed as a linear combination of these basis blades. Another example is a basis of $\mathcal{C}\ell_3$ which could be:

$$\left\{ \underbrace{1}_{\text{basis scalar}}, \underbrace{e_1, e_2, e_3}_{\text{basis vectors}}, \underbrace{e_{12}, e_{13}, e_{23}}_{\text{basis bivectors}}, \underbrace{e_{123}}_{\text{basis trivector}} \right\}$$

The total number of basis blades for an algebra can be calculated by adding the numbers required for all basis k -blades:

$$\sum_{k=0}^n \binom{n}{k} = 2^n \quad (3.1)$$

The proof relies on some combinatorial mathematics and can be found in many places. You can use the following table to check the formula for a few simple geometric algebras.

$\mathcal{C}\ell_n$	basis blades	total
$\mathcal{C}\ell_0$	$\{1\}$	$2^0 = 1$
$\mathcal{C}\ell_1$	$\{1; e_1\}$	$2^1 = 2$
$\mathcal{C}\ell_2$	$\{1; e_1, e_2; e_{12}\}$	$2^2 = 4$
$\mathcal{C}\ell_3$	$\{1; e_1, e_2, e_3; e_{12}, e_{13}, e_{23}; e_{123}\}$	$2^3 = 8$
$\mathcal{C}\ell_4$	$\{1; e_1, e_2, e_3, e_4; e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}; e_{123}, e_{124}, e_{134}, e_{234}; e_{1234}\}$	$2^4 = 16$

¹The reason we use $\mathcal{C}\ell_n$ is because geometric algebra is based on the theory of Clifford Algebras, a topic within mathematics beyond the scope of this paper

3.1 The Geometric Product

Until now we have only used the outer product. If we combine the outer product with the familiar dot product we obtain the geometric product. For arbitrary vectors a, b the geometric product can be calculated as follows:

$$ab = a \cdot b + a \wedge b \quad (3.2)$$

Wait, how is that possible? The dot product results in a scalar, and the outer product in a bivector. How does one add a scalar to a bivector?

Like complex numbers, we keep the two entities separated. The complex number $(3 + 4i)$ consists of a real and imaginary part. Likewise, $ab = a \cdot b + a \wedge b$ consists of a scalar and a bivector part. Such combinations of blades are called multivectors.

3.2 Multivectors

A multivector is a linear combination of different k -blades. In $\mathbb{R}^2$ it will contain a scalar part, a vector part and a bivector part:

$$\underbrace{\alpha_1}_{\text{scalar part}} + \underbrace{\alpha_2 e_1 + \alpha_3 e_2}_{\text{vector part}} + \underbrace{\alpha_4 I}_{\text{bivector part}}$$

Where α_i are real numbers, e.g. the components of the multivector. Note that α_i can be zero, which means that blades are multivectors as well. For example, if α_1 and α_4 are zero, we have a vector or 1-blade.

In $\mathbb{R}^2$ we need $2^2 = 4$ real numbers to denote a full multivector. A multivector in $\mathbb{R}^3$ can be defined with $2^3 = 8$ real numbers and will look like this:

$$\underbrace{\alpha_1}_{\text{scalar part}} + \underbrace{\alpha_2 e_1 + \alpha_3 e_2 + \alpha_4 e_3}_{\text{vector part}} + \underbrace{\alpha_5 e_{12} + \alpha_6 e_{13} + \alpha_7 e_{23}}_{\text{bivector part}} + \underbrace{\alpha_8 e_{123}}_{\text{trivector part}}$$

In the same way, a multivector in $\mathbb{R}^4$ will have $2^4 = 16$ components.

Unfortunately, multivectors can't be visualized easily. Vectors, bivectors and trivectors have intuitive visualizations in 2- and 3-dimensional space. Multivectors lack this way of thinking, because we have no way to visualize a scalar added to an area. However, we get something much more powerful than easy visualization. A multivector, as a linear combination of subspaces, turns out to be extremely expressive, and can be used to convey many different concepts in geometry.

3.3 The Geometric Product Continued

The generalized geometric product is an operator for multivectors. It has the following properties:

$$(AB)C = A(BC) \quad \text{associativity} \quad (3.3)$$

$$\lambda A = A\lambda \quad \text{commutative scalar multiplication} \quad (3.4)$$

$$A(B + C) = AB + AC \quad \text{distributive over addition} \quad (3.5)$$

For arbitrary multivectors A , B and C , and scalar λ . Proofs for the other properties are beyond the scope of this paper. They are not difficult per se, but it is mostly formal algebra even though all of the above intuitively feel right already. The interested reader should pick up some of the references for more information.

Note that the geometric product is, in general, not commutative:

$$AB \neq BA$$

Nor is it anticommutative. This is a direct consequence of the fact that the anticommutative outer product and the commutative dot product are both part of the geometric product.

We have seen the geometric product for vectors using the dot product and the outer product. However, since the dot product is only defined for vectors, and the outer product only for blades, we need something different for multivectors.

Consider two arbitrary multivectors A and B from $\mathcal{C}\ell_2$.

$$A = \alpha_1 + \alpha_2 e_1 + \alpha_3 e_2 + \alpha_4 I$$

$$B = \beta_1 + \beta_2 e_1 + \beta_3 e_2 + \beta_4 I$$

Multiplying A and B using the geometric product, we get:

$$AB = (\alpha_1 + \alpha_2 e_1 + \alpha_3 e_2 + \alpha_4 I)B$$

Using equation (3.5) we may rewrite this to:

$$AB = \alpha_1 B + \alpha_2 e_1 B + \alpha_3 e_2 B + \alpha_4 IB$$

Now writing out B :

$$\begin{aligned} AB &= (\alpha_1 (\beta_1 + \beta_2 e_1 + \beta_3 e_2 + \beta_4 I)) \\ &\quad + (\alpha_2 e_1 (\beta_1 + \beta_2 e_1 + \beta_3 e_2 + \beta_4 I)) \\ &\quad + (\alpha_3 e_2 (\beta_1 + \beta_2 e_1 + \beta_3 e_2 + \beta_4 I)) \\ &\quad + (\alpha_4 I (\beta_1 + \beta_2 e_1 + \beta_3 e_2 + \beta_4 I)) \end{aligned}$$

And this can be rewritten to:

$$\begin{aligned} AB &= \alpha_1 \beta_1 + \alpha_1 \beta_2 e_1 + \alpha_1 \beta_3 e_2 + \alpha_1 \beta_4 I \\ &\quad + \alpha_2 e_1 \beta_1 + \alpha_2 e_1 \beta_2 e_1 + \alpha_2 e_1 \beta_3 e_2 + \alpha_2 e_1 \beta_4 I \\ &\quad + \alpha_3 e_2 \beta_1 + \alpha_3 e_2 \beta_2 e_1 + \alpha_3 e_2 \beta_3 e_2 + \alpha_3 e_2 \beta_4 I \\ &\quad + \alpha_4 I \beta_1 + \alpha_4 I \beta_2 e_1 + \alpha_4 I \beta_3 e_2 + \alpha_4 I \beta_4 I \end{aligned}$$

And in the same way as we did when we wrote out the outer product, we may reorder the scalar multiplications (3.4) to obtain:

$$\begin{aligned}
AB = & \alpha_1\beta_1 + \alpha_1\beta_2e_1 + \alpha_1\beta_3e_2 + \alpha_1\beta_4I \\
& + \alpha_2\beta_1e_1 + \alpha_2\beta_2e_1e_1 + \alpha_2\beta_3e_1e_2 + \alpha_2\beta_4e_1I \\
& + \alpha_3\beta_1e_2 + \alpha_3\beta_2e_2e_1 + \alpha_3\beta_3e_2e_2 + \alpha_3\beta_4e_2I \\
& + \alpha_4\beta_1I + \alpha_4\beta_2Ie_1 + \alpha_4\beta_3Ie_2 + \alpha_4\beta_4II
\end{aligned} \tag{3.6}$$

This looks like a monster of a calculation at first. But if you study it for a while, you will notice that it is fairly structured. The resulting equation demonstrates that we can express the geometric product of arbitrary multivectors as a linear combination of geometric products of basis blades.

So what we need, is to understand how to calculate geometric products of basis blades. Let's look at a few different combinations. For example, using equation (3.2) we can write:

$$e_1e_1 = e_1 \cdot e_1 + e_1 \wedge e_1$$

But remember from equation (2.2) that $a \wedge a = 0$ because it has no area. Also, the dot product of a vector with itself is equal to its squared magnitude. If we choose the magnitude of the basis vectors e_1, e_2 , etc. to be 1, we may simplify the above to:

$$\begin{aligned}
e_1e_1 &= e_1 \cdot e_1 + e_1 \wedge e_1 \\
&= e_1 \cdot e_1 + 0 \\
&= 1 + 0 \\
&= 1
\end{aligned}$$

Another example is, again in $\mathcal{C}\ell_2$:

$$e_1e_2 = e_1 \cdot e_2 + e_1 \wedge e_2$$

Now remember that e_1 is perpendicular to e_2 so the dot product $e_1 \cdot e_2 = 0$. This leaves us with:

$$\begin{aligned}
e_1e_2 &= e_1 \cdot e_2 + e_1 \wedge e_2 \\
&= 0 + e_1 \wedge e_2 \\
&= 0 + I \\
&= I
\end{aligned}$$

A more complicated example involves the geometric product of e_1 and I . The previous example showed us that $I = e_{12}$ is equal to e_1e_2 . We can use this

and equation (3.3) to write:

$$\begin{aligned}
e_1 I &= e_1 e_{12} \\
&= e_1 (e_1 e_2) \\
&= (e_1 e_1) e_2 \\
&= 1 e_2 \\
&= e_2
\end{aligned}$$

You might begin to see a pattern. Because the basis blades are perpendicular, the dot and outer product have trivial results. We use this to simplify the result of a geometric product with a few rules.

1. Basis blades with grades higher than one (bivectors, trivectors, 4-blades, etc.) can be written as an outer product of perpendicular vectors. Because of this, their dot product equals zero, and consequently, we can write them as a geometric product of vectors. For example, in some high dimensional space, we could write:

$$e_{12849} = e_1 \wedge e_2 \wedge e_8 \wedge e_4 \wedge e_9 = e_1 e_2 e_8 e_4 e_9$$

2. Equation (2.1) allows us to swap the order of two non-equal basis vectors if we negate the result. This means that we can write:

$$e_1 e_2 e_3 = -e_2 e_1 e_3 = e_2 e_3 e_1 = -e_3 e_2 e_1$$

3. Whenever a basis vector appears next to itself, it annihilates itself, because the geometric product of a basis vector with itself equals one.

$$e_i e_i = 1 \tag{3.7}$$

Example:

$$e_{112334} = e_{24}$$

Using these three rules we are able to simplify any geometric product of basis blades. Take the following example:

$$\begin{aligned}
e_1 e_{23} e_{31} e_2 &= e_1 e_2 e_3 e_3 e_1 e_2 && \text{using rule one} \\
&= e_1 e_2 e_1 e_2 && \text{using rule three} \\
&= -e_1 e_1 e_2 e_2 && \text{using rule two} \\
&= -1 && \text{using rule three twice}
\end{aligned} \tag{3.8}$$

We can now create a so-called multiplication table which lists all the combinations of geometric products of basis blades. For $\mathcal{C}\ell_2$ it would look like figure 3.1.

