

Clifford Algebras and Bilinear Forms on Spinors

Honors Thesis
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Abstract

Associated with the vector space $\mathbb{R}^{p+q}$ with metric g of signature (p, q) is its Clifford algebra, denoted $Cl_{p,q}$. Inside $Cl_{p,q}$ lie the groups $Pin(p, q)$ and $Spin(p, q)$, which double cover $O(p, q)$ and $SO(p, q)$, respectively. We focus on two issues which seem to be neglected in the standard literature. The first is when $Cl_{p,q}$, $Pin(p, q)$, and $Spin(p, q)$ are isomorphic to $Cl_{q,p}$, $Pin(q, p)$, and $Spin(q, p)$. While a partial answer can be given implicitly by the representations of the various algebras, our arguments are based purely on the Clifford algebra structure. In the second section we construct natural bilinear forms on the space of spinors such that vectors are self-adjoint (up to sign). These forms are preserved (up to sign) by the Pin and Spin groups. With the Clifford action of k -forms, $0 \leq k \leq p + q$, on spinors, the bilinear forms allow us to relate spinors with elements of the exterior algebra. We then find some curious identities involving the norms of various forms.

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1 Introduction

Clifford algebras are *geometric algebras* and can be seen as generalizations of the real numbers, complex numbers, and quaternions (the free real algebra on three variables i, j, k modulo the relations $i^2 = j^2 = k^2 = ijk = -1$). As such, they have been very influential in the formulation of modern physical theories. For example the *Dirac equation*, which was the first successful description of the electron compatible with both special relativity and quantum mechanics, is a differential equation involving the elements of the Clifford algebra associated with the metric signature $(+ \ - \ - \ -)$.

1.1 Constructing the Clifford algebra

We wish to extend a real vector space with a bilinear form into an algebra by defining a notion of multiplication in a suitable way. Here, an algebra is vector space and a ring with identity. Additionally, we want multiplication of vectors to relate in some way to the geometric structure of the space given by the bilinear form. This motivates the following definition:

Definition 1.1. *Given a vector space V over the field $\mathbb{F}$ with a bilinear form g , its Clifford algebra, $Cl(V)$, is the free algebra on V modulo*

$$v^2 = g(v, v). \quad (1)$$

More formally, we can construct $Cl(V)$ by quotienting out from the tensor algebra, $T(V)$, the (two-sided) ideal generated by all elements of the form $v \otimes v - g(v, v)$, for $v \in V$.

Replacing v by $v + w$ in (1) and expanding yields

$$vw + wv = 2g(v, w), \quad (2)$$

from which we see that two vectors anti-commute if and only if they are orthogonal.

We denote the vector space $\mathbb{R}^{p+q}$ with metric signature $(\underbrace{+ \dots +}_{p\text{-times}} \underbrace{- \dots -}_{q\text{-times}})$ by $\mathbb{R}^{p,q}$. This space will be the sole focus of our work. Further, we abbreviate

$Cl(\mathbb{R}^{p,q})$ by $Cl_{p,q}$. We identify $\mathbb{R}^{p,q}$ with the image of the natural inclusion map $\mathbb{R}^{p+q} \hookrightarrow Cl_{p,q}$.

If $\{e_1, \dots, e_n\}$ is an orthonormal basis for $\mathbb{R}^{p,q}$ then by (1) and (2) these elements generate $Cl_{p,q}$ with the rules:

$$e_i^2 = \begin{cases} 1 & \text{if } 1 \leq i \leq p \\ -1 & \text{if } p+1 \leq i \leq p+q \end{cases}$$

and

$$e_i e_j = -e_j e_i \text{ if } i \neq j.$$

It turns out thatt that any algebra generated by $\mathbb{R}^{p,q}$ which satisfies (1) is unique and is *the* Clifford algebra $Cl_{p,q}$, as long as $p - q \not\equiv 1 \pmod{4}$. These algebras have dimension 2^{p+q} , a basis being $\{e_1^{i_1} e_2^{i_2} \dots e_{p+q}^{i_{p+q}} : i_j = 0 \text{ or } 1\}$. If $p - q \equiv 1 \pmod{4}$, then it is possible to have an algebra generated by $\mathbb{R}^{p,q}$ and satisfying (1) but with the property that $e_1 e_2 \dots e_{p+q} = \pm 1$. These algebras therefore have dimension 2^{p+q-1} . However, we can get the so-called universal Clifford algebra (of dimension 2^{p+q}) by taking the direct sum of these two algebras [2] [3].

As the above discussion hints, the element $e_1 e_2 \dots e_{p+q}$ is of special interest. It is called the *pseudoscalar* and is denoted by γ . Though the e_i 's are obviously basis dependent, γ is canonical in that it remains unchanged (up to sign) under any orthogonal transformation [1]. We see that

$$\begin{aligned} \gamma^2 &= (-1)^{(p+q-1)+(p+q-2)+\dots+1} \prod_{i=1}^{p+q} e_i^2 \\ &= (-1)^{\frac{(p+q-1)(p+q)}{2} + q} \\ &= (-1)^{\frac{(p+q)^2 + q - p}{2}} \end{aligned} \tag{3}$$

and

$$\gamma u = \begin{cases} u\gamma & \text{iff } p+q \text{ is odd or } u \text{ is even} \\ -u\gamma & \text{iff } p+q \text{ is even and } u \text{ is odd.} \end{cases} \tag{4}$$

In constructing isomorphisms and representations of Clifford algebras, we will be implicitly using the following universal property of Clifford algebras:

Theorem 1.1. *Let A be a real algebra and $j : \mathbb{R}^{p,q} \rightarrow A$ be linear and have the property that $j(v)^2 = g(v, v)1_A$ for all $v \in \mathbb{R}^{p,q}$, where 1_A is the identity element of A . Then there exists a unique homomorphism $h : Cl_{p,q} \rightarrow A$ such that $h(v) = j(v)$.*

The function h is given by

$$h \left(c_0 + \sum_i c_i e_{i_1} \dots e_{i_k} \right) = c_0 + \sum_i c_i j(e_{i_1}) \dots j(e_{i_k})$$

where $c_i \in \mathbb{R}$.

$Cl_{p,q}$ has a graded structure provided by the involution α , induced by $v \mapsto -v$ for $v \in \mathbb{R}^{p,q}$. We define $Cl_{p,q}^0 = \{u \in Cl_{p,q} : \alpha(u) = u\}$. It is not hard to verify that $Cl_{p,q}^0$ is a subalgebra of $Cl_{p,q}$, called the *even algebra*, of dimension 2^{p+q-1} .

1.2 The Clifford Group

Suppose u is an invertible element in $Cl_{p,q}$ such that $\rho_u(v) = uv\alpha(u^{-1}) \in \mathbb{R}^{p,q}$ for all $v \in \mathbb{R}^{p,q}$. Then ρ_u is an orthogonal (i.e. preserves g) automorphism of $\mathbb{R}^{p,q}$. To see this, note that the inverse of ρ_u is $\rho_{u^{-1}}$ and

$$\begin{aligned} g(\rho_u(v), \rho_u(v)) &= \rho_u(v)\rho_u(v) \\ &= -\alpha(\rho_u(v))\rho_u(v) \\ &= -\alpha(uv\alpha(u^{-1}))(uv\alpha(u^{-1})) \\ &= -\alpha(u)\alpha(v)u^{-1}uv\alpha(u^{-1}) \\ &= \alpha(u)(v^2)\alpha(u^{-1}) \\ &= \alpha(u)g(v, v)\alpha(u^{-1}) \\ &= g(v, v). \end{aligned}$$

The set of all such $u \in Cl_{p,q}$ forms a group called the *Clifford Group*, and is denoted by $\Gamma_{p,q}$. Define $\Gamma_{p,q}^0 = \Gamma_{p,q} \cap Cl_{p,q}^0$

Let $u \in \mathbb{R}^{p,q}$ not be null (i.e. $g(u, u) \neq 0$). Then, by 1, u has inverse $\frac{u}{g(u, u)}$. Further, we see that $\rho_u(u) = uu\alpha(\frac{u}{g(u, u)}) = -u$ and, if $g(u, v) = 0$, $\rho_u(v) = uv\alpha(u^{-1}) = -uvu^{-1} = vuu^{-1} = v$. Therefore $u \in \Gamma_{p,q}$ and ρ_u represents a reflection in the hyperplane perpendicular to u . Since any

orthogonal transformation is a composition of reflections, we see that the map $\Gamma_{p,q} \rightarrow O(p,q)$, $u \mapsto \rho_u$ is surjective. Furthermore, we claim that the map is actually a homomorphism with kernel $\mathbb{R}$. It can easily be verified that the map is a homomorphism. To see that its kernel is $\mathbb{R}$, suppose that $\rho_u(v) = v$ for all $v \in \mathbb{R}^{p,q}$. Then, for $v \in \mathbb{R}^{p,q}$, we have

$$\begin{aligned} uv\alpha(u^{-1}) &= v \\ uv &= v\alpha(u). \end{aligned}$$

Put $u = u^0 + u^1$ with $u^0 \in \Gamma_{p,q}^0$ and $u^1 \in \Gamma_{p,q} \setminus \Gamma_{p,q}^0$. Then since $\alpha(u^0) = u^0$ and $\alpha(u^1) = u^1$ we have $u^0v = vu^0$ and $u^1v = -vu^1$. We need to show that $u^0 \in \mathbb{R}$ and $u^1 = 0$. Assume that $u^0 \notin \mathbb{R}$, then there exists a basis element $e_{i_1} \dots e_{i_{2k}}$ on which u^0 has a non-zero component. Then $(e_{i_1} \dots e_{i_{2k}})e_{i_1} = -e_{i_1}^2 e_{i_2} \dots e_{i_{2k}}$ but $e_{i_1}(e_{i_1} \dots e_{i_{2k}}) = e_{i_1}^2 e_{i_2} \dots e_{i_{2k}}$ since e_{i_1} must pass an odd amount of elements with which it anti-commutes. Contradiction. A similar argument shows that $u^1 = 0$.

