Spivak's Wedge Product Definition PhL 8.25.15

I have already shown in my Spivak overview notes that

$$[\text{Alt}(T)](\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) = (1/k!)\,\Sigma_{\mathbf{abc..q}}\,\varepsilon_{\mathbf{abc..q}}\,T(\mathbf{v}_a,\mathbf{v}_b....\mathbf{v}_q) \qquad \text{// k! terms in sum}$$

If T were already AS from the start, then all those k! terms are the same, and then you would get

$$[\text{Alt}(A)](\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) = (1/k!)\,\Sigma_{\mathbf{abc..q}}\,\varepsilon_{\mathbf{abc..q}}\,A(\mathbf{v}_a,\mathbf{v}_b....\mathbf{v}_q) = A(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)$$

Better to write this instead as

$$[\text{Alt}(A)](\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) = (1/k!)\,\Sigma_{\mathbf{i_1 i_2}\,.....\mathbf{i_k}}\,\varepsilon_{\mathbf{i_1 i_2}\,.....\mathbf{i_k}}\,A(\mathbf{v}_{\mathbf{i_1}},\mathbf{v}_{\mathbf{i_2}}....\mathbf{v}_{\mathbf{i_k}}) = A(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)$$

Notice that $\det(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)$ would provide an example of $A(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)$, but he does not mention this.

Spivak notation:

Space of k-multilinear functions = $\mathcal{T}^k(\mathbf{V})$ elements are called k-tensors

Space of **alternating** k-multilinear functions = $\mathbf{\Lambda}^k(\mathbf{V})$ $\mathbf{\Lambda}^k(\mathbf{V}) \subset \mathcal{T}^k(\mathbf{V})$

The function T is in space $\mathcal{T}^k(\mathbf{V})$, but alternating T's are in subspace $\mathbf{\Lambda}^k(\mathbf{V})$. Spivak just calls these things alternating tensors, though the are also k-multilinear functionals of k vector arguments.

Now suppose we have for general integers k and $\boldsymbol{\ell}$,

ω in $\mathbf{\Lambda}^k(\mathbf{V})$ $\omega(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)$
η in $\mathbf{\Lambda}^{\boldsymbol{\ell}}(\mathbf{V})$ $\eta(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_{\boldsymbol{\ell}})$

This says that ω and η separately are AS.

In his preliminaries section he has already defined the notion of $\omega \otimes \eta$ in this manner

$$[\omega \otimes \eta](\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k,\, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell}) = \omega(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)\,\eta(\mathbf{v}_{k+1},\mathbf{v}_{k+2}....\mathbf{v}_{k+\ell}) \qquad (*)$$

so we have a *product* of two scalar functions each of which is an alternating k-multilinear, but neither of which is AS. Let's for the moment replace $[\omega \otimes \eta]$ by f and write

$$f(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k,\, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell}) = \omega(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)\,\eta(\mathbf{v}_{k+1},\mathbf{v}_{k+2}....\mathbf{v}_{k+\ell})$$

Question: Is f a $(k+\boldsymbol{\ell})$-multilinear function? My only doubt would be for something that crosses the boundary, so consider on the left side only

$$f(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k + 2\mathbf{v}_{k+2}, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell})$$

If this is (k+$\boldsymbol{\ell}$)-multilinear, then we must have

$$f(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k + 2\mathbf{v}_{k+2}, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell})$$

$$= f(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell}) \ +2\ f(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_{k+2}, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell})$$

Let's see if this works out or not. Using the product we have

$$\omega(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k + 2\mathbf{v}_{k+2})\ \eta(\mathbf{v}_{k+1},\mathbf{v}_{k+2}....\mathbf{v}_{k+\ell})$$

$$= [\ \omega(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_{k-1},\mathbf{v}_k) \ + 2\ \omega(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_{k-1},\mathbf{v}_{k+2})\ ]\ \eta(\mathbf{v}_{k+1},\mathbf{v}_{k+2}....\mathbf{v}_{k+\ell})$$

$$= \ \omega(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)\ \eta(\mathbf{v}_{k+1},\mathbf{v}_{k+2}....\mathbf{v}_{k+\ell}) \ + \ 2\ \omega(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_{k+2})\ \eta(\mathbf{v}_{k+1},\mathbf{v}_{k+2}....\mathbf{v}_{k+\ell})$$

But this last line is in fact,

$$= \ f(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell}) + 2\ \ f(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_{k+2}, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell})$$

so I have shown that

Fact: If ω in $\mathcal{T}^k(\mathbf{V})$ and η in $\mathcal{T}^{\ell}(\mathbf{V})$, then [ω ⊗ η] in $\mathcal{T}^{k+\ell}(\mathbf{V})$.

But in our case here we in fact have ω and η each being AS at the start.