	1	e_1	e_2	I
1	1	e_1	e_2	I
e_1	e_1	1	I	e_2
e_2	e_2	$-I$	1	$-e_1$
I	I	$-e_2$	e_1	-1

Figure 3.1: Multiplication Table for basis blades in $\mathcal{C}\ell_2$

According to this table, the multiplication of I and I should equal -1 , which can be calculated as follows:

$$\begin{aligned}
I^2 &= e_{12}e_{12} && \text{by definition} \\
&= e_1e_2e_1e_2 && \text{using rule one} \\
&= -e_2e_1e_1e_2 && \text{using rule two} \\
&= -e_2e_2 && \text{using rule three} \\
&= -1 && \text{using rule three}
\end{aligned}$$

Now that we have the required knowledge on geometric products of basis blades, we can return to the geometric product of arbitrary blades. Here's equation (3.6) repeated for convenience:

$$\begin{aligned}
AB &= \alpha_1\beta_1 + \alpha_1\beta_2e_1 + \alpha_1\beta_3e_2 + \alpha_1\beta_4I \\
&+ \alpha_2\beta_1e_1 + \alpha_2\beta_2e_1e_1 + \alpha_2\beta_3e_1e_2 + \alpha_2\beta_4e_1I \\
&+ \alpha_3\beta_1e_2 + \alpha_3\beta_2e_2e_1 + \alpha_3\beta_3e_2e_2 + \alpha_3\beta_4e_2I \\
&+ \alpha_4\beta_1I + \alpha_4\beta_2Ie_1 + \alpha_4\beta_3Ie_2 + \alpha_4\beta_4II
\end{aligned}$$

We can simply look up the geometric product of basis blades in the multiplication table, and substitute the results:

$$\begin{aligned}
AB &= \alpha_1\beta_1 + \alpha_1\beta_2e_1 + \alpha_1\beta_3e_2 + \alpha_1\beta_4I \\
&+ \alpha_2\beta_1e_1 + \alpha_2\beta_2 + \alpha_2\beta_3I + \alpha_2\beta_4e_2 \\
&+ \alpha_3\beta_1e_2 - \alpha_3\beta_2I + \alpha_3\beta_3 - \alpha_3\beta_4e_1 \\
&+ \alpha_4\beta_1I - \alpha_4\beta_2e_2 + \alpha_4\beta_3e_1 - \alpha_4\beta_4
\end{aligned}$$

Now the last step is to group the basis-blades together.

$$\begin{aligned}
AB &= (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 - \alpha_4\beta_4) \\
&+ (\alpha_4\beta_3 - \alpha_3\beta_4 + \alpha_1\beta_2 + \alpha_2\beta_1)e_1 \\
&+ (\alpha_1\beta_3 - \alpha_4\beta_2 + \alpha_2\beta_4 + \alpha_3\beta_1)e_2 \\
&+ (\alpha_4\beta_1 + \alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2)I
\end{aligned} \tag{3.9}$$

The final result is a linear combination of the four basis blades $\{1, e_1, e_2, I\}$ or, in other words, a multivector. This proves that the geometric algebra is closed under the geometric product.

That is to say; so far I have only showed you how the geometric product works in $\mathcal{C}\ell_2$. It is trivial to extend the same methods to $\mathcal{C}\ell_3$ or higher. The same three simplification rules apply. Figure 3.2 contains the multiplication table for $\mathcal{C}\ell_3$.

	1	e_1	e_2	e_3	e_{12}	e_{13}	e_{23}	e_{123}
1	1	e_1	e_2	e_3	e_{12}	e_{13}	e_{23}	e_{123}
e_1	e_1	1	e_{12}	e_{13}	e_2	e_3	e_{123}	$-e_{23}$
e_2	e_2	$-e_{12}$	1	e_{23}	$-e_1$	$-e_{123}$	e_3	$-e_{13}$
e_3	e_3	$-e_{13}$	$-e_{23}$	1	e_{123}	$-e_1$	$-e_2$	e_{12}
e_{12}	e_{12}	$-e_2$	e_1	e_{123}	-1	$-e_{23}$	e_{13}	$-e_3$
e_{13}	e_{13}	$-e_3$	$-e_{123}$	e_1	e_{23}	-1	$-e_{12}$	e_2
e_{23}	e_{23}	e_{123}	$-e_3$	e_2	$-e_{13}$	e_{12}	-1	$-e_1$
e_{123}	e_{123}	e_{23}	$-e_{13}$	e_{12}	$-e_3$	e_2	$-e_1$	-1

Figure 3.2: Multiplication Table for basis blades in $\mathcal{C}\ell_3$

3.4 The Dot and Outer Product revisited

We defined the geometric product for vectors as a combination of the dot and outer product:

$$ab = a \cdot b + a \wedge b$$

We can rewrite these equations to express the dot product and outer product in terms of the geometric product:

$$a \wedge b = \frac{1}{2}(ab - ba) \quad (3.10)$$

$$a \cdot b = \frac{1}{2}(ab + ba) \quad (3.11)$$

To illustrate, let us prove (3.10). Let's take two multivectors A and B $\in \mathcal{C}\ell_2$ for which the scalar and bivector parts are zero, e.g. two vectors. Using equation (3.9) and taking into account that $\alpha_1 = \beta_1 = \alpha_4 = \beta_4 = 0$ we can write AB and BA as:

$$AB = (\alpha_2\beta_2 + \alpha_3\beta_3) + (\alpha_2\beta_3 - \alpha_3\beta_2)I \quad (3.12)$$

$$BA = (\beta_2\alpha_2 + \beta_3\alpha_3) + (\beta_2\alpha_3 - \beta_3\alpha_2)I \quad (3.13)$$

Using these in equation (3.10) we get:

$$\frac{\overbrace{((\alpha_2\beta_2 + \alpha_3\beta_3) + (\alpha_2\beta_3 - \alpha_3\beta_2)I)}^{AB} - \overbrace{((\beta_2\alpha_2 + \beta_3\alpha_3) + (\beta_2\alpha_3 - \beta_3\alpha_2)I)}^{BA}}{2}$$

Reordering we get:

$$\frac{\overbrace{(\alpha_2\beta_2 + \alpha_3\beta_3) - (\beta_2\alpha_2 + \beta_3\alpha_3)}^{ScalarPart} + \overbrace{(\alpha_2\beta_3 - \alpha_3\beta_2)I - (\beta_2\alpha_3 - \beta_3\alpha_2)I}^{BivectorPart}}{2}$$

Notice the scalar part results in zero, which leaves us with:

$$\frac{(\alpha_2\beta_3 - \alpha_3\beta_2)I - (\beta_2\alpha_3 - \beta_3\alpha_2)I}{2}$$

Subtracting the two bivectors we get:

$$\frac{(\alpha_2\beta_3 - \alpha_3\beta_2 - \beta_2\alpha_3 + \beta_3\alpha_2)I}{2}$$

This may be rewritten as:

$$\frac{(2\alpha_2\beta_3 - 2\alpha_3\beta_2)I}{2}$$

And now dividing by 2 we obtain:

$$A \wedge B = (\alpha_2\beta_3 - \alpha_3\beta_2)I$$

for multivectors A and B with zero scalar and bivector part. Compare this with equation (2.6) that defines the outer product for two vectors a and b . If you remember that the vector part of a multivector $\in \mathcal{C}\ell_2$ is in the second and third component, you will realize that these equations are the same.

Note that (3.10) and (3.11) only hold for vectors. The inner and outer product of higher order blades is more complicated, not to mention the inner and outer product for multivectors. Yet, let us try to see what they could mean.

3.5 The Inner Product

I informally demonstrated what the outer product of a vector and a bivector looks like when I introduced trivectors. What about the dot product? What could the dot product of a vector and a bivector look like? Figure 3.3 depicts the result.

Notice how the inner product is the vector perpendicular to the actual projection. In more general terms, it is the *complement (within the subspace of B) of the orthogonal projection of a onto B* . [2] We will no longer call this generalization a dot product. The generic notion of projections and perpendicularity is captured by an operator called the *inner product*.

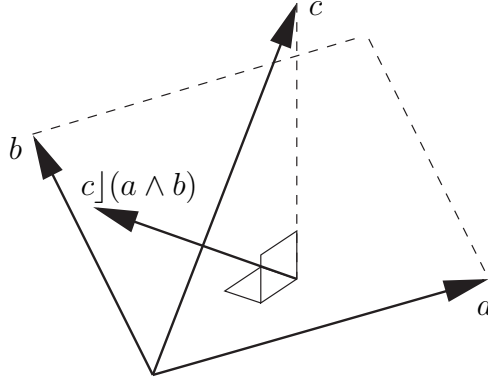


Figure 3.3: The dot product of a bivector and a vector

Unfortunately, there is not just one definition of the inner product. There are several versions floating around, their usefulness depending on the problem area. They are not fundamentally different however, and all of them can be expressed in terms of the others. In fact, one could say that the flexibility of the different inner products is one of the strengths of geometric algebra. Unfortunately, this does not really help those trying to learn geometric algebra, as it can be overwhelming and confusing.

The default and best known inner product [8] is very useful in Euclidian mechanics, whereas the *contraction inner product* [2], also known as the Lounesto inner product, is more useful in computer science. Other inner products include the semi-symmetric or semi-commutative inner product, also known as the Hestenes inner product, the modified Hestenes or (fat)dot product and the forced Euclidean contractive inner product. [13] [5]

Obviously, because of our interest for computer science, we are most interested in the contraction inner product. We will use the $\rfloor$ symbol to denote a contraction. It may seem a bit weird at first, but it will turn out to be very useful. Luckily, for two vectors it works exactly as the traditional inner product or dot product. For different blades, it is defined as follows [2]:

$$\text{scalars} \quad \alpha \rfloor \beta = \alpha \beta \quad (3.14)$$

$$\text{vector and scalar} \quad a \rfloor \beta = 0 \quad (3.15)$$

$$\text{scalar and vector} \quad \alpha \rfloor b = \alpha b \quad (3.16)$$

$$\text{vectors} \quad a \rfloor b = a \cdot b \quad (\text{the usual dot product}) \quad (3.17)$$

$$\text{vector, multivector} \quad a \rfloor (b \wedge C) = (a \rfloor b) \wedge C - b \wedge (a \rfloor C) \quad (3.18)$$

$$\text{distribution} \quad (A \wedge B) \rfloor C = A \rfloor (B \rfloor C) \quad (3.19)$$

Try to understand how the above provides a recursive definition of the contraction operator. There are the basic rules for vectors and scalars, and there is (3.18) for the contraction between a vector and the outer product of a vector and

a multivector. Because linearity holds over the contraction, we can decompose contractions with multivectors into contractions with blades. Now, remember that any blade D with grade n can be written as the outer product of a vector b and a blade C with grade $n - 1$. This means that the contraction $a \rfloor D$ can be written as $a \rfloor (b \wedge C)$ and consequently as $(a \rfloor b) \wedge C - b \wedge (a \rfloor C)$ according to (3.18). We know how to calculate $a \rfloor b$ by definition, and we can recursively solve $a \rfloor C$ until the grade of C is equal to 1, which reduces it to a contraction of two vectors.

Obviously, this is not a very efficient way of calculating the inner product. Fortunately, the inner product can be expressed in terms of the geometric product (and vice versa as we've done before), which allows for fast calculations. [12]

I will return to the inner product when I talk about grades some more in the tools chapter. In the chapter on applications we will see where and how the contraction product is useful. From now on, whenever I refer to the inner product I mean any of the generalized inner products. If I need the contraction, I will mention it explicitly. I will allow myself to be sloppy, and continue to use the $\cdot$ and $\rfloor$ symbol interchangeably.

3.6 Inner, Outer and Geometric

We saw in equation (3.2) that the geometric product for vectors could be defined in terms of the dot (inner) and outer product. What if we use (3.10) and (3.11) combined:

$$\begin{aligned} \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) &= \frac{(ab + ba) + (ab - ba)}{2} \\ &= \frac{(ab + ba + ab - ba)}{2} \\ &= \frac{2ab}{2} \\ &= ab \end{aligned}$$

This demonstrates the two possible approaches to introduce geometric algebra. Some books [7] give an abstract definition of the geometric product, by means of a few axioms, and derive the inner and outer product from it. Other material [8] starts with the inner and outer product and demonstrates how the geometric product follows from them.

You may prefer one over the other, but ultimately it is the way the geometric product, the inner product and the outer product work together that gives geometric algebra its strength. For two vectors a and b we have:

$$ab = a \cdot b + a \wedge b$$

as a result, they are orthogonal if $ab = -ba$ because the inner product of two perpendicular vectors is zero. And they are collinear if $ab = ba$ because the

wedge of two collinear vectors is zero. If the two vectors are neither collinear nor orthogonal the geometric product is able to express their relationship as ‘something in between’.

Chapter 4

Tools

Strictly speaking, all we need is an algebra of multivectors with the geometric product as its operator. Nevertheless, this chapter introduces some more definitions and operators that will be of great use in many applications. If you are tired of all this theory, I suggest you skip over this section and start with some of the applications. If you encounter unfamiliar concepts, you can refer to this chapter.

