Examples of Spivak type wedge products. PhL 9.3.15

These are the rules I shall use:

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!}\,\mathrm{Alt}(\omega \otimes \eta)$$

$$\mathrm{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} \mathrm{sgn}\,\sigma \cdot T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

Example 1. Two elements of Λ^1.

$$\omega \wedge \eta = \frac{2!}{1!1!}\,\frac{1}{2!}\,[\,\omega\otimes\eta - \eta\otimes\omega\,] = \omega\otimes\eta - \eta\otimes\omega$$

Note that $\omega\otimes\eta$ and $\eta\otimes\omega$ are elements of T^2 but not of Λ^2.
Note that $\omega\otimes\eta - \eta\otimes\omega$ is an element of both T^2 and Λ^2.

How to *verify* now that this result is really in Λ^2 ? Do I have a projection operator?

$$\mathrm{Alt}\,(\omega\otimes\eta - \eta\otimes\omega) = \mathrm{Alt}(\omega\otimes\eta) - \mathrm{Alt}(\eta\otimes\omega) = (1/2)\,[\,\omega\otimes\eta - \eta\otimes\omega\,] - (1/2)\,[\,\eta\otimes\omega - \omega\otimes\eta\,]$$

$$= (\omega\otimes\eta - \eta\otimes\omega)$$

This is like Spivak's little page 78 rule

(2) *If* $\omega \in \Lambda^k(V)$, *then* $Alt(\omega) = \omega$.

which I am running backwards.

Example 2: Element of Λ^1 with an element of Λ^2.

As our element of Λ^2 use $\alpha = \omega_2\otimes\omega_3 - \omega_3\otimes\omega_2$ where these ω_i are in Λ^1.
As our element of Λ^1 use ω_1.

Then:

$$\omega_1 \wedge \alpha = \omega_1 \wedge [\,\omega_2\otimes\omega_3 - \omega_3\otimes\omega_2]$$

$$= \frac{3!}{1!2!}\ \mathrm{Alt}\,\{\,\omega_1 \otimes [\,\omega_2\otimes\omega_3 - \omega_3\otimes\omega_2]\,\}$$

The two terms inside the Alt are really equal, so we will just double the first term, so then

$$= \frac{3!}{1!2!}\ 2\ \text{Alt}\ \{\ \omega_1 \otimes\ \omega_2\ \otimes \omega_3\ \}$$

Note added 9.9.15. Why are they equal? I ignore the 1/3! here:

$$A = \text{Alt}\ \{\ \omega_1 \otimes\ \omega_2\ \otimes \omega_3\ \} = \varepsilon_{ijk}\ \omega_i \otimes\ \omega_j\ \otimes \omega_k \quad \text{factor omitted?}$$
$$B = -\ \text{Alt}\ \{\ \omega_1 \otimes\ \omega_3\ \otimes \omega_1\ \} = -\ ????$$

Write

$$A = \text{Alt}\ \{\ \omega_1 \otimes\ \omega_2\ \otimes \omega_3\ \} = \omega_1 \otimes\ \omega_2\ \otimes \omega_3\ + \text{other terms}$$
$$B = -\ \text{Alt}\ \{\ \omega_1 \otimes\ \omega_3\ \otimes \omega_1\ \} = -\ \omega_1 \otimes\ \omega_3\ \otimes \omega_2\ + \text{other terms}$$

Maybe write in terms of a sum over permutations

$$A = \Sigma_P\ (-1)^S\ \omega_{P(1)} \otimes\ \omega_{P(2)}\ \otimes \omega_{P(3)}$$

where { P(1),P(2).....} is P {1,2...}. Then the above two lines for A,B are reasonable. (they were anyway). So we then have

$$A = \Sigma_P\ (-1)^S\ \omega_{P(1)} \otimes\ \omega_{P(2)}\ \otimes \omega_{P(3)}$$

$$B = -\ \Sigma_P\ (-1)^S\ \omega_{P(1)} \otimes\ \omega_{P(3)}\ \otimes \omega_{P(2)}\ =\ ..$$

OK stop. I know it is right, later I will write down a better tool for doing this.