Now we want to make this new (k+$\boldsymbol{\ell}$)-multilinear functional [ω ⊗ η] *be* antisymmetric (AS). Lets try this

$$[\omega \otimes \eta]_{\mathbf{AS}} = K \ \ \Sigma_{\mathbf{i_1 i_2}\ .....\mathbf{i_{k+\ell}}}\ \varepsilon_{\mathbf{i_1 i_2}\ .....\mathbf{i_{k+\ell}}}\ \ [\omega \otimes \eta]\ (\mathbf{v}_{\mathbf{i}_1},\mathbf{v}_{\mathbf{i}_2}....\mathbf{v}_{\mathbf{i}_k}, \mathbf{v}_{\mathbf{i}_{k+1}}.....\mathbf{v}_{\mathbf{i}_{k+\ell}})$$

$$= \ K \ \ \Sigma_{\mathbf{i_1 i_2}\ .....\mathbf{i_{k+\ell}}}\ \varepsilon_{\mathbf{i_1 i_2}\ .....\mathbf{i_{k+\ell}}}\ \ \omega(\mathbf{v}_{\mathbf{i}_1},\mathbf{v}_{\mathbf{i}_2}....\mathbf{v}_{\mathbf{i}_k})\ \eta(\mathbf{v}_{\mathbf{i}_{k+1}}.....\mathbf{v}_{\mathbf{i}_{k+\ell}})$$

where K is some constant. Now based on (*) it seems impossible that [ω ⊗ η] could be AS from the start. What Spivak says is

$$\omega \otimes \eta \text{ is usually not in } \Lambda^{k+l}(V).$$

I don's see how ω ⊗ η could *ever* be in $\Lambda^{k+\ell}(\mathbf{V})$. What about

$$[\omega \otimes \eta](\mathbf{v}_1,\mathbf{v}_2) = \omega(\mathbf{v}_1)\eta(\mathbf{v}_2)$$

How could the RHS here ever be AS under $\mathbf{v}_1 \leftrightarrow \mathbf{v}_2$ for all arguments? You would need

$\omega(\mathbf{v}_1)\eta(\mathbf{v}_2) = -\omega(\mathbf{v}_2)\eta(\mathbf{v}_1)$

I just don't thing that is possible for any ω and η functions, including constants. So his comment "usually" is a little strange. Well, suppose one is a constant. Then $\boldsymbol{\ell} = 0$ so not rally meaningful.

In any event, if ω in $\boldsymbol{\Lambda}^k(\mathbf{V})$ and η in $\boldsymbol{\Lambda}^{\boldsymbol{\ell}}(\mathbf{V})$, then I would say [ω ⊗ η] is definitely not in $\boldsymbol{\Lambda}^{k+\boldsymbol{\ell}}(\mathbf{V})$ due to the fact that the product is not overall TA. So he fixes up by defining the wedge product in this manner:

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!}\,\mathrm{Alt}(\omega \otimes \eta).$$

Then for sure [ω ∧ η] ($\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell}$) IS in $\boldsymbol{\Lambda}^{k+\boldsymbol{\ell}}(\mathbf{V})$. Spivak defers the explanation of the strage constant. But the main point is this:

[ω ∧ η] ($\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell}$) = K Alt(ω x η) = an element of $\boldsymbol{\Lambda}^{k+\boldsymbol{\ell}}(\mathbf{V})$

Based on this wedge definition, Spivak derives various properties.

Comments at this point. How does this compare to Sjamaar who says

[ω x η] ($\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell}$) // Spivak

"ωη" ($\mathbf{v}_1,\mathbf{v}_2$) = det(some matrix) // Sjamaar

where the matrix would be

$\omega(\mathbf{v}_1)$	$\omega(\mathbf{v}_2)$
$\eta(\mathbf{v}_1)$	$\eta(\mathbf{v}_2)$

It seems that Sjamaar's use of the wedge product is only for the latter case, where each of the two functions has only a single argument. This same is true for Sjamaar's more general a ∧ b ∧ c . So the Spivak wedge product definition is much more general, and at this point, it is not defined in terms of any determinants!

Pause: So finally at least I have "made some sense" out of the Spivak definition of the wedge product, which up to today has seemed meaningless. It does have some connection with Sjamaar.

I guess if you really want to understand things, you have to read all of Spivak Chapter 4 which is a large task. His text seems at least "readable" to me, it is not beyond me. I would then have the option to read his Chapter 5 on doing this all on manifolds.

Continue 8.26.15

Consider this Spivak section,

One important construction, familiar for the case of dual spaces, can also be made for tensors. If $f: V \to W$ is a linear transformation, a linear transformation $f^*: \mathfrak{T}^k(W) \to \mathfrak{T}^k(V)$ is defined by

$$f^*T(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k))$$

for $T \in \mathfrak{T}^k(W)$ and $v_1, \ldots, v_k \in V$. It is easy to verify that $f^*(S \otimes T) = f^*S \otimes f^*T$.