The above result also tells us that $\Gamma_{p,q}$ is the group generated by non-null vectors: since any orthogonal transformation is a product of reflections, for any $u \in \Gamma_{p,q}$ there exist $v_1, \dots, v_n \in \Gamma_{p,q}$ such that $\rho_u = \rho_{v_1 \dots v_n}$. But since the map $u \mapsto \rho_u$ is injective up to scale, we have that $u = kv_1 \dots v_n$ for some $k \in \mathbb{R}$. Note that the restriction of the homomorphism to $\Gamma_{p,q}^0$ gives a surjective homomorphism to $SO(p,q)$.

1.3 $Pin(p,q)$ and $Spin(p,q)$

To limit the kernel of the homomorphism $u \mapsto \rho_u$ from $\Gamma_{p,q}(\Gamma_{p,q}^0)$ to $O(p,q)$ ($SO(p,q)$), we define the Pin (Spin) group:

$$Pin(p,q) = \{v_1 v_2 \dots v_n | g(v_i, v_i) = \pm 1 \text{ for all } i\}$$

$$Spin(p,q) = Pin(p,q) \cap Cl_{p,q}^0.$$

The maps $Pin(p,q) \rightarrow O(p,q)$ and $Spin(p,q) \rightarrow SO(p,q)$, $u \mapsto \rho_u$, are now surjective homomorphisms with kernel $\{1, -1\}$.

1.4 Representations of Clifford Algebras

We can always represent $Cl_{p,q}$ as the set of all $n \times n$ matrices with entries in $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the quaternions). We denote the set of all $n \times n$ matrices with

entries in $\mathbb{F}$ by $\mathbb{F}[n]$. The representation space, i.e. $\mathbb{F}^n$ is called the *space of spinors*. If $p - q \not\equiv 1 \pmod{4}$, then the representation is unique. If $p - q \equiv 1 \pmod{4}$, then there are two inequivalent representations; one has $\gamma = 1$ and the other has $\gamma = -1$. Furthermore we can always chose our representations such that $e_i^\dagger = e_i$ iff $e_i^2 = 1$ and $e_i^\dagger = -e_i$ iff $e_i^2 = -1$, where $\dagger$ is the conjugate transpose.

The full matrix algebra that $Cl_{p,q}$ is isomorphic to is determined by the quantity $\tau = q - p - 1 \pmod{8}$:

$$Cl_{p,q} \simeq \begin{cases} \mathbb{R}[2^{[p+q]/2}] & \text{if } \tau = 5, 6, \text{ or } 7 \\ \mathbb{H}[2^{[p+q]/2-1}] & \text{if } \tau = 1, 2, \text{ or } 3, \\ \mathbb{C}[2^{p+q-1/2}] & \text{if } \tau = 0, \text{ or } 4 \end{cases}$$

where $[p + q]$ denotes the integer part of $p + q$. We say that $Cl_{p,q}$ is type $\mathbb{F}$ if it is isomorphic to a full matrix algebra with entries in $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$).

Given a representation of $Cl_{p,q}$ on $\mathbb{C}^n$, we can get a representation on $\mathbb{R}^{2n}$ by replacing i with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and 1 with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Note that although $Cl_{p,q}$ is isomorphic to a proper subalgebra of $\mathbb{R}[2n]$, the spin spaces, $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, are isomorphic as real vector spaces. Similarly, if we have a representation of

$Cl_{p,q}$ on $\mathbb{H}^n$, we can get a representation on $\mathbb{R}^{4n}$ by making the replacements

$$\begin{aligned} i &\rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ j &\rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ k &\rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ 1 &\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Representations are built up from lower dimensional representations using identities we will derive in later sections and tensor products of matrices. For example, $Cl_{p,q} \simeq Cl_{p-1,q-1} \otimes \mathbb{R}[2]$. To see this let $\{e_i\}$ be the standard generators for $Cl_{p-1,q-1}$. Put $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We have that $A^2 = -B^2 = C^2 = 1$, $AB = -BA$, $AC = -CA$, and $BC = -CB$. Therefore, $\{e_i \otimes A\} \cup \{I \otimes C, I \otimes B\}$ generate $Cl_{p,q}$. Additionally, a form of Bott periodicity states that $Cl_{p,q+8} \simeq Cl_{p,q} \otimes \mathbb{R}[16]$. Upon seeing that $Cl_{0,8} \simeq \mathbb{R}[16]$, we can prove this as follows: let $\{e_i\}$ and $\{E_i\}$ generate $Cl_{p,q}$ and $Cl_{0,8}$, respectively. Let γ be the $Cl_{0,8}$ pseudoscalar and note that $\gamma^2 = 1$ (3) and γ anti-commutes with E_i (4). Then it is easily seen that $\{e_i \otimes \gamma, 1 \otimes E_i\}$ generate $Cl_{p,q+8}$.

2 Space and Time Symmetry on the Clifford Algebra Level

From a purely geometric standpoint, there is no difference between $\mathbb{R}^{p,q}$ and $\mathbb{R}^{q,p}$. Indeed, $O(p, q)$ and $SO(p, q)$ are naturally isomorphic to $O(q, p)$ and $SO(q, p)$, respectively. However, the relationship with $Cl_{p,q}$ and $Cl_{q,p}$ is not immediately evident. We will see however, that $Cl_{p,q} \simeq Cl_{q,p}$ and $Pin(p, q) \simeq Pin(q, p)$ if and only if $p - q \equiv 0 \pmod{4}$ and that $Spin(p, q)$ and $Spin(q, p)$ are always isomorphic.

2.1 $Cl_{p,q}^0$ and $Cl_{q,p}^0$ and $Spin(p, q)$ and $Spin(q, p)$

We begin by proving the following proposition.

Proposition 2.1. $Cl_{p,q-1} \cong Cl_{p,q}^0 \cong Cl_{q,p-1}$, assuming for the first isomorphism that $q \geq 1$ and for the second isomorphism that $p \geq 1$.

Proof. Let $\{e_i\}$ and $\{E_i\}$ be standard generators for $Cl_{p,q-1}$ and $Cl_{p,q}^0$ respectively. We claim that the map φ from $Cl_{p,q}^0$ to $Cl_{p,q-1}$, defined on generators by, where $i < j$,

$$E_i E_j \mapsto \begin{cases} e_i & \text{if } j = p + q \\ e_i e_j & \text{otherwise} \end{cases}$$

is an isomorphism. To check that it is a homomorphism, it is sufficient, because of the universal property, to verify that the elements $\{E_i E_{p+q}\}$ anti-commute with each other and that $(E_i E_{p+q})^2 = e_i^2$. Indeed, if $i \neq j$ then we have that $(E_i E_{p+q})(E_j E_{p+q}) = -E_i E_j E_{p+q} E_{p+q} = E_j E_i E_{p+q} E_{p+q} = -(E_j E_{p+q})(E_i E_{p+q})$ and $(E_i E_{p+q})^2 = E_i E_{p+q} E_i E_{p+q} = -E_i^2 E_{p+q}^2 = e_i^2$. It remains to show that φ is a bijection. Since φ is a linear map of finite dimensional vector spaces of the same dimension, it is a bijection if and only if it is a surjection. We can easily see that φ is surjective since all of the generators of $Cl_{p,q-1}$ are in $im(\varphi)$.