4.1 Grades

I briefly talked about grades in the chapter on subspaces. The grade of a blade is the dimension of the subspace it represents. Thus multivectors have combinations of grades, as they are linear combinations of blades. We denote the blade-part with grade s of a multivector A using $\langle A \rangle_s$. For multivector $A = (4, 8, 5, 6, 2, 4, 9, 3) \in \mathcal{C}\ell_3$ we have:

$\langle A \rangle_0 = 4$	scalar part
$\langle A \rangle_1 = (8, 5, 6)$	vector part
$\langle A \rangle_2 = (2, 4, 9)$	bivector part
$\langle A \rangle_3 = 3$	trivector part

Any multivector A in $\mathcal{C}\ell_n$ can be denoted as a sum of blades, like we already did informally:

$$\sum_{k=0}^n \langle A \rangle_k = \langle A \rangle_0 + \langle A \rangle_1 + \dots + \langle A \rangle_n$$

Using this notation I can demonstrate what the inner and outer product mean for grades. For two vectors a and b the inner product $a \cdot b$ results in a scalar c . The vectors are 1-blades, the scalar is a 0-blade. This leads to:

$$\langle a \rangle_1 \cdot \langle b \rangle_1 = \langle ab \rangle_0$$

In figure 3.3 we saw that a vector a projected onto a bivector B resulted in a vector. Here, we'll be using the contraction product. So, in other words the contraction product of a 2-blade and a 1-blade results in a 1-blade. Using a multivector notation:

$$\langle a \rangle_1 \rfloor \langle B \rangle_2 = \langle aB \rangle_{2-1}$$

Generalizing this for blades A and B with grade s and t respectively:

$$\langle A \rangle_s \rfloor \langle B \rangle_t = \langle AB \rangle_u \text{ where } u = \begin{cases} s > t, & 0 \\ s \leq t, & t - s \end{cases}$$

We might say that the contraction inner product is a 'grade-lowering' operation.

And, of course, the outer product is its opposite as a grade-increasing operation. Recall that for two 1-blades or vectors the outer product resulted in a 2-blade or bivector:

$$\langle a \rangle_1 \wedge \langle b \rangle_1 = \langle ab \rangle_2$$

The outer product between a 2-blade and a 1-blade results in a $2 + 1 = 3$ -blade or trivector. Generalizing we get for two blades A and B with grade s and t :

$$\langle A \rangle_s \wedge \langle B \rangle_t = \langle AB \rangle_{s+t}$$

Note that A and B have to be blades. These equations do not hold when they are arbitrary multivectors.

4.2 The Inverse

Most multivectors have a left inverse satisfying $A^{-1L}A = 1$ and a right inverse satisfying $AA^{-1R} = 1$. We can use these inverses to divide a multivector by another. Recall that the geometric product is not commutative therefore the left and right inverse may or may not be equal. This means that the $\frac{A}{B}$ notation is ambiguous since it can mean both $B^{-1L}A$ and AB^{-1R} .

Unfortunately calculating the inverse of a geometric product is not trivial, much like calculating inverses of matrices is complicated for all but a few special cases.

Luckily there is an important set of multivectors for which calculating the inverse is very straightforward. These are called the *versors* and they have the property that they are a geometric product of vectors. A multivector A is a versor if it can be written as:

$$A = v_1 v_2 v_3 \dots v_k$$

where $v_1 \dots v_k$ are vectors, i.e. 1-blades. As a fortunate consequence, all blades are versors too.¹ For a versor A we define its *reverse*, using the $\dagger$ symbol, as:

$$A^\dagger = v_k v_{k-1} \dots v_2 v_1 \tag{4.1}$$

¹Remember that we use vectors to create subspaces of higher dimension, using the outer product.

This means that, because of equation (2.1), the reverse of a blade is only a possible sign change. Remember that each swap of indices in the product produces a sign change, thus if k is uneven the reverse of A is equal to itself, and if it's uneven the reverse of A is the $-A$. Note that this does not apply to versors in general.

The left and right inverse of a versor are the same and can be calculated as follows:

$$A^{-1} = \frac{A^\dagger}{A^\dagger A} \quad (4.2)$$

To understand this, we have to start by realizing that the denominator is always a scalar because:

$$A^\dagger A = v_1 v_2 \dots v_{k-1} v_k v_k v_{k-1} \dots v_2 v_1 = |v_1|^2 |v_2|^2 \dots |v_{k-1}|^2 |v_k|^2$$

And since a scalar divided by itself equals one, this means that:

$$A^{-1} A = \frac{A^\dagger}{A^\dagger A} A = \frac{A^\dagger A}{A^\dagger A} = 1$$

Furthermore it also proves that the left and right inverse are the same. Division by a scalar α is multiplication by $1/\alpha$, which is, according to equation (3.3) commutative, proving that the left and right inverses of a versor are indeed equal.

This means that for a versor A , we have $A^{-1L} = A^{-1R} = A^{-1}$ and therefore the following:

$$A^{-1L} A = A A^{-1R} = A^{-1} A = A A^{-1} = 1$$

It is important to notice that in the case of vectors, the scalar represents the squared magnitude of the vector. As a consequence, the inverse of a unit vector is equal to itself.

Not many people are comfortable with the idea of division by vectors, bivectors, or multivectors. They are only accustomed to division by scalars. But if we have a geometric product and a definition for the inverse, nothing stops us from division. Later we will see that this is extremely useful.

4.3 Pseudoscalars

In equation (2.3) we saw how to calculate the number of basis blades for a given grade. From this it follows that every geometric algebra has only one basis 0-blade or basis-scalar, independent of the dimension of the algebra:

$$\binom{n}{0} = \frac{n!}{(n-0)!0!} = \frac{n!}{n!} = 1$$

More interesting is the basis blade with the highest dimension. For a geometric algebra $\mathcal{C}\ell_n$ the number of blades with dimension n is:

$$\binom{n}{n} = \frac{n!}{(n-n)!n!} = \frac{n!}{n!} = 1$$

In $\mathcal{C}\ell_2$ this was $e_1e_2 = I$ as shown in figure 2.4. In $\mathcal{C}\ell_3$ this is the trivector $e_1e_2e_3 = e_{123}$.

In general every geometric algebra has a single basis blade of highest dimension. This is called the pseudoscalar.

4.4 The Dual

Traditional linear algebra uses normal vectors to represent planes. Geometric algebra introduces bivectors which can be used for the same purpose. By using the pseudoscalar we can get an understanding of the relationship between the two representations.

The dual A^* of a multivector A is defined as follows:

$$A^* = AI^{-1} \quad (4.3)$$

where I represents the pseudoscalar of the geometric algebra that is being used. The pseudoscalar is a blade (the blade with highest grade) and therefore its left and right inverse are the same, and hence the above formula is not ambiguous.

Let us consider a simple example in $\mathcal{C}\ell_3$, calculating the dual of the basis bivector e_{12} . The pseudoscalar is e_{123} . Pseudoscalars are blades and thus versors. You can check yourself that its inverse is $e_3e_2e_1$. We'll use this to calculate the dual of e_{12} :

$$\begin{aligned} e_{12}^* &= e_{12}e_3e_2e_1 \\ &= e_1e_2e_3e_2e_1 \\ &= -e_1e_3e_2e_2e_1 \\ &= -e_1e_3e_1 \\ &= e_1e_1e_3 \\ &= e_3 \end{aligned} \quad (4.4)$$

Thus, the dual is basis vector e_3 , which is exactly the normal vector of basis bivector e_{12} . In fact, this is true for all bivectors. If we have two arbitrary vectors a and $b \in \mathcal{C}\ell_3$:

$$\begin{aligned} a &= \alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3 \\ b &= \beta_1e_1 + \beta_2e_2 + \beta_3e_3 \end{aligned}$$

According to equation (2.7) their outer product is:

$$a \wedge b = (\alpha_1\beta_2 - \alpha_2\beta_1)e_{12} + (\alpha_1\beta_3 - \alpha_3\beta_1)e_{13} + (\alpha_2\beta_3 - \alpha_3\beta_2)e_{23}$$

And its dual $(a \wedge b)^*$ becomes:

$$\begin{aligned}
(a \wedge b)^* &= (a \wedge b) e_{123}^{-1} \\
&= (a \wedge b) e_3 e_2 e_1 \\
&= ((\alpha_1 \beta_2 - \alpha_2 \beta_1) e_{12} + (\alpha_1 \beta_3 - \alpha_3 \beta_1) e_{13} + (\alpha_2 \beta_3 - \alpha_3 \beta_2) e_{23}) e_3 e_2 e_1 \\
&= (\alpha_1 \beta_2 - \alpha_2 \beta_1) e_{12} e_3 e_2 e_1 + \\
&\quad (\alpha_1 \beta_3 - \alpha_3 \beta_1) e_{13} e_3 e_2 e_1 + \\
&\quad (\alpha_2 \beta_3 - \alpha_3 \beta_2) e_{23} e_3 e_2 e_1 \\
&= (\alpha_1 \beta_2 - \alpha_2 \beta_1) e_1 e_2 e_3 e_2 e_1 + \\
&\quad (\alpha_1 \beta_3 - \alpha_3 \beta_1) e_1 e_3 e_3 e_2 e_1 + \\
&\quad (\alpha_2 \beta_3 - \alpha_3 \beta_2) e_2 e_3 e_3 e_2 e_1 \\
&= (\alpha_1 \beta_2 - \alpha_2 \beta_1) e_3 - (\alpha_1 \beta_3 - \alpha_3 \beta_1) e_2 + (\alpha_2 \beta_3 - \alpha_3 \beta_2) e_1 \\
&= (\alpha_2 \beta_3 - \alpha_3 \beta_2) e_1 + (\alpha_3 \beta_1 - \alpha_1 \beta_3) e_2 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) e_3 \tag{4.5}
\end{aligned}$$

Which is exactly the traditional cross product. We conclude that in three dimensions, the dual of a bivector is its normal. The dual can be used to convert between bivector and normal representations. But the dual is even more, because it is defined for any multivector.

4.5 Projection and Rejection

If we have a vector a and bivector B we can decompose a in two parts. One part $a_{||B}$ that is collinear with B . We call this the *projection* of a onto B . The other part is $a_{\perp B}$, and orthogonal to B . We call this the *rejection*² of a from B . Mathematically:

$$a = a_{||B} + a_{\perp B} \tag{4.6}$$

This is depicted in figure 4.1. Such a decomposition turns out to be very useful and I will demonstrate how to calculate it.

First, equation (3.2) says that the geometric product of two vectors is equal to the sum of the inner and outer product. There is a generalization of this saying that for arbitrary vector a and k -blade B the geometric product is:

$$aB = a \cdot B + a \wedge B \tag{4.7}$$

²This term has been introduced by David Hestenes in his New Foundations For Classical Mechanics [8]. To quote: “The new term ‘rejection’ has been introduced here in the absence of a satisfactory standard name for this important concept.”