Now we use

$$\text{Alt}\ \{\ \omega_1 \otimes\ \omega_2\ \otimes \omega_3\ \} = \frac{1}{3!}\ [\ \omega_1 \otimes\ \omega_2\ \otimes \omega_3\ + 5\ \text{more terms}\ ] = \frac{1}{3!}\ \varepsilon_{ijk}\ \omega_i \otimes\ \omega_j \otimes \omega_k$$

We then find that

$$\omega_1 \wedge \alpha\ = \frac{3!}{1!2!}\ 2\ \frac{1}{3!}\ [\ \omega_1 \otimes\ \omega_2\ \otimes \omega_3\ + 5\ \text{more terms}\ ]$$

$$= [\ \omega_1 \otimes\ \omega_2\ \otimes \omega_3\ + 5\ \text{more terms}\ ] = \varepsilon_{ijk}\ \omega_i \otimes \omega_j \otimes \omega_k$$

This result can also be written as

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = \varepsilon_{ijk}\ \omega_i \otimes \omega_j \otimes \omega_k$$

implied sum on i,j and k. But we won't always have such nicely named vectors like ω_i so better just leave these results as

$\omega \wedge \eta = \omega\otimes\eta - \eta\otimes\omega$

$\omega \wedge (a \wedge b) = \omega\otimes a\otimes b - \omega\otimes b\otimes a$ + 4 more terms

Comment: We have to represent α as $a\otimes b - b\otimes a$ in order to talk about α being "alternating" and therefore a viable element of Λ^2. If you don't do this and just use α, you cannot write out the result. That is to say, if you are interested in $\omega \wedge \alpha$, you cannot express this as a sum of terms as I have done above just using the symbols ω and α. So you really need vectors to get explicit expressions!

Example 3: The generalization of Example 2 is going to be this:

$\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \ldots \omega_k = \varepsilon_{abc\ldots q}\,\omega_a\otimes\omega_b\otimes\omega_c\ldots\ldots\otimes\omega_q.$ k factors

Example 4. Let's now try to wedge an element of Λ^s with an element of Λ^t .

$\alpha = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \ldots \omega_s$

$\beta = \omega_{s+1} \wedge \omega_{s+2} \wedge \omega_{s+3} \wedge \ldots \omega_{s+t}$

where as usual we just represent these elements this way so we can have something to chew on. I guess we need all these vectors different so we don't zero out things. Think of a very high n for dim(V). We know that

$\alpha = \varepsilon_{ijk\ldots q}\,\omega_i\otimes\omega_j\otimes\omega_k \quad \ldots\otimes\omega_q$ s factors

$\beta = \varepsilon_{IJK\ldots Q}\ \omega_{s+I}\otimes\omega_{s+J}\otimes\omega_{s+K}\otimes \ldots \omega_{s+Q}$ t factors

And we also know that

$\alpha \wedge \beta = \varepsilon_{ijk\ldots z}\,\omega_i\otimes\omega_j\otimes\omega_k \quad \ldots\otimes\omega_z$ s+t factors

That is about all we can say.

Now would one relate $\beta \wedge \alpha$ to $\alpha \wedge \beta$?

I think we just have to slide the two groups through each other to get from one to the other!

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Maybe rename the vectors so that

$\beta = \varepsilon_{IJK...Q} \quad \omega_{s+I} \otimes \omega_{s+J} \otimes \omega_{s+K} \otimes \ldots \omega_{s+Q}$

$= \varepsilon_{IJK...Q} \quad \eta_I \otimes \eta_J \otimes \eta_K \otimes \ldots \eta_s$ s factors

$\alpha = \varepsilon_{IJK...Q} \quad \eta_{t+I} \otimes \eta_{t+J} \otimes \eta_{t+K} \otimes \ldots \omega_{t+Q}$ t factors

Then certainly we get

$\beta \wedge \alpha = \varepsilon_{ijk...z}\, \eta_i \otimes \eta_j \otimes \eta_k \quad \ldots \otimes \eta_z$ t+s factors

So how would w rearrange this thing to get the $\alpha \wedge \beta$ expression shown above?

Comments 9.9.15. I have just reread the above. Here I am scrupulously avoiding "arguments" of the Spivak type. There are no arguments at all. I find that if I want to evaluate wedge products say of an element of Λ^1 with an element of Λ^2 (Example 2 above), I can't write out the resulting wedge product unless I assume that the Λ^2 element is given by something like $\alpha = \omega_2 \otimes \omega_3 - \omega_3 \otimes \omega_2$ where ω_2 and ω_3 are elements of Λ^1. I claim that $\omega_2 \otimes \omega_3 - \omega_3 \otimes \omega_2$ is an element of Λ^2 because I interpreted Spivak's Alt formula shown at the start in a completely different way, one with no arguments. This whole approach now seems very wobbly and I have trouble really supporting it.

Plan A . Why note represent an arbitrary element of Λ^2 as an expansion on basis vectors.

Example 2A: Element of Λ^1 with an element of Λ^2 . I am NOT going to use i < j here because it makes problems down the road. So for use here I will write.