Let $\{e_i\}$ be the standard generators for $Cl_{q,p-1}$ and let $\{E_i\}$ be anti-commuting generators for $Cl_{p,q}^0$ where we break convention by having:

$$E_i^2 = \begin{cases} -1 & \text{if } 1 \leq i \leq q \\ 1 & \text{if } q + 1 \leq i \leq p + q. \end{cases}$$

Thus $e_i^2 = -E_i^2$ for $1 \leq i \leq q + p - 1$. By the same argument as before, we see that the map $\varphi : Cl_{p,q}^0 \rightarrow Cl_{q,p-1}$ defined on generators by

$$E_i E_j \mapsto \begin{cases} e_i & \text{if } j = p + q \\ e_i e_j & \text{otherwise} \end{cases}$$

is an isomorphism. $\square$

Corollary 2.1. $Cl_{p,q}^0 \cong Cl_{q,p}^0$.

Proof. One of p, q must be non-zero ($Cl_{0,0}$ is not a very interesting algebra). Without loss of generality we can assume that $p \neq 0$. Then from the previous theorem we have that $Cl_{p,q}^0 \cong Cl_{q,p-1}$. But we also have from the above theorem that $Cl_{q,p-1} \cong Cl_{q,p}^0$. Thus we have that $Cl_{p,q}^0 \cong Cl_{q,p}^0$. $\square$

Though we now know that $Cl_{p,q} \simeq Cl_{q,p}$, we construct an isomorphism from $Cl_{p,q}^0$ to $Cl_{q,p}^0$ by composing the isomorphism from $Cl_{p,q}^0$ to $Cl_{p,q-1}$ with the isomorphism from $Cl_{p,q-1}$ to $Cl_{q,p}^0$. We do this to then show that the map restricts to an isomorphism of the Spin groups.

Let $\{e_i\}$ be standard generators for $Cl_{p,q}$ and $\{E_i\}$ be generators for $Cl_{q,p}$ where $E_i^2 = -e_i^2$. Also let g and $\bar{g}$ be the metrics on $Cl_{p,q}$ and $Cl_{q,p}$, respectively. By composing the type of maps constructed in the previous corollary, we get an isomorphism $\theta : Cl_{p,q}^0 \rightarrow Cl_{q,p}^0$,

$$e_i e_j \mapsto E_i E_j \tag{5}$$

Note that (5) holds even when $i = j$ since $e_i^2 = -E_i^2$. Denote the restriction of θ to $Spin(p, q)$ by $\theta|_s$. To see that $\theta|_s$ maps into $Spin(q, p)$, it is sufficient to check that a product of two unit vectors maps to a product of two unit vectors, for then the fact that $\theta|_s$ is a group homomorphism (since θ is an algebra homomorphism) will establish that the image of a product of any even amount of unit vectors in $Spin(p, q)$ is a product of an even amount of unit vectors in $Spin(q, p)$. Thus let $u = \sum_i u_i e_i$ and $v = \sum_i v_i e_i$ be unit

vectors in $\mathbb{R}^{p,q}$. Then

$$\begin{aligned}
\theta(uv) &= \theta \left(\left(\sum_i u_i e_i \right) \left(\sum_j v_j e_j \right) \right) \\
&= \theta \left(\sum_{i,j} u_i v_j e_i e_j \right) \\
&= \sum_{i,j} -u_i v_j E_i E_j \\
&= \left(\sum_i -u_i E_i \right) \left(\sum_j v_j E_j \right)
\end{aligned}$$

is a product of two unit vectors since $\bar{g}(\sum_i -u_i E_i, \sum_i -u_i E_i) = -g(u, u) = \pm 1$ and $\bar{g}(\sum_j v_j E_j, \sum_j v_j E_j) = -g(v, v) = \pm 1$. Besides showing that $\theta|_s$ maps into $Spin(q, p)$, the above calculation makes it clear that $\theta|_s$ is surjective. Finally, $\theta|_s$ is injective since θ is injective. This establishes that $Spin(p, q) \cong Spin(q, p)$.

One may think that since $Spin(p, q) \cong Spin(q, p)$, a similar type of argument can establish that $Pin(p, q) \cong Pin(q, p)$. However, in the next section we show that this is not generally true.

2.2 $Cl_{p,q}$ and $Cl_{q,p}$ and $Pin(p, q)$ and $Pin(q, p)$

We begin by proving the only affirmative case for $Cl_{p,q} \simeq Cl_{q,p}$.

Theorem 2.1. *If $q - p \equiv 0 \pmod{4}$ then $Cl_{p,q} \cong Cl_{q,p}$.*

Proof. Let $\{e_i\}$ be the standard generators for $Cl_{p,q}$ and let $\{E_i\}$ be generators for $Cl_{q,p}$ such that $E_i^2 = -e_i^2$. Since $4|(q - p)$, $q - p$ is even so that $p + q = q - p + 2p$ is even. Therefore $4|(p + q)^2$ so that $4|((p + q)^2 + q - p) \Rightarrow \frac{(p+q)^2 + q - p}{2}$ is even. Thus, by (3) we see that $\gamma = \prod_i E_i$ squares to the identity. We claim that the map $\phi : Cl_{p,q} \rightarrow Cl_{q,p}$ given on generators by

$$\phi(e_i) = \gamma E_i$$

extends to an isomorphism. To see that ϕ is a bijection, we note that since ϕ is a linear map between two finite dimensional vector spaces with the same

dimension, it is a bijection if and only if it is a surjection. Indeed, the map is surjective since for any generator E_i of $Cl_{q,p}$, $im(\phi)$ contains

$$\begin{aligned}
\prod_{j \neq i} \gamma E_j &= \pm \gamma^{p+q-1} \prod_{j \neq i} E_j \\
&= \pm \gamma^{p+q-2} \gamma \prod_{j \neq i} E_j \\
&= \pm \gamma \prod_{j \neq i} E_j \\
&= \pm E_i
\end{aligned}$$

where the third equality follows from the (crucial) fact that $p+q$ is even. Finally, to show that ϕ is an isomorphism it is sufficient, by the universal property, to show that the elements $\{\gamma E_i\}$ obey the same Clifford relations as $\{e_i\}$, i.e. they anti-commute with each other and p of them square to the identity and q of them square to negative the identity. Since γ anti-commutes with each E_i by (4), for $i \neq j$ we have that $(\gamma E_i)(\gamma E_j) = \gamma E_j E_i \gamma = -(\gamma E_j)(\gamma E_i)$. Lastly, note that $(\gamma E_i)^2 = \gamma E_i \gamma E_i = -\gamma^2 E_i^2 = -E_i^2 = e_i^2$. $\square$

Corollary 2.2. *If $q-p \equiv 0 \pmod{4}$ then $Pin(p, q) \cong Pin(q, p)$, an isomorphism being the restriction of ϕ (as above) to $Pin(p, q)$.*

Proof. As before, let $\{e_i\}$ and $\{E_i\}$ be generators for $Cl_{p,q}$ and $Cl_{q,p}$ respectively. Let g and $\bar{g}$ be the metrics on $Cl_{p,q}$ and $Cl_{q,p}$, respectively, and let $\phi|_p$ be the restriction of ϕ to $Pin(p, q)$. Since $Pin(p, q)$ is made up of unit vectors, we need only check that $\phi|_p$ maps unit vectors to products of unit vectors and that any unit vector in $Cl_{q,p}$ is in $im(\phi_p)$.

Let $v = \sum_i v_i e_i$ be a unit vector in $\mathbb{R}^{p,q}$, i.e. $g(v, v) = \pm 1$. Then we have that

$$\phi|_p(v) = \sum_i v_i \gamma E_i = \gamma \left(\sum_i v_i E_i \right).$$

But γ is obviously a product of unit vectors and $\sum_i v_i e_i$ is a unit vector since

$$\bar{g} \left(\sum_i v_i E_i, \sum_i v_i E_i \right) = -g(v, v) = \mp 1.$$

Thus $\phi(v)$ is a product of unit vectors and so is in $Pin(q, p)$.

Finally, to see that any unit vector in $Pin(q, p)$ is in $im(\phi_p)$, we first consider $\phi|_p(\gamma')$ where $\gamma' = \prod_{i=1}^{p+q} e_i$. Since γ and E_i anti-commute by (4) and $\gamma^2 = 1$ by (3) we have

$$\begin{aligned}\phi|_p(\gamma') &= \prod_{i=1}^{p+q} \gamma E_i \\ &= (-1)^{(p+q)/2} \gamma^{p+q} \prod_{i=1}^{p+q} E_i \\ &= (-1)^{(p+q)/2} \prod_{i=1}^{p+q} E_i \\ &= \pm \gamma\end{aligned}$$

where the third equality follows from the fact that $\gamma^{p+q} = 1$ since $p + q = p - q + 2q$ is even. Now let $u = \sum_i u_i E_i$ be a unit vector in $Pin(q, p)$. Then $u' = \sum_i u_i e_i \in Pin(p, q)$ (since $g = -\bar{g}$) and

$$\begin{aligned}\phi|_p(\gamma' u') &= \phi|_p(\gamma') \phi(u') \\ &= \pm \gamma \sum_i u_i \gamma E_i \\ &= \pm \gamma^2 \sum_i u_i E_i \\ &= \pm u.\end{aligned}$$

□

Theorem 2.2. *If $p - q \equiv 1$ or $3 \pmod{4}$ then $Cl_{p,q} \not\cong Cl_{q,p}$ and $Pin(p, q) \not\cong Pin(q, p)$.*

Proof. Note that we need only prove this when $p - q \equiv 1 \pmod{4}$. For say it holds for $p - q \equiv 1 \pmod{4}$ and we have that $p - q \equiv 3 \pmod{4}$. Then $q - p \equiv 1 \pmod{4}$ so that $Cl_{q,p} \not\cong Cl_{p,q}$.