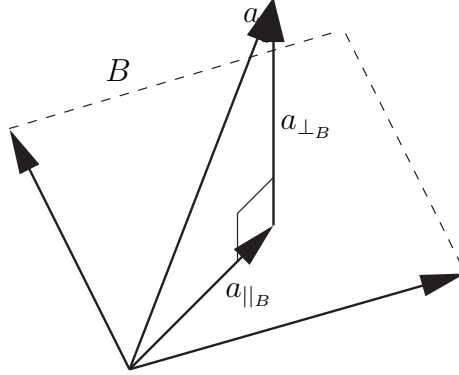


Figure 4.1: Projection and rejection of vector a in bivector B

Note that a has to be a vector, and B a blade of any grade. That is, this doesn't hold for multivectors in general. Proofs can be found in the references. [14] [8]

Using (4.7) we can calculate the decomposition (4.6) of any vector a onto a bivector B . By definition, the inner and outer product of respectively orthogonal and collinear blades are zero. In other words, the inner product of a vector orthogonal to a bivector is zero:

$$a_{\perp B} \cdot B = 0 \quad (4.8)$$

Likewise the outer product of a vector collinear with a bivector is zero:

$$a_{\parallel B} \wedge B = 0 \quad (4.9)$$

Let's see what happens if we multiply the orthogonal part of vector a with bivector B :

$$\begin{aligned} a_{\perp B} B &= a_{\perp B} \cdot B + a_{\perp B} \wedge B \\ &= a_{\perp B} \wedge B && \text{using equation (4.8)} \\ &= a_{\perp B} \wedge B + a_{\parallel B} \wedge B && \text{equation (4.9)} \\ &= (a_{\perp B} + a_{\parallel B}) \wedge B && \text{equation (2.5)} \\ &= a \wedge B && \text{equation (4.6)} \end{aligned}$$

Thus, the perpendicular part of vector a times bivector B is equal to the outer product of a and B . Now all we need to do is divide both sides of the equation by B to obtain the perpendicular part of a :

$$\begin{aligned} a_{\perp B} B &= a \wedge B \\ a_{\perp B} B B^{-1} &= (a \wedge B) B^{-1} \\ a_{\perp B} &= (a \wedge B) B^{-1} \end{aligned}$$

Notice that there is no ambiguity in using the inverse because B is a blade or versor, and its left and right inverses are therefore the same. The conclusion is:

$$a_{\perp B} = (a \wedge B)B^{-1} \quad (4.10)$$

Calculating the collinear part of vector a follows similar steps:

$$\begin{aligned} a_{||B} B &= a_{||B} \cdot B + a_{||B} \wedge B \\ &= a_{||B} \cdot B \\ &= a_{||B} \cdot B + a_{\perp B} \cdot B \\ &= (a_{\perp B} + a_{||B}) \cdot B \\ &= a \cdot B \end{aligned}$$

Again multiply both sides with the inverse bivector:

$$a_{||B} BB^{-1} = (a \cdot B)B^{-1}$$

To conclude:

$$a_{||B} = (a \cdot B)B^{-1} \quad (4.11)$$

Using these definitions, we can now confirm that $a_{||B} + a_{\perp B} = a$

$$\begin{aligned} a_{||B} + a_{\perp B} &= (a \wedge B)B^{-1} + (a \cdot B)B^{-1} && \text{equation (4.11) and (4.10)} \\ &= (a \wedge B + a \cdot B)B^{-1} && \text{equation (3.3)} \\ &= aBB^{-1} && \text{equation (4.7)} \\ &= a && \text{by definition of the inverse} \end{aligned}$$

4.6 Reflections

Armed with a way of decomposing blades in orthogonal and collinear parts we can take a look at reflections. We will get ahead of ourselves and take a specific look at the geometric algebra of the Euclidian space $\mathbb{R}^3$ denoted with $\mathcal{Cl}_3$. Suppose we have a bivector U . Its dual U^* will be the normal vector u . What if we multiply a vector a with vector u , projecting and rejecting a onto U at the same time:

$$\begin{aligned} ua &= u(a_{||U} + a_{\perp U}) \\ &= ua_{||U} + ua_{\perp U} \end{aligned}$$

Using (3.2) we write it in full:

$$ua = (u \cdot a_{||U} + u \wedge a_{||U}) + (u \cdot a_{\perp U} + u \wedge a_{\perp U})$$

Note that (because u is the normal of U) the vectors $a_{||U}$ and u are perpendicular. This means that the inner product $a_{||U} \cdot u$ equals zero. Likewise, the vectors $a_{\perp U}$ and u are collinear. This means that the outer product $a_{||U} \wedge u$ equals zero. Removing these two 0 terms:

$$\begin{aligned} ua &= u \wedge a_{||U} + u \cdot a_{\perp U} \\ &= u \cdot a_{\perp U} + u \wedge a_{||U} \end{aligned}$$

Recall that the inner product between two vectors is commutative, and the outer product is anticommutative, so we can write:

$$ua = a_{\perp U} \cdot u - a_{||U} \wedge u$$

We can now insert those 0-terms back in (putting in the form of equation (3.2)):

$$ua = (a_{\perp U} \cdot u + a_{\perp U} \wedge u) - (a_{||U} \cdot u + a_{||U} \wedge u)$$

Writing it as a geometric product now:

$$\begin{aligned} ua &= a_{||U}u - a_{\perp U}u \\ &= (a_{||U} - a_{\perp U})u \end{aligned}$$

Meaning that:

$$\begin{aligned} -ua &= -(a_{||U} - a_{\perp U})u \\ &= (a_{\perp U} - a_{||U})u \end{aligned}$$

Notice how we changed the addition of the perpendicular part into a subtraction by multiplying with $-u$. Now, if we add a multiplication with the inverse we obtain the following, depicted in figure 4.2:

$$\begin{aligned} -uau^{-1} &= -u(a_{||U} + a_{\perp U})u^{-1} \\ &= (a_{||U} - a_{\perp U})uu^{-1} \\ &= a_{||U} - a_{\perp U} \end{aligned}$$

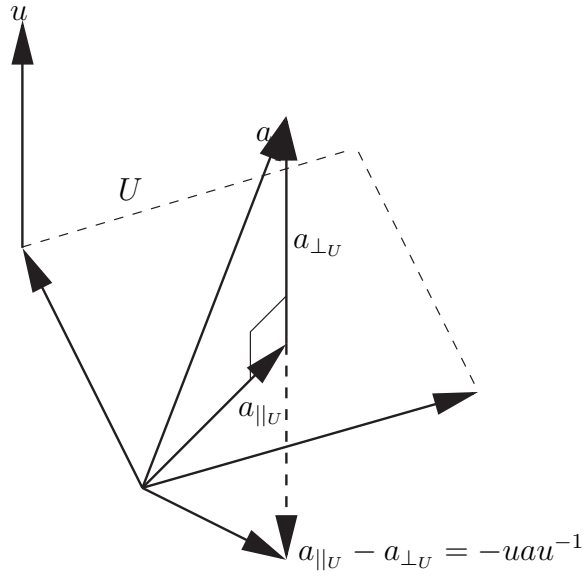


Figure 4.2: Reflection

In general, if we sandwich a vector a in between another vector $-u$ and its inverse u^{-1} , we obtain a reflection in the dual u^* .

Note that in many practical cases u will be a unit vector, which means its inverse is u itself. Thus the reflection of a vector a in a plane with unit-normal u is simply $-uau$. Later, we will see that reflections are not only useful by themselves, but combined they allow us to do rotations.

4.7 The Meet

The meet operator is an operator between two arbitrary blades A and B and defined as follows:

$$A \cap B = A^* \cdot B$$

In other words, the meet $A \cap B$ is the inner product of the dual of A with B . It is no coincidence that the $\cap$ symbol is used to denote the meet. The result of a meet represents the smallest common subspaces of blades A and B . Let's see, in an informal way, what the meet operator does for two bivectors. Looking at figure 4.3 we see two bivectors A and B .

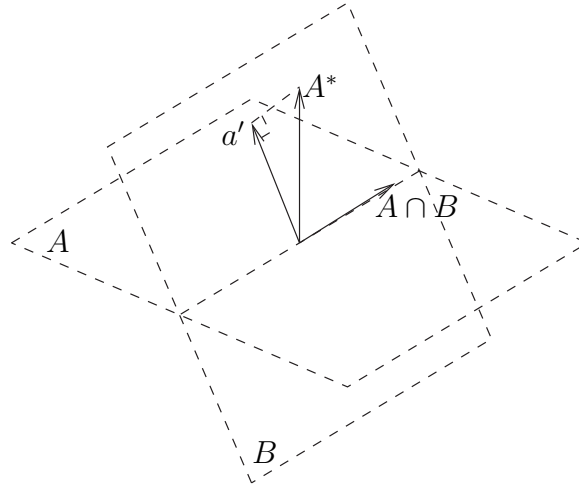


Figure 4.3: The Meet

In this figure, the dual of bivector A will be the normal vector A^* . Then, as we've already seen, the inner product of this vector with bivector B will create the vector perpendicular to the projected vector a' . This is exactly the vector that lies on the intersection of the two bivectors.

A more formal proof of the above, or even the full proof that the meet operator represents the smallest common subspace of *any* two blades, is far from trivial and beyond this paper. Here, I just want to demonstrate that there is a meet operator, and that it is easily defined using the dual and the inner product. We will be using the meet operator later when we talk about intersections between primitives.

Chapter 5

Applications

Up until now I haven't focused on the practical value of geometric algebra. With an understanding of the fundamentals, we can start applying the theory to real world domains. This is where geometric algebra reveals its power, but also its difficulty.

Geometric algebra supplies us with an arithmetic of subspaces, but it is up to us to interpret each subspace and operation and relate it to some real-life concept. This chapter will demonstrate how different geometric algebras combined with different interpretations can be used to explain traditional geometric relations.

5.1 The Euclidian Plane

For an easy start, we'll consider the two dimensional Euclidian Plane to learn about some of the things its geometric algebra $\mathcal{Cl}_2$ has to offer.

5.1.1 Complex Numbers

If you recall the geometric algebra of the Euclidian Plane, you might remember we used I to denote the basis bivector. Then, in table 3.1 we saw that $I^2 = -1$. Thus, we might say:

$$I = \sqrt{-1}$$

Suppose we interpret a multivector with a scalar and bivector blade as a complex number. The scalar corresponds to the real part, and the bivector to the imaginary part. Thus, we can interpret a multivector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ from $\mathcal{Cl}_2$ as a complex number $\alpha_1 + i\alpha_4$ as long as α_2 and α_3 are zero.

Not surprisingly, multivector addition and subtraction corresponds directly with complex number addition and subtraction. But even more so, the geometric product is exactly the multiplication of complex numbers, as the following will prove.

Recall equation (3.9); the full geometric product of multivectors $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $B = (\beta_1, \beta_2, \beta_3, \beta_4)$, repeated here:

$$\begin{aligned} AB = & ((\alpha_1\beta_1) + (\alpha_2\beta_2) + (\alpha_3\beta_3) - (\alpha_4\beta_4)) \\ & + ((\alpha_4\beta_3) - (\alpha_3\beta_4) + (\alpha_1\beta_2) + (\alpha_2\beta_1))e_1 \\ & + ((\alpha_1\beta_3) - (\alpha_4\beta_2) + (\alpha_2\beta_4) + (\alpha_3\beta_1))e_2 \\ & + ((\alpha_4\beta_1) + (\alpha_1\beta_4) + (\alpha_2\beta_3) - (\alpha_3\beta_2))I \end{aligned}$$

If A and B are complex numbers, α_2 , α_3 , β_2 and β_3 will equal zero. Thus we can discard those parts to obtain:

$$\begin{aligned} AB = & ((\alpha_1\beta_1) - (\alpha_4\beta_4)) \\ & + ((\alpha_4\beta_1) + (\alpha_1\beta_4))I \end{aligned}$$

With α_1 and β_1 being the real parts, and α_4 and β_4 being the imaginary parts, this is exactly a complex number multiplication.

5.1.2 Rotations

I will discuss rotations in two dimensions very briefly. When we return to rotations in three dimensions I will introduce the more general dimension-free theory and give several longer and more formal proofs.