This operation f* is a generalization of Sjamaar's pullback action on functions. Sja always has thing like $\varphi^* f(\mathbf{x}) = f(\varphi(\mathbf{x}))$, so Spivak is stating this as $f^*T(\mathbf{v_1}) = T(f(\mathbf{v_1}))$ where f* is his pullback. But Spivak's pullback is more general because at acts on function having k vector arguments. So our Sja/Spi connection is still doing well.

After this, Spivak says that <a,b> is his notation for the R^n inner product, and then for a while he writes that T(x,y) will be his inner product for more general vector space V. His theorem then says there is some basis v_i such that $T(v_i,v_j) = \delta_{ij}$ which seems pretty reasonable. Then he connects these last two ideas in the following manner:

$[f^*T]\,(\mathbf{x_i},\mathbf{x_j}) = T(f(\mathbf{x_i}),f(\mathbf{x_j})) = \langle \mathbf{x_i},\mathbf{x_j} \rangle$ // $\langle \mathbf{x_i},\mathbf{x_j} \rangle = I(\mathbf{x_i},\mathbf{x_j})$

It must be that f: $R^n \to V$ because the arguments of T have to be in V. He refers to f: $R^n \to V$ as an isomorphism which to me means a 1-to-1 between R^n and V. I think this means that the above statement is a definition of the mapping f. Defining the R^n inner product as operator I (my own), then the above says that $[f^*T]\,(\mathbf{x_i},\mathbf{x_j}) = I(\mathbf{x_i},\mathbf{x_j})$ so in shorthand, this says $f^*T = I = \langle , \rangle$ in his funny notation.

I have backed up for the above stuff. He then goes into his wedge product discussion as I outline above.

Let's now look at 4-4 Theorem. (he uses same strange headings as Sjamaar, I guess that is a standard.)

(1) S in $\mathcal{T}^k(\mathbf{V})$, T in $\mathcal{T}^\ell(\mathbf{V})$ [the pre-alternating adder spaces] Then

$\mathrm{Alt}(S) = 0 \implies \mathrm{Alt}(S \times T) = 0$ and $\mathrm{Alt}(T \times S) = 0$.

My proof:

$S = S(\mathbf{v_1},\mathbf{v_2}....\mathbf{v_k})$
$\mathrm{Alt}(S) = \Sigma_{\mathbf{abc..k}}\ \varepsilon_{\mathbf{abc..k}}\ S(\mathbf{v_a},\mathbf{v_b}....\mathbf{v_k})$

I think if Alt(S) = 0, then $S(\mathbf{v_1},\mathbf{v_2}....\mathbf{v_k})$ must be symmetric on at least one pair of arguments. Then when you consider

$[S \otimes T](\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k, \mathbf{v}_{k+1}.....\mathbf{v}_{k+\ell}) = S(\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k)\, T(\mathbf{v}_{k+1},\mathbf{v}_{k+2}....\mathbf{v}_{k+\ell})$

and your write out Alt[S ⊗ T] using ε as above, that same argument makes Alt[S ⊗ T] = 0. He gives a formal proof which I don't want to attempt to follow. I will accept the result as valid, and it certainly seems to be valid.

$$(2)\quad Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta)).$$

I again just accept this, it seems reasonable. He gives a tiny proof I could follow it I wanted.

(3) *If* $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, *and* $\theta \in \Lambda^m(V)$, *then*

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!\,l!\,m!}\, Alt(\omega \otimes \eta \otimes \theta).$$

This is a generalization of his earlier result that

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!}\, \mathrm{Alt}(\omega \otimes \eta).$$

It seems that since you end up with this associative rule for the wedge product, you could write

$a \wedge b \wedge c \equiv (a \wedge b) \wedge c = a \wedge (b \wedge c)$

He in fact states this his page 81. This brings us to his 4-5 Theorem. But first recall earlier result

4-1 Theorem. *Let* $v_1, \ldots, v_n$ *be a basis for* V, *and let* $\varphi_1, \ldots, \varphi_n$ *be the dual basis,* $\varphi_i(v_j) = \delta_{ij}$. *Then the set of all k-fold tensor products*

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \qquad 1 \le i_1, \ldots, i_k \le n$$

is a basis for $\mathfrak{T}^k(V)$, *which therefore has dimension* n^k.

Thus these φ_j functionals are the λ_j of Sjamaar, and as such, they have only one argument! So here is his next theorem:

4-1 Theorem. *Let $v_1, \ldots, v_n$ be a basis for V, and let $\varphi_1, \ldots, \varphi_n$ be the dual basis, $\varphi_i(v_j) = \delta_{ij}$. Then the set of all k-fold tensor products*

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \qquad 1 \leq i_1, \ldots, i_k \leq n$$

is a basis for $\mathfrak{T}^k(V)$, which therefore has dimension n^k.