Let $p - q \equiv 1 \pmod{4}$. Then $q - p \equiv 3 \pmod{4}$. We also have that $p + q = p - q + 2q$ is odd since $p - q$ is odd. Thus $p + q$ is congruent to

either 1 or 3 modulo 4. In either case, we have that $(p + q)^2 \equiv 1 \pmod{4}$ so that $(p + q)^2 + q - p \equiv 0 \pmod{4}$. Thus from (3) we have that $\gamma^2 = 1$, where γ is the pseudoscalar in $Cl_{p,q}$. However, $(p + q)^2 + p - q \equiv 2 \pmod{4}$ so that the pseudoscalar γ' in $Cl_{q,p}$ squares to -1. Since $p + q$ is odd, we see from (4) that $\gamma \in Z(Cl_{p,q})$ and $\bar{\gamma} \in Z(Cl_{q,p})$. Additionally, it is not hard to see that $Z(Cl_{p,q}) = \{a + b\gamma : a, b \in \mathbb{R}\}$ and $Z(Cl_{q,p}) = \{a + b\gamma' : a, b \in \mathbb{R}\}$. However, these centers are certainly not isomorphic since the latter contains an element (γ') which squares to -1 but the former does not. Since the centers are not isomorphic, neither are the algebras. The same argument works to show that $Pin(p, q) \not\cong Pin(q, p)$ since $Z(Pin(p, q)) = \{\gamma, -\gamma, 1, -1\}$ and $Z(Pin(q, p)) = \{\gamma', -\gamma', 1, -1\}$. $\square$

For the last case, $p - q \equiv 2 \pmod{4}$, there seems to be no natural argument and we are forced to appeal to representations of Clifford algebras. By Bott periodicity, we can know the type ($\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$) of a Clifford algebra by knowing the types of $Cl_{p,q}$ for $0 \leq p, q \leq 8$. Such tables can be found throughout the literature [3] and [2]. There we see that the types for $Cl_{p,q}$ and $Cl_{q,p}$ are indeed different if $p - q \equiv 2 \pmod{4}$. The question now becomes how do we know, for example, that $\mathbb{R}[4] \not\cong \mathbb{H}[2]$? After all, they are isomorphic as (real) vector spaces. To show that they are not isomorphic as algebras, we consider minimal left ideals.

Since the product of a matrix with a rank one matrix has rank at most one, any minimal left ideal of $\mathbb{F}[n]$ is generated by one rank one matrix. Let I be the ideal generated by the rank one matrix M and let v be a non-zero column vector of M . We claim that the map $AM \mapsto Av$ gives a vector space isomorphism from I to $\mathbb{F}^n$. Clearly the map is linear and it is surjective since for any non-zero vector $w \in \mathbb{F}^n$, there exists a matrix A such that $Av = w$. To see that it is injective, suppose that $Av = 0$. Since M has rank one, all column vectors are multiples of v . Thus A maps each column vector to 0, so that $AM = 0$. Since I is isomorphic to $\mathbb{F}^n$, any minimal left ideal has real dimension $n \dim_{\mathbb{R}} \mathbb{F}$.

Now suppose that $\mathbb{F}_1[n]$ and $\mathbb{F}_2[m]$ have the same real dimension. Then $n^2 \dim_{\mathbb{R}} \mathbb{F}_1 = m^2 \dim_{\mathbb{R}} \mathbb{F}_2$. If they are isomorphic then their minimal left ideals must also be isomorphic and, in particular, must have the same dimension. Therefore we must also have that $n \dim_{\mathbb{R}} \mathbb{F}_1 = m \dim_{\mathbb{R}} \mathbb{F}_2$. Dividing these two equations gives $n = m$ which further implies that $\dim_{\mathbb{R}} \mathbb{F}_1 = \dim_{\mathbb{R}} \mathbb{F}_2$. Since

$\mathbb{F}_i$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, we must have that $\mathbb{F}_1 = \mathbb{F}_2$.

This shows that for $p - q \equiv 2 \pmod{4}$, $Cl_{p,q} \not\cong Cl_{q,p}$, but what about the *Pin* groups? Since $Cl_{p,q}$ is isomorphic to a matrix algebra, $Pin(p, q)$ must be isomorphic to a group that is a subset of the matrix algebra. Since $Pin(p, q)$ contains all of the generators for the algebra, if it were isomorphic to $Pin(q, p)$ then we could represent $Cl_{q,p}$ and $Cl_{p,q}$ on the same space. However, since $Cl_{q,p}$ and $Cl_{p,q}$ have the same dimensions, it would then follow that they are isomorphic.

3 Bilinear Forms on Spinors

Bilinear forms on spinors are discussed in [2] but from a different perspective. Our approach is to look for bilinear forms on the space of spinors, S , such that vectors are self-adjoint, up to sign. That is, a bilinear function $(\cdot, \cdot) : S \times S \rightarrow \mathbb{R}$ such that

$$(\phi, v\psi) = \pm(v\phi, \psi) \quad (6)$$

for all $v \in \mathbb{R}^{p,q}$, $\phi, \psi \in S$. The form can be represented as $(\phi, \psi) \mapsto \phi^\dagger A \psi$, where $A \in Cl_{p,q}$. The condition (6) then becomes

$$Av = \pm v^\dagger A$$

for all $v \in \mathbb{R}^{p,q}$. If $1 \leq i \leq p$ then $e_i^\dagger = e_i$ and if $p + 1 \leq i \leq p + q$ then $e_i^\dagger = -e_i$. Therefore we must have that

$$Ae_i = \begin{cases} \pm e_i A & \text{if } 1 \leq i \leq p \\ \mp e_i A & \text{if } p + 1 \leq i \leq p + q \end{cases}.$$

Put

$$A = \sum_{I \subseteq \{1, 2, \dots, p+q\}} A_I e_I.$$

Since e_i either commutes or anti-commutes with each e_I , if $A_I \neq 0$ then we must have that

$$e_I e_i = \begin{cases} \pm e_i e_I & \text{if } 1 \leq i \leq p \\ \mp e_i e_I & \text{if } p + 1 \leq i \leq p + q \end{cases}$$

It follows that if $A_I \neq 0$ then e_I must be either $e_1 e_2 \dots e_p$ or $e_{p+1} e_{p+2} \dots e_{p+q}$. Thus A is, up to scale, either $e_1 e_2 \dots e_p$ or $e_{p+1} e_{p+2} \dots e_{p+q}$. We denote the former element by γ_p and the latter by γ_q . We define two real bilinear forms

$$\begin{aligned}(\phi, \psi)_+ &= \text{Re}(\phi^\dagger \gamma_p \psi) \\ (\phi, \psi)_- &= \text{Re}(\phi^\dagger \gamma_q \psi).\end{aligned}$$

We see that

$$\gamma_p^2 = (e_1 e_2 \dots e_p)(e_1 e_2 \dots e_p) = (-1)^{(p-1)+(p-2)+\dots+1} e_1^2 e_2^2 \dots e_p^2 = (-1)^{p(p-1)/2}$$

and, similarly,

$$\gamma_q^2 = (-1)^{q(q-1)/2} (-1)^q = (-1)^{q(q+1)/2}.$$

Thus $(\cdot, \cdot)_+$ is symmetric if $p = 0$ or $1 \pmod{4}$ and anti-symmetric if $p = 2$ or $3 \pmod{4}$ and $(\cdot, \cdot)_-$ is symmetric if $q = 0$ or $3 \pmod{4}$ and anti-symmetric if $p = 1$ or $2 \pmod{4}$.