If we want to rotate a vector $a = \alpha_2 e_1 + \alpha_3 e_2$ over an angle θ into vector $a' = \alpha'_2 e_1 + \alpha'_3 e_2$, we can employ the following well known formulas:

$$\alpha'_2 = \cos(\theta)\alpha_2 - \sin(\theta)\alpha_3 \quad (5.1)$$

$$\alpha'_3 = \sin(\theta)\alpha_2 + \cos(\theta)\alpha_3 \quad (5.2)$$

Or, more commonly we employ the matrix formula $a' = Ma$ where M is:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Returning to geometric algebra, let us see what happens if we multiply vector a with a complex number $B = \beta_1 + \beta_4 I$ to obtain:

$$\begin{aligned} a'' = aB & = (\alpha_2 e_1 + \alpha_3 e_2)(\beta_1 + \beta_4 I) \\ & = \alpha_2 e_1(\beta_1 + \beta_4 I) + \alpha_3 e_2(\beta_1 + \beta_4 I) \\ & = \alpha_2 e_1 \beta_1 + \alpha_2 e_1 \beta_4 I + \alpha_3 e_2 \beta_1 + \alpha_3 e_2 \beta_4 I \\ & = \beta_1 \alpha_2 e_1 + \beta_4 \alpha_2 e_1 I + \beta_1 \alpha_3 e_2 + \beta_4 \alpha_3 e_2 I \\ & = \beta_1 \alpha_2 e_1 + \beta_4 \alpha_2 e_1 e_1 e_2 + \beta_1 \alpha_3 e_2 + \beta_4 \alpha_3 e_2 e_1 e_2 \\ & = \beta_1 \alpha_2 e_1 + \beta_4 \alpha_2 e_2 + \beta_1 \alpha_3 e_2 - \beta_4 \alpha_3 e_1 \\ & = (\beta_1 \alpha_2 - \beta_4 \alpha_3) e_1 + (\beta_4 \alpha_2 + \beta_1 \alpha_3) e_2 \end{aligned}$$

Thus we see that the geometric product of a complex number and a vector results in a vector with components:

$$\begin{aligned}\alpha_2'' &= \beta_1\alpha_2 + \beta_4\alpha_3 \\ \alpha_3'' &= \beta_4\alpha_2 - \beta_1\alpha_3\end{aligned}\tag{5.3}$$

Compare this with equations (5.1) and (5.2). If we take $\beta_1 = \cos(\theta)$ and $\beta_4 = \sin(\theta)$ we can use complex numbers to do rotations, because then:

$$\begin{aligned}\alpha_2' &= \cos(\theta)\alpha_2 - \sin(\theta)\alpha_3 = \beta_1\alpha_2 - \beta_4\alpha_3 = \alpha_2'' \\ \alpha_3' &= \sin(\theta)\alpha_2 + \cos(\theta)\alpha_3 = \beta_4\alpha_2 + \beta_1\alpha_3 = \alpha_3''\end{aligned}$$

At this point we no longer talk about complex numbers but we call B a spinor. In general, spinors are n -dimensional rotators, and in $\mathcal{Cl}_2$ they are represented by a linear combination of a scalar plus a bivector.

Equation (3.2) says that the geometric product of two vectors is a scalar plus a bivector. So let's find a p and q that generate B :

$$B = \beta_1 + \beta_4 I = p \cdot q + p \wedge q = pq$$

Traditional vector algebra tells us that the angle between two unit vectors can be expressed using the dot product, i.e. $p \cdot q = \cos(\theta)$. It also tells us that the magnitude of the cross product of two unit vectors is equal to the sine of the same angle, $|p \times q| = \sin(\theta)$.

I already demonstrated that the cross product is related to the outer product through the dual. In fact, it turns out that the outer product between two unit vectors is exactly the sine of their angle times the basis bivector I . In other words:

$$\begin{aligned}p \cdot q &= \cos(\theta) \\ p \wedge q &= \sin(\theta)I\end{aligned}$$

A thorough proof can be found in Hestenes's New Foundations For Classical Mechanics [8]. The consequence is that, because of equation (3.2), for unit vectors p and q :

$$pq = \cos(\theta) + \sin(\theta)I$$

with θ being the angle between the two vectors.

Concluding, a spinor in $\mathcal{Cl}_2$ is a scalar plus a bivector. Its components correspond directly to the sine and cosine of the angle. The geometric product of two unit vectors generates a spinor.

We can derive this in a similar way by creating the following identity:

$$\frac{p}{q} = \frac{a'}{a}$$

Which is not ambiguous, and can also be written as:

$$pq^{-1} = a'a^{-1}$$

Basically, this says that “what p is to q , is a' to a .” We can rewrite this to:

$$a' = pq^{-1}a$$

But the inverse of a unit vector is equal to itself, and thus:

$$a' = \underbrace{pq}_{\text{spinor}} a$$

If the spinor pq represents a clockwise rotation, then qp represents a counter-clockwise rotation. This is thanks to the fact that the geometric product is not commutative or anticommutative. As a result it can convey more information removing much of the ambiguities of traditional methods where certain representations can only identify the rotation over the smallest angle.

It's interesting to look at the rotation over 180 degrees. We can do it by constructing a spinor out of the basis vector e_1 and its negation $-e_1$. Obviously there is a 180 degree angle between them. If we multiply them (see table 3.1 for a refresher), we get:

$$e_1(-e_1) = -(e_1e_1) = -1$$

which makes sense because a rotation over 180 degrees reverses the signs. But this becomes more interesting if we do it through two successive rotations over 90 degrees. A rotation by 90 degrees is a multiplication with the basis bivector I . As an example, consider the following geometric products between the basis blades e_1 , e_2 and I , taken directly from the multiplication table for $\mathcal{Cl}_2$ as in figure 3.1.

$$\begin{aligned} Ie_1 &= -e_2 \\ Ie_2 &= e_1 \\ I(-e_1) &= e_2 \\ I(-e_2) &= -e_1 \end{aligned}$$

Now doing two successive rotations:

$$\begin{aligned} I^2e_1 &= I(Ie_1) = I(-e_2) = -e_1 \\ I^2e_2 &= I(Ie_2) = I(e_1) = -e_2 \end{aligned}$$

This provides yet another demonstration that I^2 equals -1 . But it also demonstrates how we can combine rotations. It turns out that, as our intuition dictates, that the geometric product of two rotations form a new rotation. This demonstrates that we can use the geometric product to concatenate rotations, and that spinors form a subgroup of $\mathcal{Cl}_2$.

To prove this we need to show that the multiplication of two spinors is another spinor. The general form of a spinor is a scalar and a bivector, so let's multiply:

$$\begin{aligned} A &= \alpha_1 + \alpha_4 I \\ B &= \beta_1 + \beta_4 I \end{aligned}$$

Which is easy:

$$\begin{aligned} AB &= (\alpha_1 + \alpha_4 I)(\beta_1 + \beta_4 I) \\ &= (\alpha_1 + \alpha_4 I)\beta_1 + (\alpha_1 + \alpha_4 I)\beta_4 I \\ &= \alpha_1\beta_1 + \alpha_4\beta_1 I + \alpha_1\beta_4 I + \alpha_4\beta_4 II \\ &= \alpha_1\beta_1 + \alpha_4\beta_1 I + \alpha_1\beta_4 I - \alpha_4\beta_4 \\ &= \underbrace{(\alpha_1\beta_1 - \alpha_4\beta_4)}_{\text{scalar part}} + \underbrace{(\alpha_4\beta_1 + \alpha_1\beta_4)I}_{\text{bivector part}} \end{aligned}$$

And we conclude, as expected, that the multiplication of two spinors results in a spinor in $\mathcal{C}\ell_2$.

5.1.3 Lines

Equation (2.2) told us that the outer product of a vector with itself equals zero. We can use this to construct a line through the origin at $(0,0)$. For a given direction vector u all points x on the line satisfy:

$$x \wedge u = 0$$

The proof is easy if you realize that every x can be written as a scalar multiple of u . For lines through an arbitrary point a we can write the following:

$$(x - a) \wedge u = 0$$

We can rewrite this equation in several different forms:

$$\begin{aligned} (x - a) \wedge u &= 0 \\ (x \wedge u) - (a \wedge u) &= 0 \\ (x \wedge u) &= (a \wedge u) \\ (x \wedge u) &= U \end{aligned}$$

Where u is the direction vector of the line and U is the bivector $a \wedge u$. This is depicted in figure 5.1.

Again, note that the bivector has no specific shape. It is only the area that indirectly defines the distance from origin to the line. If we want to calculate the distance directly, we need the vector d as illustrated in figure 5.1. We can calculate this vector easily by doing:

$$d = \frac{U}{u}$$

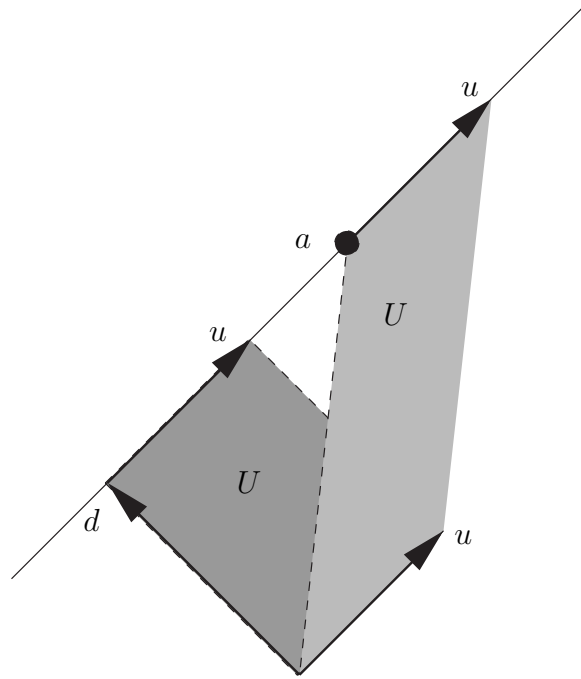


Figure 5.1: Lines in the Euclidian plane

The magnitude of this vector will be the distance from the line to the origin. This is trivial to prove. From the above we see that

$$du = U$$

From equation (3.2) we can write:

$$du = d \cdot u + d \wedge u = U$$

But remember that U is a bivector equal to $d \wedge u$ and that consequently $d \cdot u$ must be zero. Well, if the inner product of two vectors equals zero, they are perpendicular, and hence d is perpendicular to the line, and thus the shortest vector from the origin to the line.

Things are even better if u is a unit vector. In applications this is often the case, and it allows us to write $d = Uu$ because the inverse of a unit vector is the unit vector itself, hence $d = \frac{U}{u} = Uu^{-1} = Uu$. Even more so, in the case that u is a unit vector, then $|U| = |d|$, exactly the distance from the origin to the line.

5.2 Euclidian Space

For those with an interest in computer graphics, the world of three dimensions is the most interesting of course. I will look at some common 3D operations and concepts, and show how they relate to traditional methods. However, we will learn about homogeneous space later, which deals with a four dimensional model of three dimensional space. This model provides significant advantages over $\mathcal{C}\ell_3$. Fortunately, most of the theory presented in this section will be presented in a dimension free manner, and can readily be applied there as well.

5.2.1 Rotations

When I discussed rotations in the Euclidian plane, I showed you that a spinor $\cos \theta + \sin \theta I$ rotates a vector over an angle θ . I then showed that the geometric product of two unit vectors s and t generates a spinor because $s \cdot t = \cos \theta$ and $s \wedge t = \sin \theta I$.

I will now prove that such a *rotation in a plane* works for any dimension. After that I will extend it so we can do arbitrary rotations. Finally, we will see how this relates to a traditional method for rotations. You might be surprised.

Rotations in a plane

We will use a unit-bivector B to define our plane of rotation. Given a vector v lying in the same plane as the bivector, we want to rotate it over an angle θ into a vector v' .

Using a spinor R of the form $\cos \theta + \sin \theta B$, the proposition is:

$$v' = Rv \quad (5.4)$$

We need to prove that v' lies in the same plane as B and v , by demonstrating that

$$v' \wedge B = 0 \quad (5.5)$$

And we need to prove that the angle between v and v' equals θ by showing that the well known dot product equality holds:

$$v' \cdot v = |v'| |v| \cos \theta \quad (5.6)$$

Note that nothing is said about the dimension of the space these vectors and bivector reside in. This makes sure our proof holds for any dimensions

Let's start out by writing the spinor in full:

$$\begin{aligned} v' &= (\cos \theta + \sin \theta B)v \\ &= \cos \theta v + \sin \theta Bv \end{aligned}$$

Because v is a vector and B is a bivector, we can write:

$$v' = \cos \theta v + \sin \theta (B \cdot v + B \wedge v)$$

Because v lies in the same plane as B , we know that $B \wedge v = 0$, resulting in:

$$v' = \cos \theta v + \sin \theta B \cdot v$$

If you refer back to image 3.3 you will see that the inner product between a vector and a bivector returns the complement of the projection. In this case v is already in the bivector plane, and thus its own projection. The resulting complement $B \cdot v$ is a vector in the same plane, but perpendicular to v . We will denote this vector with $v_{\perp}$. This leads to the final result, which will allow us to prove (5.5) and (5.6):

$$v' = \cos \theta v + \sin \theta v_{\perp}$$

The proof of (5.5) is easy. If you look at the last equation, you will see that v' is an addition of two vectors v and $v_{\perp}$. Both these vectors are in the B plane, and hence any addition of these vectors will also be in the same plane. Hence $v' \wedge B = 0$.

Understanding the proof of (5.6) will be easier if you take a look at figure 5.2. There, I've depicted the three relevant vectors in the B plane.