I am happy with this at once. Remember that there is no alternation concept for $\mathcal{T}^k(V)$. Spivak is using the black rectangles for his QED. I am now on page 77, and this Chapter 4 started on page 75, so I have not gone very far! Oops, the above is earlier, I am out of order. I am really on page 81 and we have -- but wait, more back and fill. Earlier he says

following properties of $\wedge$ are left as an exercise for the reader:

$$\begin{aligned}
(\omega_1 + \omega_2) \wedge \eta &= \omega_1 \wedge \eta + \omega_2 \wedge \eta, \\
\omega \wedge (\eta_1 + \eta_2) &= \omega \wedge \eta_1 + \omega \wedge \eta_2, \\
a\omega \wedge \eta &= \omega \wedge a\eta = a(\omega \wedge \eta), \\
\omega \wedge \eta &= (-1)^{kl} \eta \wedge \omega, \\
f^*(\omega \wedge \eta) &= f^*(\omega) \wedge f^*(\eta).
\end{aligned}$$

Keep in mind that this is for Spivak's very general wedge product. If you were to take functions of only a single variable, you would have k = $\boldsymbol{\ell}$ = 1 and then the second last result says $\omega \wedge \eta = - \eta \wedge \omega$ which is a form more familiar to me! But the more general result is like $\beta\alpha = (-1)^{k\ell}\alpha\beta$ of Sja page 19 Ch 2. For Sja this was shown where α and β are "forms". Sja defined "forms" by assuming that in dx_I if you swap any pair of dx_i and dx_i you get a - sign, see top of page 18. This would certainly be true if he had inserted ∧ symbols and used the Spivak stuff. So we seem to have this association implied

$dx_K dx_L \quad \leftrightarrow \quad dx_K \wedge dx_L \quad dx_K(\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_k) \quad dx_L(\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_\ell)$

so Sja has this implied $dx_K \wedge dx_L$ which in Spivak world IS in fact the wedge of two functions having many vector arguments. So if you think of $\alpha = dx_K = \omega$ and also $\beta = dx_L = \eta$, you have a correspondence between Sja and Spi. BUT, Spivak at this point has not mentioned any dx objects!!

Now we continue on page 81 with this:

4-5 Theorem. *The set of all*

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

is a basis for $\Lambda^k(V)$, *which therefore has dimension*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This replicates a theorem of Sja for the λ_i coordinate functions, so I accept 4-5. Note that this is a basis for the *alternating* space! The dimension is easy to see since for k = ℓ = 1 you have $\omega \wedge \omega = 0$. But still we have no dx_i objects. I am awaiting their appearance!

The above is a k-wedge-product in R^n and if k = n, the dimension is 1. I agree.

Next:

4-6 Theorem. *Let* $v_1, \ldots, v_n$ *be a basis for* V, *and let* $\omega \in \Lambda^n(V)$. *If* $w_i = \Sigma_{j=1}^n a_{ij} v_j$ *are n vectors in* V, *then*

$$\omega(w_1, \ldots, w_n) = \det(a_{ij}) \cdot \omega(v_1, \ldots, v_n).$$

How are ω and ω_i related here?? OK, ω is a function name, $\boldsymbol{\omega_i}$ is one of the arguments of ω. So we have a vector change of variables indicated here by the matrix a. This is a linear change of variables, not a differentials thing as in the R matrix. In the claim, ω appears on both sides. This result seems reasonable, but it is a completely new result to me! And ω is in the alternating space. Note that $\omega_i = \mathbf{r_i} \bullet \mathbf{v}$ is a vector in V (since the v_i are in V) and here $\mathbf{r_i}$ is the i^{th} row of matrix a. I will draw vectors in V in red.

The proof is very brief. He defines a certain η (which has n vector arguments which are the rows of matrix a) in terms of ω evaluated at sums

Proof. Define $\eta \in \mathfrak{I}^n(\mathbf{R}^n)$ by

$$\eta((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{nn})) = \omega(\Sigma a_{1j} v_j, \ldots, \Sigma a_{nj} v_j).$$

I can write this as (writing $\mathbf{r_i}$ as the i^{th} row of matrix a)

$$\eta(\mathbf{r_1}, \mathbf{r_2} \ldots \mathbf{r_n}) = \omega\,(\,(\mathbf{r_1} \bullet \mathbf{v}), (\mathbf{r_2} \bullet \mathbf{v}) \ldots (\mathbf{r_n} \bullet \mathbf{v})\,) \qquad (*)$$

Notice that η has n vector arguments which live in R^n, whereas ω has n vector arguments which live in V. So ω is in $\Lambda^n(V)$?