Given any vector $v \in \mathbb{R}^{p,q}$, we can put $v = v_+ + v_-$, with $v_+ \in \text{span}\{e_1, e_2, \dots, e_p\}$ and $v_- \in \text{span}\{e_{p+1}, e_{p+2}, \dots, e_{p+q}\}$. We then have

$$\begin{aligned}(\phi, v\psi)_+ &= \phi^\dagger \gamma_p (v_+ + v_-) \psi \\ &= \phi^\dagger ((-1)^{p+1} v_+ + (-1)^p v_-) \gamma_p \psi \\ &= \phi^\dagger ((-1)^{p+1} v_+^\dagger + (-1)^{p+1} v_-^\dagger) \gamma_p \psi \\ &= (-1)^{p+1} (v\phi, \psi)_+.\end{aligned}\tag{7}$$

A similar calculation yields

$$(\phi, v\psi)_- = (-1)^q (v\phi, \psi)_-.\tag{8}$$

Recall that the action of the Pin and Spin groups on vectors preserves the metric. We also see that the action of the pin and spin groups on spinors (which is just left multiplication) preserves $(\cdot, \cdot)_\pm$ up to sign:

$$(u\phi, u\psi)_\pm = \pm (\phi, \psi)_\pm,$$

for all $u \in \text{Pin}(p, q)$, $\phi, \psi \in S$. This is evident from (7), (8), and the fact that for $u \in \text{Pin}(p, q)$, $u\tilde{u} = \pm 1$. We can define a subgroup $\text{Pin}_+(p, q)$ of $\text{Pin}(p, q)$ by

$$\text{Pin}_+(p, q) = \{u \in \text{Pin}(p, q) : u\tilde{u} = 1\}.$$

Then we see that the action of $\text{Pin}_+(p, q)$ on spinors preserves $(\cdot, \cdot)_+$ if p is odd and preserves $(\cdot, \cdot)_-$ if q is even. Furthermore, $\text{Spin}_+(p, q) = \text{Pin}_+(p, q) \cap \text{Spin}(p, q)$ always preserves $(\cdot, \cdot)_\pm$.

3.1 Relations Between Spinors and Forms

Denote the exterior algebra on $\mathbb{R}^{p,q}$ by $\Lambda(\mathbb{R}^{p,q})$. The metric g on $\mathbb{R}^{p,q}$ induces a metric on $\Lambda(\mathbb{R}^{p,q})$ by

$$g(e_{i_1}e_{i_2}\dots e_{i_m}, e_{j_1}e_{j_2}\dots e_{j_n}) = \prod_{i_k=j_l} g(e_{i_k}, e_{j_l}).$$

We can use the bilinear forms to associate an element v in the dual space of $\Lambda(\mathbb{R}^{p,q})$ with spinors ϕ, ψ by

$$v(u) = (\phi, u\psi)_\pm, u \in \Lambda(\mathbb{R}^{p,q}).$$

Since the metric provides an identification of forms with dual forms, we can associate two spinors with an element of $\Lambda(\mathbb{R}^{p,q})$. Denote the k -form associated to the spinors ϕ, ψ using $(\cdot, \cdot)_\pm$ by $v_\pm^k$. Under this identification, the component of $v_\pm^k$ along $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$ is $(\phi, e_{i_1}e_{i_2}\dots e_{i_k}\psi)_\pm$ and

$$g(v_\pm^k, v_\pm^k) = \sum_{1 \leq i_1 < \dots < i_k \leq p+q} \epsilon_{i_1, \dots, i_k} (\phi, e_{i_1} \dots e_{i_k} \psi)^2$$

where

$$\epsilon_{i_1, \dots, i_k} = g(e_{i_1} \dots e_{i_k}, e_{i_1} \dots e_{i_k}) = e_{i_1}^2 \dots e_{i_k}^2.$$

3.2 Identities

Upon playing around with these constructions in certain dimensions, we found some identities relating the norms of various forms associated with spinors. We then used the scripting language Python and Mathematica to search for similar identities in general. The identities resemble some of the Fierz identities, as seen in the context of Clifford algebras, for example, in [2].

We use a Monte-Carlo method whereby we created random spinors and saw if there were any identities which held in those special cases. We then checked those to see if they held in general. We first give the cases when the Clifford algebra is isomorphic to a real matrix algebra.

3.2.1 Real Clifford algebras

We first consider the real corner algebras (non-universal Clifford algebras where $\gamma = \pm 1$) up to dimension 13, from which we can derive identities in the subordinate algebras. Recall that for corner algebras $\gamma_p = \pm \gamma_q$ so that there is only one bilinear form, which we denote by $(\cdot, \cdot)$. We denote the k -form which acts on a k -form u as $(\phi, u\psi)$ as v^k .

- $Cl_{3,2}$:

$$\begin{aligned} 2g(v, v) + g(v^2, v^2) &= 0 \\ -(\phi, \psi)^2 + g(v, v) &= 0 \end{aligned}$$

- $Cl_{4,3}$ and $Cl_{0,7}$:

$$\begin{aligned} 3g(v, v) + g(v^2, v^2) &= 0 \\ -7(\phi, \psi)^2 + 4g(v, v) - g(v^3, v^3) &= 0 \\ 7(\phi, \phi)(\psi, \psi) + 3g(v, v) + g(v^3, v^3) &= 0 \end{aligned}$$

- $Cl_{9,0}$, $Cl_{5,4}$, and $Cl_{1,8}$:

$$\begin{aligned} 28g(v, v) + 7g(v^2, v^2) - 3g(v^3, v^3) - 2g(v^4, v^4) &= 0 \\ -6(\phi, \psi)^2 + 6g(v, v) + g(v^2, v^2) - g(v^3, v^3) &= 0 \\ -24(\phi, \phi)(\psi, \psi) + 24g(v, v) + 7g(v^2, v^2) - g(v^3, v^3) &= 0 \end{aligned}$$

- $Cl_{10,1}$, $Cl_{6,5}$, and $Cl_{2,9}$:

$$\begin{aligned} -5(\phi, \psi)^2 + 5g(v, v) + g(v^2, v^2) - g(v^3, v^3) &= 0 \\ 75g(v, v) + 21g(v^2, v^2) - 16g(v^3, v^3) - 5g(v^5, v^5) &= 0 \\ 15g(v, v) + 3g(v^2, v^2) - 2g(v^3, v^3) - g(v^4, v^4) &= 0 \end{aligned}$$

- $Cl_{11,2}$, $Cl_{7,6}$, and $Cl_{3,10}$:

$$\begin{aligned}
30g(v, v) + 5g(v^2, v^2) - 5g(v^3, v^3) - 2g(v^4, v^4) &= 0 \\
60g(v, v) + 21g(v^2, v^2) - 21g(v^3, v^3) - 4g(v^5, v^5) &= 0 \\
32g(v, v) + 9g(v^2, v^2) - 7g(v^3, v^3) + g(v^6, v^6) &= 0 \\
-4(\phi, \psi)^2 + 4g(v, v) + g(v^2, v^2) - g(v^3, v^3) &= 0
\end{aligned}$$

From the identities on corner algebras, we can get identities on the subordinate algebras. Consider the corner algebra $Cl_{p,q}$, generated by $\{e_i | 1 \leq i \leq p+q\}$. To pass to $Cl_{p,q-1}$ we use generators $\{E_i | 1 \leq i \leq p+q-1\}$ where $E_i = e_i$. Letting γ be the pseudoscalar in $Cl_{p,q-1}$, we have that $\gamma = \pm e_{p+q}$. Denote by g the metric on $Cl_{p,q}$, $\bar{g}$ the metric on $Cl_{p,q-1}$, v^k the form $(\phi, \psi) \in \Lambda^k(\mathbb{R}^{p,q})$, and $v_{\pm}^k$ the form $(\phi, (\cdot)\psi)_{\pm} \in \Lambda^k(\mathbb{R}^{p,q-1})$. Note that $(\cdot, \cdot)_+ = (\cdot, \cdot)$ since the γ_p of the two algebras are the same.