$\alpha = \Sigma_{i,j}\, \alpha_{ij}\ \mathbf{e}_i \wedge \mathbf{e}_j$

Then it really is in terms of those required vectors, but it is at the same time a general element of Λ^2. Then let ω_1 be the element of Λ^1 same as in example 2. Now we already know that

$\omega \wedge \eta = \omega \otimes \eta - \eta \otimes \omega$ for any two vectors ω and η each of which is in Λ^1

Then we can say $\omega = e_i$ and $= e_j$ and we have

$\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i$

And then

$\alpha = \Sigma_{i,j}\, \alpha_{ij}\ [\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i]$

$= \Sigma_{i,j}\, \alpha_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j - \Sigma_{i,j}\, \alpha_{ij}\ \mathbf{e}_j \otimes \mathbf{e}_i$

$= \Sigma_{i,j}\, \alpha_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j - \Sigma_{j,i}\, \alpha_{ji}\ \mathbf{e}_i \otimes \mathbf{e}_j$

$= \Sigma_{i,j} (\alpha_{ij} - \alpha_{ji})\ \mathbf{e}_i \otimes \mathbf{e}_j$

This shows that only the antisymmetric component of α_{ij} makes a contribution to α. We can maintain a symmetric component of α if we wish.

How do I verify this to be an element of Λ^2? I just repeat what I did in Example 1. Apply the Alt operator and find that Alt(q) = q.

Now we are ready to continue with this Example 2a:

$\omega_1 \wedge \alpha = \omega_1 \wedge \{ \Sigma_{i,j}\ \alpha_{ij}\ [\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i] \}$

$= \Sigma_{i,j} \alpha_{ij} \{ \omega_1 \wedge [\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i] \}$

$= \Sigma_{i,j}\ \alpha_{ij}\ \frac{3!}{1!2!}\ \text{Alt} \{ \omega_1 \otimes [e_i \otimes e_j - e_j \otimes e_i] \}$

$= \Sigma_{i,j} \alpha_{ij}\ \frac{3!}{1!2!}\ 2\ \text{Alt} \{ \omega_1 \otimes e_i \otimes e_j \}$

$= \Sigma_{i,j} \alpha_{ij} [\ \omega_1 \otimes e_i \otimes e_j - \omega_1 \otimes e_j \otimes e_i\ + 4 \text{ more signed terms }]$

So at least I have a result! Can this be simplified? I don't think so.

Example 2B: Let's repeat the above, but write

$\omega_1 = \Sigma_k a_k e_k$

Then we start over again:

$\omega_1 \wedge \alpha = (\Sigma_k a_k e_k) \wedge \{ \Sigma_{i,j}\ \alpha_{ij}\ [\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i] \}$

$= \Sigma_k a_k\ \ \Sigma_{i,j} \alpha_{ij}\ \ e_k \wedge (e_i \otimes e_j - e_j \otimes e_i)$

$= \Sigma_k a_k\ \ \Sigma_{i,j} \alpha_{ij}\ \frac{3!}{1!2!}\ \text{Alt} \{ e_k \otimes [e_i \otimes e_j - e_j \otimes e_i] \}$

$= \Sigma_k a_k\ \ \Sigma_{i,j} \alpha_{ij}\ \frac{3!}{1!2!}\ 2\ \text{Alt} \{ e_k \otimes e_i \otimes e_j \}$

$= \Sigma_k a_k\ \ \Sigma_{i,j} \alpha_{ij}\ [\ e_k \otimes e_i \otimes e_j\ + 5 \text{ other signed terms }]$

$= \Sigma_k a_k\ \ \Sigma_{i,j} \alpha_{ij}\ [\ e_i \otimes e_j \otimes e_k\ + 5 \text{ other signed terms }]$

Now write this as

$$\omega_1 \wedge \alpha = \Sigma_{ijk}\, T_{ijk}\, \Sigma_P\, (-1)^S\, e_{P(i)} \otimes e_{P(j)} \otimes e_{P(k)} \qquad T_{ijk} \equiv \alpha_{ij} a_k$$

Now use the result of "theorem on sums" to rewrite this as

$$\omega_1 \wedge \alpha = \Sigma_{ijk}\ [\ \Sigma_P\, (-1)^S T_{P(i)P(j)P(k)}\ ]\ e_i \otimes e_j \otimes e_k$$

where now the coefficient is

$$\Sigma_P\, (-1)^S T_{P(i)P(j)P(k)} = T_{ijk} - T_{ikj} + T_{kij} - T_{kji} + T_{jki} - T_{jik}$$

$$= \alpha_{ij} a_k - \alpha_{ik} a_j + \alpha_{ki} a_j - \alpha_{kj} a_i + \alpha_{jk} a_i - \alpha_{ji} a_k$$

