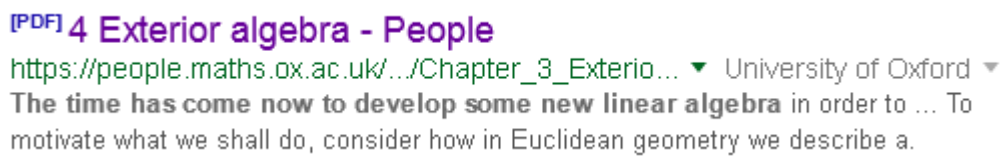
Notes on the Hitchin Exterior PDF PhL 8.31.15

1. Who wrote the PDF?

He has no name on his doc, but I found it this way:

When I get a Google link to a PDF, how do I find the source. For example,



The link is some google mess,

https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=1&cad=rja&uact=8&ved=0CCAQFjAAahUKEwiH09zS5NPHAhVMFpIKHYyYANo&url=https%3A%2F%2Fpeople.maths.ox.ac.uk%2Fhitchin%2Fhitchinnotes%2FProjective_geometry%2FChapter_3_Exterior.pdf&ei=1IbkVYeZG8ysyASMsYLQDQ&usg=AFQjCNExgeyztIomrtqUzocjV5AmpnreUw&sig2=MmsCmmYaQAwOJXoSjeGJeQ

and if I copy the green link it is incomplete:

https://people.maths.ox.ac.uk/.../Chapter_3_Exterio...

BUT, look at the google link and you see a name Fhitchin. But there is no-one there by that name. Oh, duh, it is F Hitchin and we have a nice picture, he is even an FRS, prof of geometry.

http://people.maths.ox.ac.uk/hitchin/

and now we see his full set of notes, 4 chapters. So I will adjust my PDF doc name .

Projective Geometry

- Chapter 1: Projective Geometry
- Chapter 2: Quadrics
- Chapter 3: Exterior
- Chapter 4: The Klein programme

Geometry of Surfaces

- Notes

Differentiable Manifolds

- Notes

Algebraic Curves

- Notes

2. Notes on his PDF.

How can I get Maple cmul to work better without having to state some K matrix? What is going on here?

Question: What does Maple mean by saying B is "the bilinear form" ? I think this is just a 2-multilinear function. Here is something from my "chapter_3.."doc: (now known as Hitchin, and really Ch 4)

Definition 12 *An alternating bilinear form on a vector space V is a map $B : V \times V \to F$ such that*

- $B(v, w) = -B(w, v)$
- $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1, w) + \lambda_2 B(v_2, w)$

This is the skew-symmetric version of the symmetric bilinear forms we used to define quadrics. Given a basis $\{v_1, \ldots, v_n\}$, B is uniquely determined by the skew symmetric matrix $B(v_i, v_j)$. We can add alternating forms and multiply by scalars so they form a vector space, isomorphic to the space of skew-symmetric $n \times n$ matrices. This has dimension $n(n-1)/2$, spanned by the basis elements E^{ab} for $a < b$ where $E^{ab}_{ij} = 0$ if $\{a, b\} \neq \{i, j\}$ and $E^{ab}_{ab} = -E^{ab}_{ba} = 1$.

OK, so B is a linear mapping from VxV into some F which I guess is a field, take to be reals for now. The basis vectors of VxV can be taken to be $v_i \otimes v_j$. This particular basis "vector" of V^2 could be used to evaluate

$$B_{ij} \equiv B(\mathbf{v}_i, \mathbf{v}_j)$$

Now suppose we have some vectors **a** and **b** in V. Then

$\mathbf{a} = \Sigma_i a_i \mathbf{v}_i$
$\mathbf{b} = \Sigma_j b_j \mathbf{v}_j$

$B(\mathbf{a},\mathbf{b}) = B(\Sigma_i a_i \mathbf{v}_i, \Sigma_j b_j \mathbf{v}_j) \quad = \Sigma_{ij} a_i b_j \quad B(\mathbf{v}_i, \mathbf{v}_j)$

So I agree with the claim that $B(\mathbf{v}_i,\mathbf{v}_j)$ completely determines B. I agree that adding two such "forms" gives another bilinear form, and scalar stuff works right, so I agree that the set of such B forms comprises a vector space. The added alternating property does not change this fact. Since $B_{ij} \equiv B(\mathbf{v}_i,\mathbf{v}_j)$ is a matrix, and in fact a "skew symmetric matrix", I agree that the forms are in isomorphism with these matrices. Such B matrices have no diagonal elements and have $(n^2 - n)/2$ in a triangle so you can specify this number of coefficients in writing $B = \Sigma_{ij} a_{ij} B_{ij}$ and this the dimension is n(n-1)/2, I agree. Here are some "basis matrices" for n =3 where n(n-1)/2 = 3*2/2 = 3:

$$E^{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E^{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad E^{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

So everything said above makes sense to me.