Define

$$\begin{aligned}
\epsilon_{i_1, \dots, i_k} &= g(e_{i_1} \dots e_{i_k}, e_{i_1} \dots e_{i_k}) = e_{i_1}^2 \dots e_{i_k}^2 \\
\bar{\epsilon}_{i_1, \dots, i_m} &= \bar{g}(E_{i_1} \dots E_{i_m}, E_{i_1} \dots E_{i_m}) = E_{i_1}^2 \dots E_{i_m}^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
g(v^k, v^k) &= \sum_{1 \leq i_1 < \dots < i_k \leq p+q} \epsilon_{i_1, \dots, i_k} (\phi, e_{i_1} \dots e_{i_k} \psi)^2 \\
&= \sum_{1 \leq i_1 < \dots < i_k \leq p+q-1} \epsilon_{i_1, \dots, i_k} (\phi, E_{i_1} \dots E_{i_k} \psi)_+^2 + \sum_{1 \leq i_1 < \dots < i_{k-1} \leq p+q-1} \epsilon_{i_1, \dots, i_{k-1}, p+q} (\phi, \gamma E_{i_1} \dots E_{i_{k-1}})_+^2 \\
&= \sum_{1 \leq i_1 < \dots < i_k \leq p+q-1} \bar{\epsilon}_{i_1, \dots, i_k} (\phi, E_{i_1} \dots E_{i_k} \psi)_+^2 - \sum_{1 \leq i_1 < \dots < i_{k-1} \leq p+q-1} \bar{\epsilon}_{i_1, \dots, i_{k-1}} (\phi, E_{i_1} \dots E_{i_{k-1}})_-^2 \\
&= \bar{g}(v_+^k, v_+^k) - \bar{g}(v_-^{k-1}, v_-^{k-1}). \tag{9}
\end{aligned}$$

A similar argument yields the following equation for passing from $Cl_{p,q}$ to $Cl_{p-1,q}$

$$g(v^k, v^k) = \bar{g}(v_-^k, v_-^k) + \bar{g}(v_+^{k-1}, v_+^{k-1}). \tag{10}$$

Using (9) and (10) we get the following identities

- $Cl_{3,1}$:

$$-2(\phi, \psi)_-^2 + 2g(v_+, v_+) - g(v_-, v_-) + g(v_+^2, v_+^2) = 0$$

$$-(\phi, \psi)_+^2 - (\phi, \psi)_-^2 + g(v_+, v_+) = 0.$$

- $Cl_{2,2}$:

$$2(\phi, \psi)_+^2 + g(v_+, v_+) + 2g(v_-, v_-) + g(v_-^2, v_-^2) = 0$$

$$(\phi, \psi)_+^2 - (\phi, \psi)_-^2 + g(v_-, v_-) = 0.$$

- $Cl_{4,2}$ and $Cl_{0,6}$:

$$-3(\phi, \psi)_-^2 + 3g(v_+, v_+) - g(v_-, v_-) + g(v_+^2, v_+^2) = 0$$

$$-7(\phi, \psi)_+^2 - 4(\phi, \psi)_-^2 + 4g(v_+, v_+) + g(v_-^2, v_-^2) - g(v_+^3, v_+^3) = 0$$

$$7(\phi, \phi)_+(\psi, \psi)_+ - 3(\phi, \psi)_-^2 + 3g(v_+, v_+) - g(v_-^2, v_-^2) + g(v_+^3, v_+^3) = 0.$$

- $Cl_{3,3}$:

$$3(\phi, \psi)_+^2 + g(v_+, v_+) + 3g(v_-, v_-) + g(v_-^2, v_-^2) = 0$$

$$4(\phi, \psi)_+^2 - 7(\phi, \psi)_-^2 + 4g(v_-, v_-) - g(v_+^2, v_+^2) - g(v_-^3, v_-^3) = 0$$

$$3(\phi, \psi)_+^2 + 7(\phi, \phi)_-(\psi, \psi)_- + 3g(v_-, v_-) + g(v_+^2, v_+^2) + g(v_-^3, v_-^3) = 0.$$

- $Cl_{5,3}$ and $Cl_{1,7}$:

$$-28(\phi, \psi)_-^2 + 28g(v_+, v_+) - 7g(v_-, v_-) + 7g(v_+^2, v_+^2) + 3g(v_-^2, v_-^2) - 3g(v_+^3, v_+^3) + 2g(v_-^3, v_-^3) + 2g(v_+^4, v_+^4) = 0$$

$$-6(\phi, \psi)_+^2 - 6(\phi, \psi)_-^2 + 6g(v_+, v_+) - g(v_-, v_-) + g(v_+^2, v_+^2) + g(v_-^2, v_-^2) - g(v_+^3, v_+^3) = 0$$

$$-24(\phi, \phi)_+(\psi, \psi)_+ - 24(\phi, \psi)_-^2 + 24g(v_+, v_+) - 7g(v_-, v_-) + 7g(v_+^2, v_+^2) + g(v_-^2, v_-^2) - g(v_+^3, v_+^3) = 0.$$

- $Cl_{8,0}$, $Cl_{4,4}$, and $Cl_{0,8}$:

$$28(\phi, \psi)_+^2 + 7g(v_+, v_+) + 28g(v_-, v_-) - 3g(v_+^2, v_+^2) + 7g(v_-^2, v_-^2) - 2g(v_+^3, v_+^3) - 3g(v_-^3, v_-^3) - 2g(v_-^4, v_-^4) = 0$$

$$6(\phi, \psi)_+^2 - 6(\phi, \psi)_-^2 + g(v_+, v_+) + 6g(v_-, v_-) - g(v_+^2, v_+^2) + g(v_-^2, v_-^2) - g(v_-^3, v_-^3) = 0$$

$$-24(\phi, \phi)_-(\psi, \psi)_- + 24(\phi, \psi)_+^2 + 7g(v_+, v_+) + 24g(v_-, v_-) - g(v_+^2, v_+^2) + 7g(v_-^2, v_-^2) - g(v_-^3, v_-^3) = 0.$$

- $Cl_{10,0}$, $Cl_{6,4}$, and $Cl_{2,8}$:

$$-5(\phi, \psi)_+^2 - 5(\phi, \psi)_-^2 + 5g(v_+, v_+) - g(v_-, v_-) + g(v_+^2, v_+^2) + g(v_-^2, v_-^2) - g(v_-^3, v_-^3) = 0$$

$$-75(\phi, \psi)_-^2 + 75g(v_+, v_+) - 21g(v_-, v_-) + 21g(v_+^2, v_+^2) + 16g(v_-^2, v_-^2) - 16g(v_+^3, v_+^3) + 5g(v_-^4, v_-^4) - 5g(v_+^5, v_+^5) = 0$$

$$-15(\phi, \psi)_-^2 + 15g(v_+, v_+) - 3g(v_-, v_-) + 3g(v_+^2, v_+^2) + 2g(v_-^2, v_-^2) - 2g(v_+^3, v_+^3) + g(v_-^3, v_-^3) - g(v_+^4, v_+^4) = 0.$$

- $Cl_{9,1}$, $Cl_{5,5}$, and $Cl_{1,9}$:

$$5(\phi, \psi)_+^2 - 5(\phi, \psi)_-^2 + g(v_+, v_+) + 5g(v_-, v_-) - g(v_+^2, v_+^2) + g(v_-^2, v_-^2) - g(v_-^3, v_-^3) = 0$$

$$75(\phi, \psi)_+^2 + 21g(v_+, v_+) + 75g(v_-, v_-) - 16g(v_+^2, v_+^2) + 21g(v_-^2, v_-^2) - 16g(v_-^3, v_-^3) - 5g(v_+^4, v_+^4) - 5g(v_-^5, v_-^5) = 0$$

$$15(\phi, \psi)_+^2 + 3g(v_+, v_+) + 15g(v_-, v_-) - 2g(v_+^2, v_+^2) + 3g(v_-^2, v_-^2) - g(v_+^3, v_+^3) - 2g(v_-^3, v_-^3) - g(v_-^4, v_-^4) = 0.$$

- $Cl_{11,1}$, $Cl_{7,5}$, and $Cl_{3,9}$:

$$-30(\phi, \psi)_-^2 + 30g(v_+, v_+) - 5g(v_-, v_-) + 5g(v_+^2, v_+^2) + 5g(v_-^2, v_-^2) - 5g(v_+^3, v_+^3) + 2g(v_-^3, v_-^3) - 2g(v_+^4, v_+^4) = 0$$

$$-60(\phi, \psi)_-^2 + 60g(v_+, v_+) - 21g(v_-, v_-) + 21g(v_+^2, v_+^2) + 21g(v_-^2, v_-^2) - 21g(v_+^3, v_+^3) + 4g(v_-^4, v_-^4) - 4g(v_+^5, v_+^5) = 0$$

$$-32(\phi, \psi)_-^2 + 32g(v_+, v_+) - 9g(v_-, v_-) + 9g(v_+^2, v_+^2) + 7g(v_-^2, v_-^2) - 7g(v_+^3, v_+^3) - g(v_-^5, v_-^5) + g(v_+^6, v_+^6) = 0$$

$$-4(\phi, \psi)_+^2 - 4(\phi, \psi)_-^2 + 4g(v_+, v_+) - g(v_-, v_-) + g(v_+^2, v_+^2) + g(v_-^2, v_-^2) - g(v_+^3, v_+^3) = 0.$$