But Spivak then makes this claim which I do not follow:

Clearly $\eta \in \Lambda^n(\mathbf{R}^n)$ so $\eta = \lambda \cdot \det$ for some $\lambda \in \mathbf{R}$ and $\lambda = \eta(e_1, \ldots, e_n) = \omega(v_1, \ldots, v_n)$. ∎

It is not at all "clear" to me what he is saying. I guess he is claiming that η = K det(a) for some constant K. That is to say, he claims that η($\mathbf{r}_1$, $\mathbf{r}_2$.....$\mathbf{r}_n$) = K det($\mathbf{r}_1$, $\mathbf{r}_2$.....$\mathbf{r}_n$) where I show the det argument as the rows of matrix a. More specifically he is claiming that

η($\mathbf{r}_1$, $\mathbf{r}_2$.....$\mathbf{r}_n$) = η($\mathbf{e}_1$, $\mathbf{e}_2$.....$\mathbf{e}_n$) det($\mathbf{r}_1$, $\mathbf{r}_2$.....$\mathbf{r}_n$) (**)

This result is certainly reasonable, and it seems to have nothing to do with (*) above. I guess I might have to lean on Sja's notion of the det in terms of the axioms. Since ω in Λ^n(V), we know it changes sign on any swap, and that means η in (*) also changes sign on any argument swap. That is really "most of the way" to a determinant. We need the scalar rule so if $\mathbf{r}_1 \rightarrow \alpha\mathbf{r}_1$ then from ω we know things overall scale by α, and that fact is then induced into η. Basically the "multilinearity" properties of ω are induced into η. The final normalization axiom just sets the value for K.

Now, given the result, how does this proof our theorem? Well (*) really says this

η($\mathbf{r}_1$, $\mathbf{r}_2$.....$\mathbf{r}_n$) = ω ($\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ ..... $\boldsymbol{\omega}_n$)

So then (**) reads

ω ($\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ ..... $\boldsymbol{\omega}_n$) = η($\mathbf{e}_1$, $\mathbf{e}_2$.....$\mathbf{e}_n$) det(a)

But note that

η($\mathbf{e}_1$, $\mathbf{e}_2$.....$\mathbf{e}_n$) = ω (($\mathbf{e}_1$•v), ($\mathbf{e}_2$•v) ..... ($\mathbf{e}_n$•v)) = ω($\mathbf{v}_1$, $\mathbf{v}_2$.....$\mathbf{v}_n$)

and therefore

ω ($\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ ..... $\boldsymbol{\omega}_n$) = ω($\mathbf{v}_1$, $\mathbf{v}_2$.....$\mathbf{v}_n$) det(a)

which involves only the function ω and vectors in V. So this proof is really quite long as measured by me, and his super dense proof is deceptive (but only for me based on what I know). Note that

$\boldsymbol{\omega}_i = \Sigma_j a_{ij} \mathbf{v}_j$ all vectors are in V

I think that as long as det(a) ≠0, then if $\mathbf{v}_i$ is a basis, then $\boldsymbol{\omega}_i$ is also a basis. I would have to prove this to myself but won't do that right now.

If deta > 0, Spivak says that both the v and ω bases have the same "orientation" and he uses exactly the Sjamaar notation for orientation.

The rest of this chapter *section* comments on "volume element" and seems a bit obscure, so I skip for now. We then have a batch of problems, and then the next section starts, same chapter 4. I am still looking for mention of dx_i.

[86] He opens with a strange description of "tangent space". Here **p** is a point of interest where you form the tangent space, which Sjamaar calls **x** on manifold M. He writes $(R^n)_p$ as the tangent space, given that point **p** lies in R^n . I think he just means here that $(R^n)_p$ is a version of R^n which has its origin at p instead of at 0, and this relates to vectors "floating in space" rather than "tethered to the origin" as his picture shows. Then he says

4-2). To be precise, a vector field is a function F such that $F(p) \in \mathbf{R}^n{}_p$ for each $p \in \mathbf{R}^n$. For each p there are numbers



and I understand this completely. Here we have those Maple style floating vectors. I normally write a vector field as **F**(**x**) so there is an arrow **F** at each point x in space. Fine.

Now finally we get around to "**differential form**" :

Many similar considerations may be applied to a function ω with $\omega(p) \in \Lambda^k(\mathbf{R}^n{}_p)$; such a function is called a ***k*-form** on $\mathbf{R}^n$, or simply a **differential form.** If $\varphi_1(p), \ldots, \varphi_n(p)$ is the dual basis to $(e_1)_p, \ldots, (e_n)_p$, then

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k}(p) \cdot [\varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p)]$$

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so we still have no dx_i objects, just those Sjamaar λ_i basis functions of V*, called here φ_i. Fine. Notice that the φ_i basis of V* here is related not to a general basis $\mathbf{v}_i$ of V, but to the specific basis $\mathbf{e}_i$ of V. That may be an important point, keep it in mind. Note that $(\mathbf{e}_i)_p$ have tails at point **p**.