Let's call this vector space W for the moment. The elements of W are functions $B(\mathbf{v}_1,\mathbf{v}_2)$. If W' is the matrix space, then its elements are matrices B. The functions are bilinear and alternating, and the matrices are skew matrices.

Continuing on,

Definition 13 *The second exterior power $\Lambda^2 V$ of a finite-dimensional vector space is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-vectors.*

Parse please: "the vector space of alternating bilinear forms on V" is the vector space W just discussed above, having dimension n(n-1)/2. It is isomorphic to a matrix space W' as noted. The "dual space" is a set of *functionals* which can act on the vectors of W or W' to give (for me) a real number. What are some examples of such functionals?

$\mathbf{x} \bullet B \bullet \mathbf{y}$: $B \rightarrow R$ $\qquad x^T B x = s$ $\qquad B$ = matrix.

Is this linear? I think so: $\mathbf{x} \bullet (B_1 + 5B_2) \bullet \mathbf{y} = \mathbf{x} \bullet B_1 \bullet \mathbf{y} + 5\, \mathbf{x} \bullet B_2 \bullet \mathbf{y}$

I may be way off base here. In the W space you could have

$\int dudv\, w(u,v)\, B(u,v)$ = linear functional of B

So this very abstract idea is used to define the space $\Lambda^2(V)$. Now is a matrix a 2-vector? I think he means a bivector or a 2-blade. I know that a 1D matrix can represent a vector, so maybe a 2D matrix can represent a bivector. This idea did not appear in either Suter or Denker.

Next:

Definition 14 *Given $u, v \in V$ the exterior product $u \wedge v \in \Lambda^2 V$ is the linear map to F which, on an alternating bilinear form B, takes the value*

$$(u \wedge v)(B) = B(u, v).$$

So you see here how it is a functional of B and hence the dual space thing. Note that u and v are vectors, meaning they are 1-blades ONLY.

I guess he is saying that ANY alternating bilinear functional B defines a wedge product, so there are then lots of different wedge products possible. This might be the matrix B that is involved when Maple does its cmul operation. Now one of my Spivak equations (see "wedge product")

$$[f \wedge g](\mathbf{v_1},\mathbf{v_2}) = f(\mathbf{v_1})\, g(\mathbf{v_2}) \;-\; g(\mathbf{v_1})\, f(\mathbf{v_2}) \;=\; \det \begin{pmatrix} f(\mathbf{v_1}) & g(\mathbf{v_1}) \\ f(\mathbf{v_2}) & g(\mathbf{v_2}) \end{pmatrix}$$

So maybe write this as

$$[f \wedge g] = fg - gf = B(f,g)$$

Maybe the products are geo products and don't commute. Recall also that for vectors a and b

$$a \wedge b = \frac{1}{2}(ab - ba) \qquad (3.10)$$

so we are pretty close apart from factor of 2. Maybe this is the only B possible apart from a constant? It has to be linear in each argument separately. Let's continue,

From this definition follows some basic properties:

- $(u \wedge v)(B) = B(u, v) = -B(v, u) = -(v \wedge u)(B)$ so that

  $$v \wedge u = -u \wedge v$$

  and in particular $u \wedge u = 0$.
- $((\lambda_1 u_1 + \lambda_2 u_2) \wedge v)(B) = B(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v)$ which implies

  $$(\lambda_1 u_1 + \lambda_2 u_2) \wedge v = \lambda_1 u_1 \wedge v + \lambda_2 u_2 \wedge v.$$
- if $\{v_1, \ldots, v_n\}$ is a basis for V then $v_i \wedge v_j$ for $i < j$ is a basis for $\Lambda^2 V$.

This last property holds because $v_i \wedge v_j(E^{ab}) = E^{ab}_{ij}$ and in facts shows that $\{v_i \wedge v_j\}$ is the dual basis to the basis $\{E^{ab}\}$.