- $Cl_{10,2}$, $Cl_{6,6}$, and $Cl_{2,10}$:

$$30(\phi, \psi)_+^2 + 5g(v_+, v_+) + 30g(v_-, v_-) - 5g(v_+^2, v_+^2) + 5g(v_-^2, v_-^2) - 2g(v_+^3, v_+^3) - 5g(v_-^3, v_-^3) - 2g(v_-^4, v_-^4) = 0$$

$$60(\phi, \psi)_+^2 + 21g(v_+, v_+) + 60g(v_-, v_-) - 21g(v_+^2, v_+^2) + 21g(v_-^2, v_-^2) - 21g(v_-^3, v_-^3) - 4g(v_+^4, v_+^4) - 4g(v_-^5, v_-^5) = 0$$

$$32(\phi, \psi)_+^2 + 9g(v_+, v_+) + 32g(v_-, v_-) - 7g(v_+^2, v_+^2) + 9g(v_-^2, v_-^2) - 7g(v_-^3, v_-^3) + g(v_+^5, v_+^5) + g(v_-^6, v_-^6) = 0$$

$$4(\phi, \psi)_+^2 - 4(\phi, \psi)_-^2 + g(v_+, v_+) + 4g(v_-, v_-) - g(v_+^2, v_+^2) + g(v_-^2, v_-^2) - g(v_-^3, v_-^3) = 0.$$

Since $(\cdot, \cdot)_+$ is symmetric if $p = 0$ or $1 \pmod{4}$ and anti-symmetric otherwise, we have that

$$(\phi, \psi)_+ = (-1)^{p(p+3)/2}(\psi, \phi)_+.$$

Similarly, since $(\cdot, \cdot)_-$ is symmetric if $q = 0$ or $3 \pmod{4}$ and anti-symmetric otherwise, we can write

$$(\phi, \psi)_- = (-1)^{q(q+1)/2}(\psi, \phi)_-.$$

Combining this with (7) and (8), we can find the conditions under which $v_{\pm}^k \neq 0$ when $\phi = \psi$ (for $k \geq 0$):

$$\begin{aligned} (\phi, e_{i_1} \dots e_{i_k} \phi)_+ &= (-1)^{k(p+1)}(e_{i_k} \dots e_{i_1} \phi, \phi)_+ \\ &= (-1)^{k(p+1)+k(k-1)/2}(e_{i_1} \dots e_{i_k} \phi, \phi)_+ \\ &= (-1)^{k(p+1)+k(k-1)/2+p(p+3)/2}(\phi, e_{i_1} \dots e_{i_k} \phi)_+ \end{aligned}$$

Thus for v_+^k to be nonzero it is necessary to have $k(p+1) + k(k-1)/2 + p(p+3)/2 \equiv 0 \pmod{2}$. This is equivalent to

$$(k+p)^2 - p + k \equiv 0 \pmod{4}.$$

A similar calculation shows that for v_-^k to be nonzero we need

$$(k+q)^2 + q - k \equiv 0 \pmod{4}.$$

Therefore many of the terms in the above identities vanish when we specialize to the case where $\phi = \psi$. The simplified identities for corner algebras are:

- $Cl_{3,2}$:

$$g(v^2, v^2) = 0$$

- $Cl_{4,3}$ and $Cl_{0,7}$:

$$7(\phi, \phi)^2 + g(v^3, v^3) = 0$$

- $Cl_{9,0}$, $Cl_{5,4}$, and $Cl_{1,8}$:

$$\begin{aligned} 14g(v, v) - g(v^4, v^4) &= 0 \\ -(\phi, \psi)^2 + g(v, v) &= 0 \end{aligned}$$

- $Cl_{10,1}$, $Cl_{6,5}$, and $Cl_{2,9}$:

$$\begin{aligned} 5g(v, v) + g(v^2, v^2) &= 0 \\ 75g(v, v) + 21g(v^2, v^2) - 5g(v^5, v^5) &= 0 \end{aligned}$$

- $Cl_{11,2}$, $Cl_{7,6}$, and $Cl_{3,10}$:

$$\begin{aligned} g(v^2, v^2) - g(v^3, v^3) &= 0 \\ 9g(v^2, v^2) - 7g(v^3, v^3) + g(v^6, v^6) &= 0 \end{aligned}$$

and for subordinate algebras are:

- $Cl_{3,1}$:

$$g(v_-, v_-) - g(v_+^2, v_+^2) = 0$$

- $Cl_{2,2}$:

$$g(v_+, v_+) + g(v_-^2, v_-^2) = 0$$

- $Cl_{4,2}$ and $Cl_{0,6}$:

$$-7(\phi, \phi)_+^2 + g(v_-^2, v_-^2) - g(v_+^3, v_+^3) = 0$$

- $Cl_{3,3}$:

$$7(\phi, \phi)_-^2 + g(v_+^2, v_+^2) + g(v_-^3, v_-^3) = 0$$

- $Cl_{5,3}$ and $Cl_{1,7}$:

$$\begin{aligned} -28(\phi, \phi)_-^2 + 14g(v_+, v_+) + g(v_-^3, v_-^3) + g(v_+^4, v_+^4) &= 0 \\ (\phi, \phi)_+^2 + (\phi, \phi)_-^2 - g(v_+, v_+) &= 0 \end{aligned}$$

- $Cl_{8,0}$, $Cl_{4,4}$, and $Cl_{0,8}$:

$$\begin{aligned} 14(\phi, \phi)_+^2 + 14g(v_-, v_-) - g(v_+^3, v_+^3) - g(v_-^4, v_-^4) &= 0 \\ (\phi, \phi)_+^2 - (\phi, \phi)_-^2 + g(v_-, v_-) &= 0 \end{aligned}$$

- $Cl_{10,0}$, $Cl_{6,4}$, and $Cl_{2,8}$:

$$\begin{aligned} -5(\phi, \phi)_-^2 + 5g(v_+, v_+) - g(v_-, v_-) + g(v_+^2, v_+^2) &= 0 \\ -75(\phi, \phi)_-^2 + 75g(v_+, v_+) - 21g(v_-, v_-) + 21g(v_+^2, v_+^2) + 5g(v_-^4, v_-^4) - 5g(v_+^5, v_+^5) &= 0 \end{aligned}$$

- $Cl_{9,1}$, $Cl_{5,5}$, and $Cl_{1,9}$:

$$\begin{aligned} 5(\phi, \phi)_+^2 + g(v_+, v_+) + 5g(v_-, v_-) + g(v_-^2, v_-^2) &= 0 \\ 75(\phi, \phi)_+^2 + 21g(v_+, v_+) + 75g(v_-, v_-) + 21g(v_-^2, v_-^2) - 5g(v_+^4, v_+^4) - 5g(v_-^5, v_-^5) &= 0 \end{aligned}$$

- $Cl_{11,1}$, $Cl_{7,5}$, and $Cl_{3,9}$:

$$\begin{aligned} g(v_-, v_-) - g(v_+^2, v_+^2) - g(v_-^2, v_-^2) + g(v_+^3, v_+^3) &= 0 \\ -9g(v_-, v_-) + 9g(v_+^2, v_+^2) + 7g(v_-^2, v_-^2) - 7g(v_+^3, v_+^3) - g(v_-^5, v_-^5) + g(v_+^6, v_+^6) &= 0 \end{aligned}$$

- $Cl_{10,2}$, $Cl_{6,6}$, and $Cl_{2,10}$:

$$\begin{aligned} g(v_+, v_+) - g(v_+^2, v_+^2) + g(v_-^2, v_-^2) - g(v_-^3, v_-^3) &= 0 \\ 9g(v_+, v_+) - 7g(v_+^2, v_+^2) + 9g(v_-^2, v_-^2) - 7g(v_-^3, v_-^3) + g(v_+^5, v_+^5) + g(v_-^6, v_-^6) &= 0. \end{aligned}$$

3.2.2 Quaternionic Clifford algebras

We consider first the quaternionic corner algebras, from which we can deduce identities in the subordinate algebras using the same argument as the previous section. Recall that if we have a representation of $Cl_{p,q}$ on $\mathbb{H}^n$, we can get a representation on $\mathbb{R}^n$. Given a type $\mathbb{H}$ corner algebra, $Cl_{p,q}$, we can find generators for it using $p+q-1$ many generators from a type $\mathbb{R}$ corner algebra in dimensions $p+q+2$. By taking products of the three unused we generators, we give a quaternionic structure on the spinors of $Cl_{p,q}$, i.e. three maps I, J, K such that $I^2 = J^2 = K^2 = IJK = -1$ by which we can define an action of $\mathbb{H}$ on the space of spinors by $(q_0 + q_1I + q_2J + q_3K)\phi = q_0\phi + q_1I\phi + q_2J\phi + q_3K\phi$. Furthermore, these maps commute with $Cl_{p,q}$. Lastly, this process is such that if $E_1, \dots, E_{p+q}$ are generators for $Cl_{p,q}$ then there is one of them, E_i , such that $E_1, \dots, E_{p+q}, iE_i, jE_i, kE_i$ generate the type $\mathbb{R}$ algebra.

In this section e_i and E_i will be the generators, $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ the scalar products, and g and $\bar{g}$ the metrics for type $\mathbb{R}$ and $\mathbb{H}$ algebras, respectively. We denote the m -forms corresponding to $\phi, I\psi$, $\phi, J\psi$, and $\phi, K\psi$ by ${}^i v_{\pm}^m$, ${}^j v_{\pm}^m$, and ${}^k v_{\pm}^m$.