We are getting closer now however to mention of dx_i !! It is going to happen very fast. First we have to parse the following coded statement:

If $f: \mathbf{R}^n \to \mathbf{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbf{R}^n)$. By a minor modification we therefore obtain a 1-form df, defined by

$$df(p)(v_p) = Df(p)(v).$$

The first sentence: Df(**p**) is a function of only one vector variable so that explains $\Lambda^1(R^n)$. But is this a vector valued function? If we go back to all that ω and η stuff, ω and η were always called "functions" and page 87 says they are scalar functions. So I guess he is staying that each *component* of [Df](**p**) is such a scalar function of one vector variable.

Now for the second sentence. I usually write

$df = \Sigma_j(\partial_j f)\, dx_j = (Df) \bullet d\mathbf{x} = (\nabla f) \bullet d\mathbf{x}$ where here Df is a vector, not a matrix.

Now using Spivak's differential form general case above, I would say for a 1-form

$\omega(\mathbf{p}) = \Sigma_i\, \omega_i\, \varphi_i(\mathbf{p})$ Spivak 1-form

Now with this prep, consider the "sentence" above,

$df(p)(\mathbf{v}_p) = Df(p)(\mathbf{v})$ = "a 1-form"

First we have to decipher his notation. If we just regard p as a "tag"

$[df_p](\mathbf{v}_p) = (Df)(\mathbf{v})$

But what does $\mathbf{v}_p$ really mean here? Top page 87 just says this is a vector **v** with tail at p. So perhaps the above is just saying that the thing on the left is (Df)(**v**) translated so tails on are point p. That is probably it. Here is a clip from nearby

Proof. $df(p)(v_p) = Df(p)(v) = \Sigma_{i=1}^n v^i \cdot D_i f(p)$
$= \Sigma_{i=1}^n dx^i(p)(v_p) \cdot D_i f(p).$ ∎

which I interpret to say

$(Df)(\mathbf{v}) = \mathbf{v} \bullet D\mathbf{f} = d\mathbf{x} \bullet D\mathbf{f}$ where he just selects $\mathbf{v} = d\mathbf{x}$

Idea 1. You can regard (Df) above as a vector of linear functionals on V where **v** lives. So $(Df)_i$ in V*. Since the φ_j form a basis in V* you can write

$(Df)_i = \Sigma_j\, c_{ij}\, \varphi_j$

When you write it this way, you can claim now that $(Df)_i$ is indeed a 1-form. So I have filled in a lot of implied content here.

Now suppose we write [here d**x** is a differential in the calculus sense !!!]

$$df = \Sigma_j(\partial_j f)\, dx_j = (Df) \bullet d\mathbf{x} = (\nabla f) \bullet d\mathbf{x} = (Df)^T d\mathbf{x} \equiv (Df)(d\mathbf{x})$$

In the previous line $[df_p](\mathbf{v}_p) = (Df)(\mathbf{v})$ the vector **v** lies in R^n and $\mathbf{v}_p$ in $R^n{}_p$. Note that d**x** also lies in R^n.

So I guess I can evaluate the linear functional $(Df)_i = \Sigma_j\, c_{ij}\, \varphi_j$ at any vector **v** I want, and so then for example we can say'

$$(Df)_i(d\mathbf{x}) = \Sigma_j\, c_{ij}\, \varphi_j(d\mathbf{x}) = \text{a 1-form}$$

So this is the very first time I am showing the calculus differential vector d**x** appearing in a 1-form.

But I don't think Idea 1 is correct! Let's go back a bit to review his notation in earlier chapters:

often called the line, the plane, and space, respectively. If x denotes an element of $\mathbf{R}^n$, then x is an n-tuple of numbers, the ith one of which is denoted x^i; thus we can write

$$x = (x^1, \ldots, x^n).$$

So OK, without explanation he uses UPPER indices on coordinates of x. Maybe he is thinking contravariant? Or maybe he has some other plan for lower indices. Next,

point of $\mathbf{R}^m$, so the norm signs are essential. The linear transformation λ is denoted $Df(a)$ and called the **derivative** of f at a. The justification for the phrase "*the* linear transformation λ" is

So this seems to say that Df is just the derivative functional for a scalar function. Chain rule

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a).$$

But eventually I find this

(3) *If* $f: \mathbf{R}^n \to \mathbf{R}^m$, *then* f *is differentiable at* $a \in \mathbf{R}^n$ *if and only if each* f^i *is, and*

$$Df(a) = (Df^1(a), \ldots, Df^m(a)).$$

so now finally Df is a vector and f^i is a component of the vector function **f**. Still using superscripts for components of vectors in R^n at least. More:

2-5 Theorem. *If $D_{i,j}f$ and $D_{j,i}f$ are continuous in an open set containing a, then*

$$D_{i,j}f(a) = D_{j,i}f(a).$$

and then

2-8 Theorem. *If $f: \mathbf{R}^n \to \mathbf{R}^m$, then $Df(a)$ exists if all $D_j f^i(x)$ exist in an open set containing a and if each function $D_j f^i$ is continuous at a.*
(Such a function f is called **continuously differentiable** at a.)