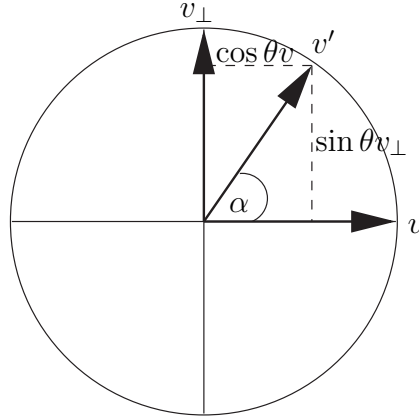


Figure 5.2: Rotation in an arbitrary plane

To recap, we need to show that:

$$v' \cdot v = |v||v'| \cos \theta$$

Vectors v' and v will have equal length. I won't prove this, hoping the picture will suffice. Using this, the equation becomes:

$$v' \cdot v = |v|^2 \cos \theta$$

Writing out v' we get:

$$(\cos \theta v + \sin \theta v_{\perp}) \cdot v = |v|^2 \cos \theta$$

The dot product is distributive over vector addition:

$$\cos \theta v \cdot v + \sin \theta v_{\perp} \cdot v = |v|^2 \cos \theta$$

Because $v_{\perp}$ and v are perpendicular, their dot product equals zero:

$$\cos \theta v \cdot v = |v|^2 \cos \theta$$

And finally, the dot product of a vector with itself is equal to its squared magnitude:

$$\cos \theta |v|^2 = |v|^2 \cos \theta$$

This concludes our proof that a spinor $\cos \theta + \sin \theta B$ rotates any vector in the B -plane over an angle θ . Since we made no assumptions on the dimension anywhere, this shows that spinors work in any dimension. In the Euclidian Plane we didn't have much choice regarding the B -plane, because there is only one plane, but in three or more dimensions we can use any arbitrary bivector.

Yet, the above only works when the vector we wish to rotate lies in the actual plane. Obviously we also want to rotate arbitrary vectors. I will now demonstrate how we can use reflections to achieve this in a dimension free way.

Arbitrary Rotations

Let's see what happens if we perform two reflections on any vector v . One using unit vector s and then another using unit vector t :

$$v' = -t(-svs)t = tsvt$$

We will denote st using R , and because it is a versor we can write $ts = (st)^{\dagger} = R^{\dagger}$:

$$v' = R^{\dagger}vR$$

Note that R and $R^{\dagger}$ have the form:

$$R = st = s \cdot t + s \wedge t$$

$$R^{\dagger} = ts = s \cdot t - s \wedge t$$

Hence, they are spinors -scalar plus bivector- and can be used to rotate in the $s \wedge t$ plane. It is the combination with the reverse that allows us to do arbitrary rotations, removing the restriction that v has to lie in the rotation plane.

Let's denote the plane of rotation that is specified by the bivector $s \wedge t$, with A . Now, decompose v into a part perpendicular to A and a part collinear with A :

$$v = v_{||A} + v_{\perp A}$$

Now the following relations hold:

$$\begin{aligned}
v_{||A}A &= v_{||A} \cdot A + \overbrace{v_{||A} \wedge A}^{=0} \\
&= v_{||A} \cdot A \\
&= -A \cdot v_{||A} \\
&= -A \cdot v_{||A} - A \wedge v_{||A} \\
&= -Av_{||A}
\end{aligned}$$

It may be worth pointing out why $v_{||A} \cdot A = -A \cdot v_{||A}$. The inner product for vectors is commutative whereas the inner product between a vector and a bivector is anticommutative. It turns out that the sign of the commutativity depends on the grade of the blade. For vector p and blade Q with grade r we have:

$$p \cdot Q = (-1)^{r+1} Q \cdot p$$

I will not prove this here. It can be found in Hestenes New Foundations for Classical Mechanics [8] as well as other papers.

For the perpendicular part of v , we have:

$$\begin{aligned}
v_{\perp A}A &= \overbrace{v_{\perp A} \cdot A}^{=0} + v_{\perp A} \wedge A \\
&= v_{\perp A} \wedge A \\
&= A \wedge v_{\perp A} \\
&= A \cdot v_{\perp A} + A \wedge v_{\perp A} \\
&= Av_{\perp A}
\end{aligned}$$

If you are confused by the fact that the outer product between a vector and a bivector turns out to be commutative, then remember that the outer product between a vector and a vector was anticommutative. Hence:

$$\begin{aligned}
v_{\perp A} \wedge A &= v_{\perp A} \wedge s \wedge t \\
&= -s \wedge v_{\perp A} \wedge t \\
&= s \wedge t \wedge v_{\perp A} \\
&= A \wedge v_{\perp A}
\end{aligned}$$

In fact, the commutativity of the outer product between a vector p and a blade Q with grade r is related much like the inner product:

$$p \wedge Q = (-1)^r Q \wedge p$$

For this, the proof is easy. The sign simply depends on the amount of vector-swaps necessary, and hence on the number of vectors in the blade. This is directly related to the grade of the blade.

Here's a quick recap of the identities we just established.

$$v_{||A}A = -Av_{||A} \quad (5.7)$$

$$v_{\perp A}A = Av_{\perp A} \quad (5.8)$$

We will now use these in the rest of the proof.

$$\begin{aligned} R^\dagger v_{||A} &= (s \cdot t - s \wedge t)v_{||A} \\ &= (s \cdot t)v_{||A} - (s \wedge t)v_{||A} \\ &= (s \cdot t)v_{||A} - Av_{||A} \end{aligned}$$

The inner product $s \cdot t$ is just a scalar, and hence the geometric product $(s \cdot t)v_{||A}$ is commutative. This, and equation (5.7) allows us to write:

$$\begin{aligned} R^\dagger v_{||A} &= v_{||A}(s \cdot t) + v_{||A}A \\ &= v_{||A}(s \cdot t) + v_{||A}(s \wedge t) \\ &= v_{||A}(s \cdot t + s \wedge t) \\ &= v_{||A}R \end{aligned}$$

In the same way, using (5.8) this time, we can write:

$$\begin{aligned} R^\dagger v_{\perp A} &= (s \cdot t - s \wedge t)v_{\perp A} \\ &= (s \cdot t)v_{\perp A} - (s \wedge t)v_{\perp A} \\ &= (s \cdot t)v_{\perp A} - Av_{\perp A} \\ &= v_{\perp A}(s \cdot t) - v_{\perp A}A \\ &= v_{\perp A}(s \cdot t) - v_{\perp A}(s \wedge t) \\ &= v_{\perp A}(s \cdot t - s \wedge t) \\ &= v_{\perp A}R^\dagger \end{aligned}$$

To summarize the newly established identities:

$$R^\dagger v_{||A} = v_{||A}R \quad (5.9)$$

$$R^\dagger v_{\perp A} = v_{\perp A}R^\dagger \quad (5.10)$$

We can now return to our original equation, stating that:

$$v' = R^\dagger v R$$

Decomposing v and using (5.9) and (5.10) we get:

$$\begin{aligned} v' &= R^\dagger(v_{||A} + v_{\perp A})R \\ &= R^\dagger v_{||A}R + R^\dagger v_{\perp A}R \\ &= v_{||A}RR + v_{\perp A}R^\dagger R \end{aligned}$$

Remember that s and t are unit vectors and hence:

$$R^\dagger R = tsst = 1$$

This allows us to write:

$$v' = v_{||A}RR + v_{\perp A}$$

Now, notice that we are multiplying the collinear part $v_{||A}$ by R twice, and that the perpendicular part $v_{\perp A}$ is untouched.

Recall that R was a scalar and a bivector and thus a spinor that rotates vectors in the bivector plane. Well, $v_{||A}$ lies in that plane. This means that if we make sure that R rotates over an angle $\frac{1}{2}\theta$, a double rotation will rotate over angle θ . Thus the operation $R^\dagger v R$ does exactly what we want. This is illustrated in figure 5.3.

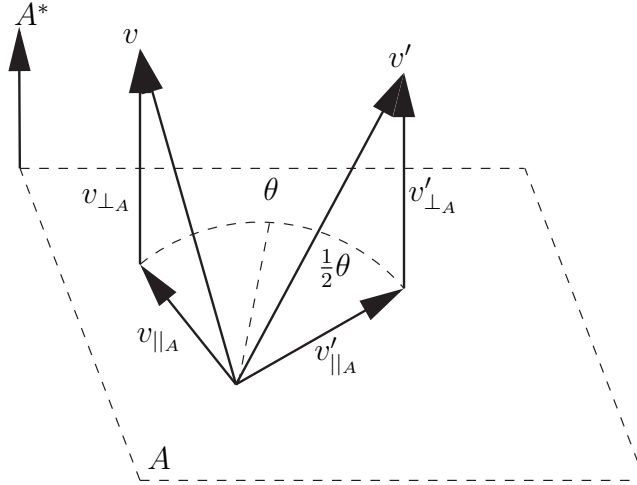


Figure 5.3: An arbitrary rotation

It is worth mentioning that the dual of the A plane, denoted by A^* , is obviously the normal to the plane, and hence the rotation axis.

Conclusion

If, by now, you think that rotations are incredibly complicated in geometric algebra you are in for a surprise. All of the above was just to prove that for any angle θ and bivector A , the spinor:

$$R = \cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta A$$

can perform a rotation of an arbitrary vector relative to the A plane, by doing:

$$v' = R^\dagger v R$$

In other words, the spinor performs a rotation around the normal of the A plane, denoted by the dual A^* . Best of all, spinors don't just work for vectors, but for any multivector. This means that we can also use spinors to rotate bivectors, trivectors or complete multivectors. For example, if we have a bivector B , we can rotate it by θ degrees around an axis a or relative to a bivector $a^* = A$ by :

$$B' = R^\dagger B R$$

$$B' = (\cos \frac{1}{2}\theta - \sin \frac{1}{2}A)B(\cos \frac{1}{2}\theta + \sin \frac{1}{2}A)$$

Proofs of this generalization of spinor theory can be found in most of the references.

Now that we've seen the dimension free definition of spinors, and a proof on why they perform rotations, we can return to Euclidian space and see what it means in three dimensions.

Recall the bivector representation in $\mathcal{C}\ell_3$. We used three basis bivectors e_{12} , e_{13} and e_{23} and denoted any bivector B using a triplet $(\beta_1, \beta_2, \beta_3)$. If B is a unit-bivector and introduce the γ symbol for the scalar, then we can represent a spinor $R \in \mathcal{C}\ell_3$ over an angle θ as follows:

$$R = \underbrace{\cos \frac{1}{2}\theta}_{(\gamma)} + \underbrace{\sin \frac{1}{2}\theta B}_{(\beta_1, \beta_2, \beta_3)}$$

As you see, in three dimensions a spinor consists of four components. Now take a look at the following multiplications (taken from the sign table in figure (3.2)):

$$\begin{aligned} e_{12}e_{12} &= e_{13}e_{13} = e_{23}e_{23} = -1 \\ e_{12}e_{13} &= -e_{23} \\ e_{13}e_{12} &= e_{23} \\ e_{12}e_{23} &= e_{13} \\ e_{23}e_{12} &= -e_{13} \\ e_{13}e_{23} &= -e_{12} \\ e_{23}e_{13} &= e_{12} \end{aligned}$$

And denote the basis bivectors e_{12} , e_{23} , e_{13} using i , j and k respectively. We end up with:

$$\begin{aligned} i^2 &= k^2 = j^2 = -1 \\ ik &= -j \\ ki &= j \\ ij &= k \\ ji &= -k \\ kj &= -i \\ jk &= i \end{aligned}$$

Allowing us to write a spinor $\in \mathcal{C}\ell_3$ as:

$$R = (w, xi, yj, zk)$$

where

$$w = \cos \frac{1}{2}\theta$$

$$x = \sin \frac{1}{2}\theta\beta_1$$

$$y = \sin \frac{1}{2}\theta\beta_2$$

$$z = \sin \frac{1}{2}\theta\beta_3$$

Hopefully this is starting to look familiar by now. Compare my definition of a spinor, including the way i , j and k multiply, with any textbook on quaternions and you will easily see the resemblance. In fact, in three dimensions, spinors are exactly quaternions. A quaternion rotation is exactly a vector sandwiched in between the quaternion and its inverse. But because we are using unit-quaternions (or unit-spinors) the inverse is the same as the reverse.