Next,

Proposition 15 *Let $u \in V$ be a non-zero vector. Then $u \wedge v = 0$ if and only if $v = \lambda u$ for some scalar λ.*

Proof both ways is simple. I now skip some stuff and get around to

4.2 Higher exterior powers

Definition 15 *An alternating multilinear form of degree p on a vector space V is a map $M : V \times \ldots \times V \to F$ such that*

- $M(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_p) = -M(u_1, \ldots, u_j, \ldots, u_i, \ldots, u_p)$
- $M(\lambda_1 v_1 + \lambda_2 v_2, u_2, \ldots, u_p) = \lambda_1 M(v_1, u_2, \ldots, u_p) + \lambda_2 M(v_2, u_2, \ldots, u_p)$

Example: Let $u_1, \ldots, u_n$ be column vectors in $\mathbf{R}^n$. Then

$$M(u_1, \ldots, u_n) = \det(u_1 u_2 \ldots u_n)$$

is an alternating multilinear form of degree n.

So he is just extending the B(u,v) idea to more arguments and calling it M. His second line does not say that this must be true for all arguments, but I guess it follows from the first line. And we have the usual example of a viable M.

Definition 16 *The p-th exterior power $\Lambda^p V$ of a finite-dimensional vector space is the dual space of the vector space of alternating multilinear forms of degree p on V. Elements of $\Lambda^p V$ are called p-vectors.*

Just extending the earlier bilinear claim.

Definition 17 *Given $u_1, \ldots, u_p \in V$ the exterior product $u_1 \wedge u_2 \wedge \ldots \wedge u_p \in \Lambda^p V$ is the linear map to F which, on an alternating multilinear form M takes the value*

$$(u_1 \wedge u_2 \wedge \ldots \wedge u_p)(M) = M(u_1, u_2, \ldots, u_p).$$

Now again, (....) is the name of a functional which acts on the space of possible M functions. Now maybe Spivak has taken this one step further where the u_i as vectors in V become *functions* in V which is a function vector space. Then you could show each u_i having a little argument, or perhaps several arguments. But Hitchins is not doing that here.

Proposition 16 *The exterior product $u_1 \wedge u_2 \wedge \ldots \wedge u_p$ of p vectors $u_i \in V$ vanishes if and only if the vectors are linearly dependent.*

Now in his next act, Hitchins wants to talk about applying a certain transformation T* onto one of our M functions,

The exterior powers $\Lambda^p V$ have natural properties with respect to linear transformations: given a linear transformation $T : V \to W$, and an alternating multilinear form M on W we can define an induced one T^*M on V by

$$T^*M(v_1, \ldots, v_p) = M(Tv_1, \ldots, Tv_p)$$

and this defines a dual linear map

$$\Lambda^p T : \Lambda^p V \to \Lambda^p W$$

with the property that

$$\Lambda^p T(v_1 \wedge v_2 \wedge \ldots \wedge v_p) = Tv_1 \wedge Tv_2 \wedge \ldots \wedge Tv_p.$$

Why does he refer to T* as mapping $\Lambda^p T$? Well, M is associated with Λ^p, it is an element of Λ^p I guess, the way B was an element of the vector space Λ^2. So I will just regard "$\Lambda^p T$" as the name of this mapping. Now notice

$(u_1 \wedge u_2)(M) = M(u_1,u_2)$

$T^*M(u_1,u_2) = M(Tu_1,Tu_2) = (Tu_1 \wedge Tu_2)(M)$

$T^*M(u_1,u_2) = T^* (u_1 \wedge u_2)(M)$

so since true for all M, we get

$T^* (u_1 \wedge u_2) = Tu_1 \wedge Tu_2$

and so I agree with his last line, where we call T* = "$\Lambda^p T$" . This is a transformation acting on a wedge product, perhaps this is the pullback business somehow.

Special case: Suppose p = n = dimension of V. He then shows that you arrive at this result,

$$\Lambda^n T(v_1 \wedge \ldots \wedge v_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) T_{\sigma(1)1} T_{\sigma(2)2} \ldots T_{\sigma(n)n} v_1 \wedge \ldots \wedge v_n$$

where T_{ij} is the matrix for the linear operator T acting on V as in Tu_1 above. Thus he shows that

$T^*(v_1 \wedge \ldots \wedge v_n) = \det(T)$

Not clear why this is useful. Perhaps this is the only T* possible for order n. Hence $T^* = \Lambda^n T$ a better name.

He goes on to derive various properties of ^ products. I am now losing interest in this paper.

I was hoping to make a connection to the B bilinear form of Maple! Hitchin did use such a form B(u,v), but I see no connection between this B and the geometric product, so a dead end I am afraid.