But here he is not really saying what Df is! I presume it is a matrix.

Through all Chapters 1 and 2 there is no mention of a basis $\mathbf{e_i}$ so I don't yet know his definition of these symbols. The first use seems to be in Chap 4 which I reviewed above,

other orientation is denoted $-[v_1, \ldots, v_n]$. In $\mathbf{R}^n$ we define the **usual orientation** as $[e_1, \ldots, e_n]$.

but no definition of e_j. Unlike Sjamaar, this book has no symbols index. Another hint

4-9.* Deduce the following properties of the cross product in $\mathbf{R}^3$:

(a) $e_1 \times e_1 = 0$ $\quad e_2 \times e_1 = -e_3$ $\quad e_3 \times e_1 = e_2$
$e_1 \times e_2 = e_3$ $\quad e_2 \times e_2 = 0$ $\quad e_3 \times e_2 = -e_1$
$e_1 \times e_3 = -e_2$ $\quad e_2 \times e_3 = e_1$ $\quad e_3 \times e_3 = 0.$

$$\nabla = \sum_{i=1}^{n} D_i \cdot e_i,$$

this last on page 88

OK, I guess I will really assume that the e_i are the standard **unit vectors** or R^n . Nowhere in the book does he refer to e_i as "unit vectors". But my search is working very poorly, a bad OCR I think.

Now after this digression, backtrack to page . The nubbins is really all in this one paragraph:

If $f: \mathbf{R}^n \to \mathbf{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbf{R}^n)$. By a minor modification we therefore obtain a 1-form df, defined by

$$df(p)(v_p) = Df(p)(v).$$

Let us consider in particular the 1-forms $d\pi^i$. It is customary to let x^i denote the *function* π^i. (On $\mathbf{R}^3$ we often denote x^1, x^2, and x^3 by x, y, and z.) This standard notation has obvious disadvantages but it allows many classical results to be expressed by formulas of equally classical appearance. Since $dx^i(p)(v_p) = d\pi^i(p)(v_p) = D\pi^i(p)(v) = v^i$, we see that $dx^1(p), \ldots, dx^n(p)$ is just the dual basis to $(e_1)_p, \ldots, (e_n)_p$. Thus every k-form ω can be written

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

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I discussed the first three lines already above. What on earth does me mean by $d\pi^i$ which he just pulls out of his hat from nowhere? And why is $d\pi^i$ a 1-form when we don't know what $d\pi^i$ even means? Well let's just assume that he means dx^i . But then he has put the rabbit in the hat. Every sentence here is a mystery that I will have to decode with many minutes of labor. Well consider

$$(Dx^2)(\mathbf{v}) = (Dx^2)\bullet \mathbf{v} = (Dx^2)^i v^i = \delta_{2,i}\, v^i = v^2.$$

or more generally

$$(Dx^j)(\mathbf{v}) = (Dx^j)\bullet \mathbf{v} = (Dx^j)^i v^i = \delta_{j,i}\, v^i = v^j.$$

So I agree that $(Dx^i)(\mathbf{v}) = v^i$.

Certainly on is allowed to set $\mathbf{v} = d\mathbf{x}$ and write this equation out since $d\mathbf{x}$ is in R^n.

$$(Dx^j)(\mathbf{dx}) = (Dx^j)\bullet \mathbf{dx} = (Dx^j)^i dx^i = \delta_{j,i}\, v^i = dx^j.$$

Why does this say the dx^i is the dual basis of $\mathbf{e}_i$? In this claim, I think I have to regard dx^i not as a differential, but as some kind of functional. How can I interpret this as a functional?

The object (Dx^j) is a vector, so I could define this functional

$$C^j(\mathbf{v}) \equiv (\mathbf{D}x^j) \bullet \mathbf{v} = (\mathbf{D}x^j)^T \mathbf{v}$$

Then you say that the scalar functional C^j in V^* is associated with the row vector $(\mathbf{D}x^j)^T$ in V^T. Aha, this looks good

Now suppose we assign the name "dx^j" $\equiv C^j$ as the *name* of this functional. It is not a bad name because it reminds us of itself a little bit. Then the above says

$dx^j(\mathbf{v}) \equiv (\mathbf{Dx}^j) \bullet \mathbf{v} = (\mathbf{Dx}^j)^T \mathbf{v}$ = a functional acting on one vector in R^n.