Don't assume that all of this is a coincidence, or merely a striking resemblance. Quaternions and spinors are exactly the same:

A quaternion is a scalar plus a bivector

Apart from the fact that quaternions have four components, there is nothing four-dimensional or imaginary about a quaternion. The first component is a scalar, and the other three components form the bivector-plane relative to which the rotation is performed.

If you would look at the conversion from an axis-angle representation to a quaternion, you would see that the angle is divided by two. This makes sense now, because the spinor rotates the collinear part of a vector twice. Also, the conversion takes the dual of the axis and multiplies it by $\sin \frac{1}{2}\theta$ to create exactly the bivector we need. If you would write out the geometric product of two spinors, you would notice that it is exactly the same as a quaternion multiplication. Finally, the inverse of a spinor (w, x, y, z) is $(w, -x, -y, -z)$ which follows from the fact that the inverse rotates over the opposite angle combined with the fact that $\cos(-\theta) = \cos \theta$ yet $\sin(-\theta) = -\sin \theta$.

Spinors may be quaternions in 3D, but geometric algebra has given us much more:

- There is nothing four dimensional or imaginary to a quaternion. It is simply a scalar plus a real bivector. This gives us a way to actually visualize quaternions. Compare this to the many failed attempts to draw four dimensional unit hyperspheres in textbook examples on quaternions.
- Spinors rotate more than just vectors. They can rotate bivectors, trivectors, n -blades and any multivector. If you ever tried rotating the normal of plane, you will readily appreciate the ability to rotate the plane directly.

- Unlike quaternions, spinors are dimension-free and work just as well in 2D, 4D and any other dimension. Their theory extends to any space without changes.

Back to Reflections

At the start of this section we combined two reflections to generate a rotation. Let's look at that again, to get another way of looking at rotations. So, we perform two reflections on any vector v . One using unit vector s and then another using unit vector t :

$$v' = -t(-svs)t = tsvt$$

We denoted st using R and ts using $R^\dagger$ and worked with those from then. Instead, we will now look at what those two reflections do in a geometric interpretation. This is illustrated in figure 5.4 using a top down view.

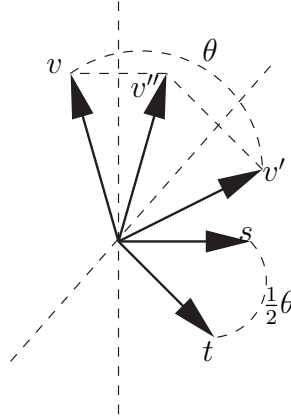


Figure 5.4: A rotation using two reflections

In this figure, we reflect v in the plane perpendicular to s to obtain vector v'' . Then we reflect that vector in the plane perpendicular to t to obtain the final rotated vector v' .

We can actually easily prove this by adding a few angles. We will measure the angles using the bivector dual of s as the frame of reference. Then we denote the angle of vector v as $\theta(v)$. Thus, if we reflect v in this bivector dual we obtain vector v'' with an angle of $\theta(v'') = -\theta(v)$. Then we mirror this vector v'' in the bivector dual of t to obtain an angle $\theta(v')$ equal to:

$$\begin{aligned}\theta(v') &= \theta(t) + \theta(t) - \theta(v'') \\ &= \theta(t) + \theta(t) + \theta(v) \\ &= 2\theta(t) + \theta(v)\end{aligned}$$

This proves that the angle between v and v' is twice the angle between s and t . That this works for vectors even outside of the $s \wedge t$ plane is left as an exercise for the reader. Also, try to convince yourself that reflecting in t followed by s is precisely the opposite rotation over the angle $-\theta$.

5.2.2 Lines

We've discussed lines before when we looked at the Euclidian plane. The interesting thing is that no reference to any specific dimension was made. As a result, we can use the same equations to describe a line in 3D. That is, for a line through point a with direction-vector u , every point x on that line will satisfy all of the following:

$$\begin{aligned}(x - a) \wedge u &= 0 \\ (x \wedge u) - (a \wedge u) &= 0 \\ (x \wedge u) &= (a \wedge u) \\ (x \wedge u) &= U\end{aligned}$$

Where U is the bivector $a \wedge u$. Also, the shortest distance from the origin is again given by vector $d = \frac{U}{u}$. The short proof given in the previous section on lines is still valid. Again, if u is a unit vector, then $d = Uu$ because $u^{-1} = u$.

5.2.3 Planes

Talking about planes in the Euclidian plane wasn't very useful as there is only one plane. In three dimensions, things are more interesting. Of course, the concept we use to model a plane should not come as a surprise. We've been using it implicitly in our talk about rotations already. We can model a plane by using a bivector. That is, any point x on the plane will satisfy:

$$x \wedge B = 0$$

Remember that the outer product between a bivector and a vector collinear with the bivector will equal zero, and the above will be obvious. If we wish to describe a plane through a point a , we can use the following:

$$(x - a) \wedge B = 0$$

Compare this with the equations for describing a line. It's interesting to see that the outer product with a vector describes a line, and the outer product with a bivector describes a plane.

In three dimensions the equation above requires six components. Three for bivector B , and three for vector a . Fortunately, we can rewrite it, so it requires four components, comparable to the traditional way of storing a plane where you use the normal and the shortest distance to the origin. It's as follows:

$$\begin{aligned}(x - a) \wedge B &= 0 \\ x \wedge B - a \wedge B &= 0 \\ x \wedge B &= a \wedge B \\ x \wedge B &= U\end{aligned}$$

Notice that the part on the right side is a trivector (denoted by U). In 3D trivectors are represented by just one scalar, allowing us to store a plane using

four components. This scalar represents the volume of the pyramid spanned by bivector B and the origin. In most applications, we can easily make sure that B is a unit-bivector. The consequence is that the scalar is exactly the distance from the origin to the plane.

5.3 Homogeneous Space

So far, we have used a point on a plane and point on a line to characterize the actual position of the plane and line in the Euclidian Plane and Space. For example, to describe a plane in three dimensions we needed a bivector to define its orientation and a coordinate-vector to define its position. In this section, we will embed an n -dimensional space in a higher $n + 1$ -dimensional space. This builds upon the traditional homogeneous model and it will enable us to use pure blades to describe elements from the n -space without the need for a supporting position.

5.3.1 Three dimensional homogeneous space

Let's look at the Euclidian plane and embed it in a three dimensional space. All we have to do is extend our vector basis e_1, e_2 with a third unit vector e that is perpendicular to the other two:

$$\begin{aligned}e_1 \cdot e_2 &= 0 \\e_1 \cdot e &= 0 \\e_2 \cdot e &= 0 \\e_1 \cdot e_1 &= |e_1| = 1 \\e_2 \cdot e_2 &= |e_2| = 1 \\e \cdot e &= |e| = 1\end{aligned}$$

We will then describe every coordinate (x, y) in the euclidian plane as $(\frac{x}{w}, \frac{y}{w}, \frac{w}{w})$. More commonly we make sure that w equals one so we can simply write $(x, y, 1)$. By doing this, we have effectively taken the origin $(0, 0)$ and placed it a dimension higher. This means that the origin is no longer a special case, allowing us to do *positional geometry* in easier ways, without the need for supporting coordinates.

Lines

Let's construct the line through points P and Q . Instead of taking the direction vector $P - Q$ and a supporting coordinate P or Q , we can simply take the bivector $P \wedge Q$ and that's it. This is shown in figure 5.5. The resulting bivector uniquely defines the line through P and Q and conversely $Q \wedge P$ defines the line with opposite direction.

Recall that a bivector has no shape, only orientation and area. Hence, figure 5.5 displays three possible representations for the bivector $P \wedge Q$ which are all the same, and all define the same line. Simply put, the bivector defines a plane, and the intersection of the plane with the Euclidian plane (depicted with the dashed plane) defines the line.

The question is what this representation buys us. First, we can transform lines a lot easier now. For example, instead of having to rotate two entities (the position vector and the direction vector) we simply rotate the bivector.

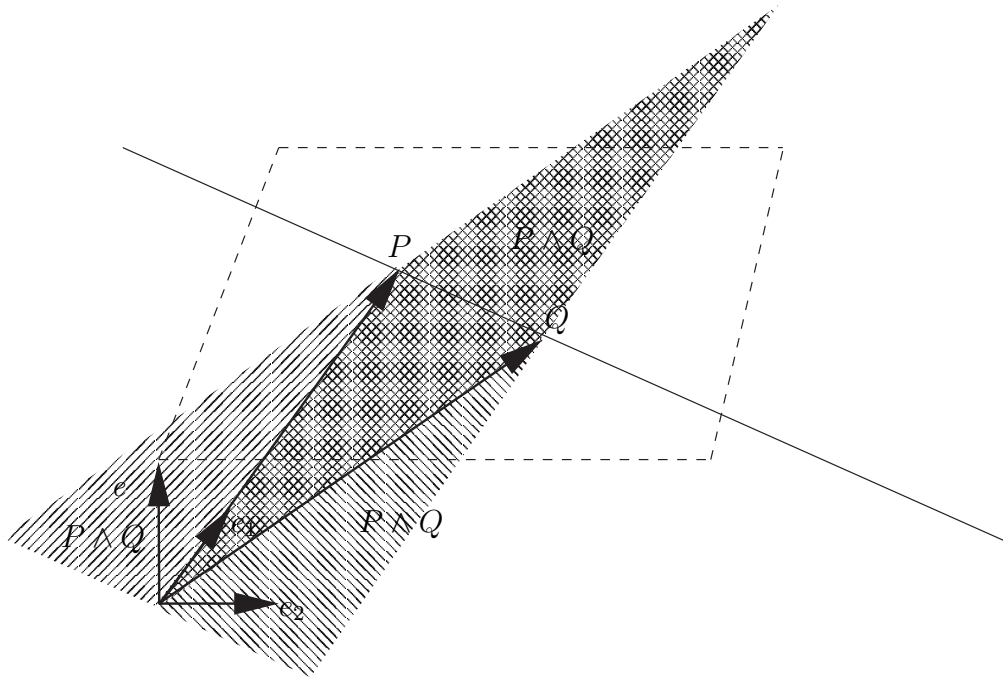


Figure 5.5: A two dimensional line in the homogeneous model

Another benefit shows up when we want to calculate the intersection point of two lines. This is where the meet operator becomes incredibly useful. Let's look at two lines in the homogeneous model, and see what their meet results in. We have a line through points P and Q defined by the bivector $P \wedge Q$, and a line through points N and M defined by the bivector $N \wedge M$. The intersection point will be exactly at:

$$(P \wedge Q) \cap (N \wedge M) = (P \wedge Q)^* \cdot (N \wedge M)$$

Instead of proving this, I will give a numerical example. Let's suppose our coordinates are:

$$\begin{aligned} P &= (2, 0, 1) = 2e_1 + 0e_2 + 1e \\ Q &= (2, 4, 1) = 2e_1 + 4e_2 + 1e \\ N &= (0, 2, 1) = 0e_1 + 2e_2 + 1e \\ M &= (4, 2, 1) = 4e_1 + 2e_2 + 1e \end{aligned}$$

This is shown in figure 5.6. Obviously we can already reason that the intersection of the two lines is at $(2, 2, 1)$. The bivectors that correspondingly define

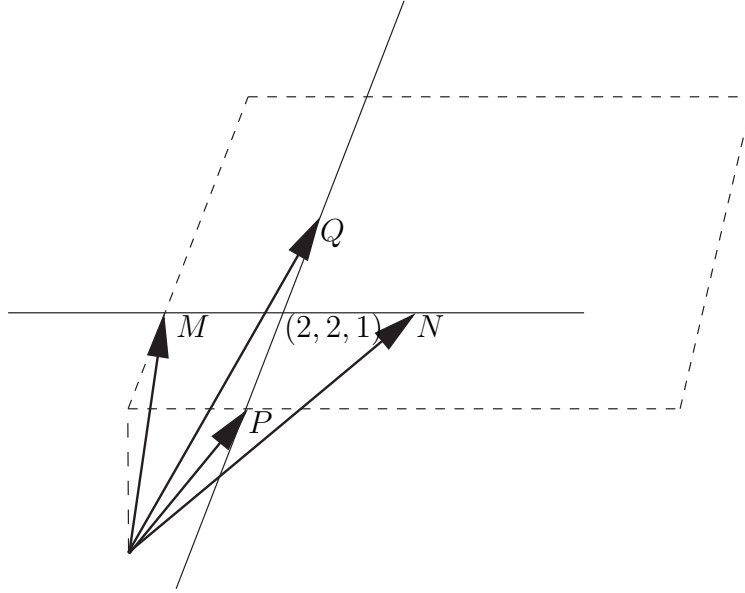


Figure 5.6: An homogeneous intersection

the lines are:

$$\begin{aligned} P \wedge Q &= (2, 0, 1) \wedge (2, 4, 1) \\ N \wedge M &= (0, 2, 1) \wedge (4, 2, 1) \end{aligned}$$