$Cl_{0,3}$ via $Cl_{3,2}$: $E_1 = e_4, E_2 = e_5$, and $E_3 = e_4e_5$. Note that since $e_1e_2e_3e_4e_5 = 1$, E_3 is equal, up to sign, to $e_1e_2e_3$. Quaternionic structure is given by $I = e_1e_2$, $J = e_3e_1$, and $K = e_2e_3$. We see that $e_1 = KE_3, e_2 = JE_3, e_3 = IE_3, e_4 = E_1, e_5 = E_2$. We see that (where everything is modulo sign)

$$(\phi, \psi) = \phi^t e_4 e_5 \psi = \phi^t E_3 \psi = \langle \phi, E_3 \psi \rangle.$$

Therefore

$$g(v, v) = \sum_{i=1}^3 (\phi, e_i \psi)^2 - \sum_{i=4}^5 (\phi, e_i \psi)^2$$

$$= \langle \phi, I\psi \rangle^2 + \langle \phi, J\psi \rangle^2 + \langle \phi, K\psi \rangle^2 - \langle \phi, E_1\psi \rangle^2 - \langle \phi, E_2\psi \rangle^2.$$

Hence the identity $-(\phi, \psi)^2 + g(v, v) = 0$ becomes

$$\langle \phi, I\psi \rangle^2 + \langle \phi, J\psi \rangle^2 + \langle \phi, K\psi \rangle^2 + \bar{g}(v, v) = 0.$$

We have

$$g(v^2, v^2) = \sum_{1 \leq i < j \leq 3} (\phi, e_i e_j \psi)^2 + (\phi, e_4 e_5 \psi)^2 - \sum_{1 \leq i \leq 3} \sum_{4 \leq j \leq 5} (\phi, e_i e_j \psi)^2$$

$$= \langle \phi, IE_3\psi \rangle^2 + \langle \phi, JE_3\psi \rangle^2 + \langle \phi, KE_3\psi \rangle^2 + \langle \phi, \psi \rangle^2$$

$$- \sum_{i=1}^2 (\langle \phi, IE_i\psi \rangle^2 + \langle \phi, JE_i\psi \rangle^2 + \langle \phi, KE_i\psi \rangle^2)$$

So by the $Cl_{3,2}$ identity $2g(v, v) + g(v^2, v^2) = 0$, we have that

$$2\langle \phi, I\psi \rangle^2 + 2\langle \phi, J\psi \rangle^2 + 2\langle \phi, K\psi \rangle^2 - 2\langle \phi, E_1\psi \rangle^2 - 2\langle \phi, E_2\psi \rangle^2$$

$$+ \langle \phi, IE_3\psi \rangle^2 + \langle \phi, JE_3\psi \rangle^2 + \langle \phi, KE_3\psi \rangle^2 + \langle \phi, \psi \rangle^2$$

$$- \sum_{i=1}^2 (\langle \phi, IE_i\psi \rangle^2 + \langle \phi, JE_i\psi \rangle^2 + \langle \phi, KE_i\psi \rangle^2) = 0$$

By symmetry, the above equation must be valid if we permute E_i, E_j, E_k . Thus we get three equations which, when added together, give

$$3\langle \phi, \psi \rangle^2 + 6\langle \phi, I\psi \rangle^2 + 6\langle \phi, J\psi \rangle^2 + 6\langle \phi, K\psi \rangle^2 +$$

$$4\bar{g}(v, v) + \bar{g}({}^i v, {}^i v) + \bar{g}({}^j v, {}^j v) + \bar{g}({}^k v, {}^k v) = 0$$

If we go down to $Cl_{0,2}$ then $\langle \phi, \psi \rangle_+ = \langle \phi, \psi \rangle$ and $\langle \phi, \psi \rangle_- = \langle \phi, E_3\psi \rangle$. Hence

$$g(v, v) = \langle \phi, I\psi \rangle^2 + \langle \phi, J\psi \rangle^2 + \langle \phi, K\psi \rangle^2 - \bar{g}(v, v)$$

$$g(v^2, v^2) = \langle \phi, I\psi \rangle_-^2 + \langle \phi, J\psi \rangle_-^2 + \langle \phi, K\psi \rangle_-^2 + \langle \phi, \psi \rangle_+^2$$

$$- \bar{g}({}^i v, {}^i v) - \bar{g}({}^j v, {}^j v) - \bar{g}({}^k v, {}^k v).$$

$Cl_{1,4}$ via $Cl_{4,3}$: $E_1 = e_4, E_2 = e_5, E_3 = e_6, E_4 = e_7, E_5 = e_4 e_5 e_6 e_7, I = e_1 e_2, J = e_3 e_1, K = e_2 e_3$. Hence $e_1 = KE_5, e_2 = JE_5, e_3 = IE_5, e_4 = E_1, e_5 = E_2, e_6 = E_3, e_7 = E_4$.

For $Cl_{1,3}$ we take E_1, E_2, E_3, E_4 and E_5 becomes γ . Then $(\phi, \psi) = \phi^t e_5 e_6 e_7 \psi = \phi^t E_2 E_3 E_4 \psi = \langle \phi, \psi \rangle_-$. We have

$$\begin{aligned} g(v, v) &= \sum_{i=1}^4 (\phi, e_i \psi)^2 - \sum_{i=5}^7 (\phi, e_i \psi)^2 \\ &= \langle \phi, K\gamma\psi \rangle_-^2 + \langle \phi, J\gamma\psi \rangle_-^2 + \langle \phi, I\gamma\psi \rangle_-^2 + \langle \phi, E_1\psi \rangle_- - \sum_{i=2}^4 \langle \phi, E_i\psi \rangle_- \\ &= \langle \phi, I\psi \rangle_+^2 + \langle \phi, J\psi \rangle_+^2 + \langle \phi, K\psi \rangle_+^2 + \bar{g}(v_-, v_-) \end{aligned}$$

and

$$\begin{aligned} g(v^2, v^2) &= \sum_{1 \leq i < j \leq 4} (\phi, e_i e_j \psi)^2 + \sum_{5 \leq i < j \leq 7} (\phi, e_i e_j \psi)^2 - \sum_{i=1}^4 \sum_{j=5}^7 (\phi, e_i e_j \psi)^2 \\ &= \langle \phi, I\psi \rangle_-^2 + \langle \phi, J\psi \rangle_-^2 + \langle \phi, K\psi \rangle_-^2 + \langle \phi, \gamma I E_1 \psi \rangle_-^2 + \langle \phi, \gamma J E_1 \psi \rangle_-^2 \\ &\quad + \langle \phi, \gamma K E_1 \psi \rangle_-^2 + \sum_{2 \leq i < j \leq 4} \langle \phi, E_i E_j \psi \rangle_-^2 - \sum_{i=2}^4 (\langle \phi, \gamma I E_i \psi \rangle_-^2 + \langle \phi, \gamma J E_i \psi \rangle_-^2 \\ &\quad + \langle \phi, \gamma K E_i \psi \rangle_-^2 + \langle \phi, E_1 E_i \psi \rangle_-^2) \\ &= \langle \phi, I\psi \rangle_-^2 + \langle \phi, J\psi \rangle_-^2 + \langle \phi, K\psi \rangle_-^2 + \bar{g}({}^i v_+, {}^i v_+) + \bar{g}({}^j v_+, {}^j v_+) + \bar{g}({}^k v_+, {}^k v_+) + \bar{g}(v_-^2, v_-^2) \end{aligned}$$

and

$$\begin{aligned} g(v^3, v^3) &= \langle \phi, \psi \rangle_+^2 + \bar{g}({}^i v_-, {}^i v_-) + \bar{g}({}^j v_-, {}^j v_-) + \bar{g}({}^k v_-, {}^k v_-) \\ &\quad + \bar{g}({}^i v_+^2, {}^i v_+^2) + \bar{g}({}^j v_+^2, {}^j v_+^2) + \bar{g}({}^k v_+^2, {}^k v_+^2) - \bar{g}(v_+, v_+) \end{aligned}$$

4 Conclusion

In the area of bilinear forms on spinors, there is still much work to be done. This includes identifying a pattern with the identities, studying the relationships between algebras of different type, and considering more general expressions, like $g(v, w)$ where $v(u) = (\phi, u\psi)_\pm$ and $w(u) = (\alpha, u\beta)_\pm$.

5 Acknowledgements

I am greatly indebted to my advisor Dr. George Sparling who has spent countless hours with me. Additionally, I am grateful for the support of the University of Pittsburgh Honors College through two fellowships, during which most of this work was done: the Chancellor's Undergraduate Research Fellowship in Spring 2008 and the Brackenridge Fellowship in Summer 2008.

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