If we consider α_{ij} to be antisymmetric as noted above, then we can combine terms to get

$$\Sigma_P\, (-1)^S T_{P(i)P(j)P(k)} = 2\ [\ \alpha_{ij} a_k + \alpha_{ki} a_j + \alpha_{jk} a_i\ ]$$

and then our result is

$$\omega_1 \wedge \alpha = 2\ \Sigma_{ijk}\ [\ \alpha_{ij} a_k + \alpha_{ki} a_j + \alpha_{jk} a_i\ ]\ e_i \otimes e_j \otimes e_k$$

Is this coefficient really antisymmetric under i↔j? The first term is. The second term does this

$$\alpha_{ki} a_j \rightarrow \alpha_{kj} a_i = -\alpha_{jk} a_i = \text{minus the third term}$$

So yes, the coefficient is totally antisym!

Comment: This is the first time ever that I have computed the wedge product of a general vector with a general bivector using a non-function-based definition of the wedge product:

$$\omega \wedge \eta = \frac{(k+l)!}{k!\, l!}\, \mathrm{Alt}(\omega \otimes \eta)$$

I first considered this for the case ω and η were vectors so $k = \boldsymbol{\ell} = 1$.

I then considered this for the case that ω was a vector and η was a general bivector!

Here I will do the first case again showing general vectors:

$$\omega = \Sigma_i a_i e_i$$
$$\eta = \Sigma_j b_j e_j$$

$$\omega \wedge \eta = \Sigma_{ij} a_i b_j \frac{2!}{1!1!} \frac{1}{2!}\ [\ e_i \otimes e_j - e_j \otimes e_i\ ] = \Sigma_{ij} a_i b_j\ [\ e_i \otimes e_j - e_j \otimes e_i\ ]$$

$= \Sigma_{ij} [a_i b_j - a_j b_i] e_i \otimes e_j$

I think I can extend this method to compute the wedge product of any two multivectors.

Maybe the function-based approach is the dual space?

Consider the simple space V with elements like $\mathbf{e_i}$ or **x**.
The space V* is the dual space of functionals on V, such as f(**x**).

Now consider the direct product space V x V which has elements like (**x**,**y**).
The space V* x V* would be the space of linear functionals on V x V such as f(**x**,**y**)

In general we can define $V^{\mathbf{k}}$ with elements like ($\mathbf{v_1}$, $\mathbf{v_2}$....$\mathbf{v_k}$) .
The dual space would be $V^{*\mathbf{k}}$ with functionals like f($\mathbf{v_1}$, $\mathbf{v_2}$....$\mathbf{v_k}$).

Do any of my authors talk about things in this way?

Spivak: Well, he is in fact quite clear on this. Right on his opening page:

ALGEBRAIC PRELIMINARIES

If V is a vector space (over $\mathbf{R}$), we will denote the k-fold product $V \times \cdots \times V$ by V^k. A function $T: V^k \to \mathbf{R}$ is called **multilinear** if for each i with $1 \leq i \leq k$ we have

$$T(v_1, \ldots, v_i + v_i', \ldots, v_k) = T(v_1, \ldots, v_i, \ldots, v_k) + T(v_1, \ldots, v_i', \ldots, v_k),$$
$$T(v_1, \ldots, av_i, \ldots, v_k) = aT(v_1, \ldots, v_i, \ldots, v_k).$$

A multilinear function $T: V^k \to \mathbf{R}$ is called a ***k*-tensor** on V and the set of all k-tensors, denoted $\mathfrak{T}^k(V)$, becomes a vector space (over $\mathbf{R}$) if for $S, T \in \mathfrak{T}^k(V)$ and $a \in \mathbf{R}$ we define

$$(S + T)(v_1, \ldots, v_k) = S(v_1, \ldots, v_k) + T(v_1, \ldots, v_k),$$
$$(aS)(v_1, \ldots, v_k) = a \cdot S(v_1, \ldots, v_k).$$

There is also an operation connecting the various spaces $\mathfrak{T}^k(V)$. If $S \in \mathfrak{T}^k(V)$ and $T \in \mathfrak{T}^l(V)$, we define the **tensor product** $S \otimes T \in \mathfrak{T}^{k+l}(V)$ by

$$S \otimes T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = S(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+l}).$$

So for Spivak, a k-tensor is a functional in this dual space, though he does not mention dual spaces.

Benn-Tucker. They thing the same way

maps on ordered sets of elements from V^*. The F-valued multilinear maps on $V^* \times V^* \times \ldots \times V^*$ (r times) are called *tensors* of degree r.

except they have their perverse thing about V ↔ V*.