We use equation (2.7) to calculate the actual bivectors:

$$\begin{aligned} P \wedge Q &= (2 \cdot 4 - 2 \cdot 0, \\ &\quad 2 \cdot 1 - 2 \cdot 1, \\ &\quad 0 \cdot 1 - 4 \cdot 1) = (8, 0, -4) \\ N \wedge M &= (0 \cdot 2 - 4 \cdot 2, \\ &\quad 0 \cdot 1 - 4 \cdot 1, \\ &\quad 2 \cdot 1 - 2 \cdot 1) = (-8, -4, 0) \end{aligned}$$

It is worth stressing that the bivector triplets are again linear combinations of the basis bivectors. However, instead of the basis vector e_3 we use the special symbol vector e . This creates the following bivectors:

$$\begin{aligned} P \wedge Q &= (8, 0, -4) = 8e_{12} + 0(e_1 \wedge e) - 4(e_2 \wedge e) \\ N \wedge M &= (-8, -4, 0) = -8e_{12} - 4(e_1 \wedge e) + 0(e_2 \wedge e) \end{aligned}$$

With these two definitions of the lines trough P and Q and trough N and M we can use the meet operator to find their intersection. First we calculate the

dual of $P \wedge Q$ using equation (4.5):

$$\begin{aligned}(P \wedge Q)^* &= (8, 0, -4)^* \\ &= (-4, 0, 8) \\ &= -4e_1 + 0e_2 + 8e_3\end{aligned}$$

Now to complete the meet operator:

$$\begin{aligned}(P \wedge Q)^* \cdot (N \wedge M) &= (8, 0, -4)^* \cdot (-8, -4, 0) \\ &= (-4, 0, 8) \cdot (-8, -4, 0)\end{aligned}$$

It is important to realize that the right hand side of the inner product is a bivector and not a vector. Thus, we have the inner product between a vector (on the left hand side) and a bivector. We will use the earlier definition of the contraction inner-product to calculate this. To be precise, we employ definition (3.18). The meet of the two lines is:

$$(P \wedge Q)^* \rfloor (N \wedge M) = ((P \wedge Q)^* \rfloor N) \wedge M - N \wedge ((P \wedge Q)^* \rfloor M)$$

Now we notice two inner products between vectors, which we know how to calculate. They will result in scalars. Then we are left with outer products between scalars and bivectors. As it happens to be, the outer product of a scalar and a bivector is simply the intuitive scalar product. Hence, we calculate:

$$\begin{aligned}((-4, 0, 8) \rfloor (0, 2, 1)) \wedge (4, 2, 1) &- (0, 2, 1) \wedge ((-4, 0, 8) \rfloor (4, 2, 1)) = \\ 8 \wedge (4, 2, 1) &- (0, 2, 1) \wedge -8 = \\ (32, 16, 8) &- (0, -16, -8) = \\ (32, 32, 16)\end{aligned}$$

Now remember that in homogeneous space every coordinate is defined as $(\frac{x}{w}, \frac{y}{w}, w)$, hence the intersection point of the two lines is:

$$\left(\frac{32}{16}, \frac{32}{16}, \frac{16}{16}\right) = (2, 2, 1)$$

Which obviously, and according to figure 5.6, is correct.

The complete calculation we just did is rather verbose and not something you would usually do. Instead, we just accept the meet operator and its definition using the inner product and the dual operator. Using the meet we can simply denote the intersection of two lines using the $\cap$ symbol.

5.3.2 Four dimensional homogeneous space

We have used a three dimensional space to model the Euclidean plane. We will now use $\mathcal{C}\ell_4$ to embed Euclidian space. Once again, we use a fourth basis vector e orthogonal to the other three basis vectors e_1 , e_2 and e_3 . All coordinates (x, y, z) will be denoted using $(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}, \frac{w}{w})$. Again, if w equals one we can write $(x, y, z, 1)$.

Lines in 4D homogeneous space

Constructing a line in this model is no different than we did before. For a line through points P and Q we take the outer product to construct a bivector that uniquely defines the line in three dimensions. It's rather complicated to depict this in a figure since we have no means to effectively display a four dimensional space. Fortunately, the same rules still apply. If we want to rotate a line we use a spinor. Of course such a spinor would now be a spinor in $\mathcal{Cl}_4$. Also, if we want the intersection of two lines we simply take the meet of the two bivectors.

Let us take a brief moment to look at the bivector representation in the homogeneous model. As discussed before, there are six basis bivectors in $\mathcal{Cl}_4$. Instead of the basis vector e_4 we have the special basis vector e . This creates the following set of basis bivectors:

$$e_1 \wedge e_2 = e_{12}$$

$$e_1 \wedge e_3 = e_{13}$$

$$e_2 \wedge e_3 = e_{23}$$

$$e_1 \wedge e$$

$$e_2 \wedge e$$

$$e_3 \wedge e$$

Thus, in a four dimensional geometric algebra, a bivector forms a linear combination of the six basis bivectors. Hence we need six components to define a three dimensional line. As it turns out, these six scalars are what other theories commonly refer to as Plückerspace coordinates. However, the only thing that is six-dimensional about Plückerspace is the fact that we use six scalars. Everything becomes much more intuitive once we realize that in the four dimensional homogeneous model bivectors define lines and arbitrary bivectors are linear combinations of the six basis bivectors. In fact, the well known Plückerspace operations to define relations among lines can be expressed in terms of the *meet* and the related *join* operator which defines the union of two subspaces. Unfortunately, a more detailed discussion of the join and meet operators is beyond this paper.

Planes

Lines are defined by two coordinates, and we can express them by taking the outer product of the two coordinates. We can easily extend this. Planes are defined by three coordinates, and the homogeneous representation of a plane is simply the outer product of these three points. For example, the plane through points P , Q and R is defined by the trivector:

$$P \wedge Q \wedge R$$

For example, given a triangle defined by those three points there is no need to calculate the edge vectors to perform a cross product, after which the distance

to the origin still needs to be calculated. We simply take the outer product of the three vertices to obtain the bivector which defines the plane.

As with lines before, transforming the plane can simply be done by transforming the associated trivector. And not surprising, and very elegantly, the intersection of two planes or the intersection of a plane and a line can simply be done using the meet operator. The actual proof is beyond this paper, but trying out some numerical examples yourself should easily convince you that the meet operator does not discriminate among subspaces.

5.3.3 Concluding Homogeneous Space

Traditional Euclidian vector algebra always uses a supporting position vector to define the location of elements in space. This complicates all calculations since they have to act on two separate concepts (position and orientation) instead of just one. Furthermore, because the origin is treated as a special kind of position vector, we often run into special cases and exceptions in our algorithms.

Homogeneous space allows us to use k -blades in an n -dimensional space to define positional $(k - 1)$ -blades in an $(n - 1)$ -dimensional space. This greatly simplifies our algorithms and it gives us the ability to reason about subspaces without the need for special cases or exceptions.

At first sight, this elegance and ease of use comes with a considerable overhead; namely the extra storage required for the added dimension. However, at a second glance we notice that it's not entirely true.

A vector takes three components in $\mathcal{Cl}_3$ and four in $\mathcal{Cl}_4$. However, most often the last scalar is equal to one and could be optimized away. There does not seem to be much need for this though. The advantages of homogeneous vectors have already been acknowledged in many traditional applications. An obvious example is matrix-algebra where four-by-four matrices are used to store rotations and translations combined.

In $\mathcal{Cl}_4$ a bivector takes six scalars, and a trivector takes four scalars. Refer back to figure 2.3 for a reminder. We've used a bivector to represent a line, and a trivector to represent a plane. The amount of components required is therefore exactly the same as it would be in Euclidian space. There, we use two three-component vectors to represent a line, and four scalars to represent a plane.

Admittedly, the homogeneous model is not optimal in all cases. There are certain operations and algorithms where a traditional approach is more suitable and other contexts where the homogeneous model is advantageous. Geometric algebra allows us to use both models alongside each other and use the one that best fits the situation.

Chapter 6

Conclusion

Geometric Algebra introduces two new concepts. First, it acknowledges that a vector is a one-dimensional subspace and hence that there should be higher-dimensional subspaces as well. It then defines bivectors, trivectors and k -blades as the generalization of subspaces. Multivectors are linear combinations of several blades of different grades.

Secondly, it defines the geometric product for multivectors. Because the geometric product embodies both the inner and outer product it combines the notion of orthogonality and collinearity into one operation. Combined with the fact that the geometric product gives most multivectors an inverse it becomes an extremely powerful operator capable of expressing many different geometric relations and ideas.

Using blades and multivectors and the inner, outer and geometric product we can build up a set of tools consisting of the dual, inverses, projections and rejections, reflections, the meet and potentially much more.

The fundamentals and the tools give us new ways of defining common elements like lines and planes. Both in traditional models and in the homogeneous model. It also gives us new ways to reason about the relations between elements.

Furthermore, geometric algebra provides the full background for quaternion theory, finally demonstrating that there is nothing four-dimensional about them. Instead, they are just a linear combination of a scalar and a bivector. It also explains Plücker space in terms of the homogeneous model and the join and meet operators.

Best of all, the entire theory is independent of any dimension. This allows us to provide independent proofs and algorithms that can be applied to any situation. In fact, it is often intuitive to sketch a proof in a lower dimension and then extend it to the general case. Geometric algebra greatly simplifies this process.

Still, it is not just the mere elegance and power of geometric algebra that makes it interesting, but it is the simple fact that we don't lose any of our traditional methods. Geometric algebra explains and embraces them, and then enriches and unifies those existing theories into one.

6.1 The future of geometric algebra

Currently, there are two major obstacles that prohibit the mainstream acceptance of geometric algebra as the language of choice.

First of all, the learning curve for geometric algebra is too steep. Existing introductory material assumes the reader has a certain mathematical knowledge that many doing geometry in practical environments lack. Furthermore, old habits die hard. So instead of presenting geometric algebra as a replacement of traditional methods, it should simply be a matter of language. Geometric algebra is a language to express old and new concepts in easier ways. Hopefully this paper succeeded in providing a relatively easy introduction and demonstrating that it can explain and expand on existing theories.

Secondly, there is still a lot of work to be done when it concerns implementations and applications. Even though geometric algebra looks promising for computational geometry at first sight, it turns out that a mapping from theory to practice is not as straightforward as one would hope. Obviously the old methods and theories have been used in practice for years and have undergone severe tweaking and tuning. Geometric algebra still has a long way to go before we will see implementations that allow us to benefit from the theoretical expressiveness and elegance without sacrificing performance. At the moment it is best used in theory, after which actual implementations give up on the generality and simply provide specialized algorithms for specific problem domains.

Fortunately, there are several initiatives to work on this second obstacle. The first is a library by Daniel Fontijne, Leo Dorst and Tim Bouma called Gaigen [11]. This library comes in the form of a code generator that generates optimized code given a certain algebra signature. Another initiative is the Clifford library [12] which uses meta programming techniques to transform generic function calls into specialized and optimized implementations.

6.2 Further Reading

This paper merely tries to provide a simple introduction to geometric algebra and some of its applications. Hopefully it gives the reader a small foundation on which to pursue more detailed and thorough material.

I can strongly recommend reading chapters one, two and five of Hestenes' *New Foundations for Classical Mechanics* [8], which is still considered the best and most intuitive introduction to geometric algebra by many. All of the material on Leo Dorst's website [6] is very informative and useful, especially the two papers published in the IEEE Computer Graphics and Applications [2] [3]. Finally, the material on Hestenes' website [10] provides some rather advanced, but useful, material. Especially the *Old Wine in New Bottles* paper which introduces a new and very elegant model called the conformal split. If you wonder why planes are spheres through infinity and how this can be useful, this [9] is the paper to read.

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