To show that dx^j is a dual basis to $\mathbf{e}_i$, I would work with the vector representation $(\mathbf{Dx}^j)$ and I would have to prove that this was true:

$(\mathbf{Dx}^j) \bullet \mathbf{e}_i = \delta_{j,i}$

See my earlier notes "the meaning of λ..." , section "Plan A3 restated", for why this means dx_j is a basis that is dual to the basis $\mathbf{e}_i$ in R^n. Now is the above true??

$(\mathbf{Dx}^j) \bullet \mathbf{e}_i = \Sigma_k (\mathbf{Dx}^j)_k (\mathbf{e}_i)_k = \Sigma_k (\mathbf{Dx}^j)_k \delta_{i,k} = (\mathbf{Dx}^j)_i = \partial_i(x^j) = \delta_{i,j}$

so yes it is true.

Fact: The functional "dx^j" is that functional associated with the vector $(\mathbf{Dx}^j)$ and the action of this functional dx^j on a general vector $\mathbf{v}$ in R_n is this:

$dx^j(\mathbf{v}) = (\mathbf{Dx}^j) \bullet \mathbf{v} = v^j$

$dx^j(\mathbf{e}_i) = \delta_{i,j}$

But this is the same as Sjamaar's original λ_j *where* you select e_i as the V space basis.

$\lambda_j(\mathbf{e}_i) = \delta_{i,j}$

Probably the functional so defined is unique.

Now all this is fine. $d\mathbf{x}^j$ is the functional associated with the vector $(\mathbf{Dx}^j)$, and

$(\mathbf{Dx}^j)_i = \partial_i x^j = \delta^j{}_i = (\mathbf{e}^j)_i$

So this vector $(\mathbf{Dx}^j)$ is exactly the vector $\mathbf{e}_j$ (I don't think Spivak does metric tensors). Note also:

$dx^j(\mathbf{dx}) = (Dx^j)(\mathbf{dx}) = (Dx^j) \bullet \mathbf{dx} = (Dx^j)^i dx^i = \delta_{j,i} v^i = dx^j$.

This then makes a connection between the functional "dx^j" and the differential dx^j !!!!

Now let's try to assemble all these pieces in a more logical order.

1. A differential k-form is defined in terms of the 1-variable functional λ_i in this manner

$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\ldots,i_k}(p) \cdot [\varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p)]$$

Here ω is the differential form, φ_i are those functionals, ω_I is a coefficient function, and **p** is a "tag" indicating a point in space for your effective origin. Following Sjamaar, I would alter the appearance of the above to say

$\omega_p = \Sigma_I\, \omega_I(p)\, d\varphi_{I,p}$ $\qquad$ $d\varphi_{I,p} = \varphi_{1,p} \wedge \varphi_{1,p}$ $\qquad$ for example

This notation is bad for me in terms of connecting with Sjamaar. I would say instead

$\alpha_x = \Sigma_I\, f_I(x)\, \lambda_{I,x}$ $\qquad$ x = that tag, formerly point p.

How if you add the arguments, you get this

$\alpha_x(v_1,v_2,\ldots v_k) = \Sigma_I\, f_I(x)\, \lambda_{I,x}(v_1,v_2,\ldots v_k)$

$= \Sigma_I\, f_I(x)\, \lambda_{i1}(v_1) \wedge \lambda_{i2}(v_2)\ \ldots\ldots \wedge \lambda_{ik}(v_k)$

This I think is how Spivak would translate his stuff to Sjamaar notation. Let's just write down an example

$\alpha_x(v_1,v_2) = f_{12}(x)\ \lambda_1(v_1) \wedge \lambda_2(v_3)$

Now let's compare how Sjamaar and how Spivak would evaluate this wedge product:

Spivak: $\lambda_1(v_1) \wedge \lambda_2(v_2) = 2\ \mathrm{Alt}[[\lambda_1 \otimes \lambda_2](v_1,v_2)] = 2\ \mathrm{Alt}\,[\,\lambda_1(v_1)\lambda_2(v_2)]$

$= 2\ (1/2!)\,[\,\lambda_1(v_1)\lambda_2(v_2) - \lambda_1(v_2)\lambda_2(v_1)\,] = \det\begin{pmatrix} \lambda_1(v_1) & \lambda_1(v_2) \\ \lambda_2(v_1) & \lambda_2(v_2) \end{pmatrix} = \det(\lambda_i(v_j))$

Sjamaar: $"\lambda_1\lambda_2"(v_1,v_2) = \det(\lambda_i(v_j))$