

Clifford Algebras as Filtered Algebras

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1 The Tensor Algebra

Definition. *An algebra is a vector space V over a field F with a multiplication. The multiplication must be distributive and, for every $f \in F$ and $x, y \in V$ must satisfy $f(xy) = (fx)y = x(fy)$.*

Our favorite example of an algebra is $M_n(F)$ — n by n matrices over F . It has the vector space structure over $\mathbb{F}$ of $\mathbb{F}^{n^2}$, and the usual matrix multiplication makes it an algebra over F .

There are very general ways of defining the tensor product of two vector spaces V and W , but here I will stick to a very concrete basis-dependent one, for finite dimensional vector spaces V and W with bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$, respectively.

Definition. *The tensor product of V and W is the vector space $V \otimes W$ is spanned by elements of the form $v \otimes w$, where the following rules are satisfied, with a an element of the field $\mathbb{F}$:*

1. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
2. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
3. $a(v \otimes w) = (av) \otimes w = v \otimes (aw)$

Aside: For those who know about dual spaces, $W \otimes V^*$ is the space of all linear maps from V to W . So our most familiar example of an algebra — $n \times n$ matrices — is in fact the tensor product $V \otimes V^*$. One of the canonical ways of defining the tensor product $V \otimes W$ is in fact as the space of linear maps from V to W^* .

We can certainly take the tensor product of a vector space V with itself. We can also do this as many times as we feel like.

Definition. *For $k \geq 1$, define*
 $T^k(V) = V \otimes V \dots \otimes V$ (k factors)
and $T^0(V) = \mathbb{F}$

Now, we define the tensor algebra as

Definition. $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$

2 The Exterior Algebra

The tensor algebra is a ring - it has a multiplication which is distributive over the vector space addition. In this ring is an ideal $A(V)$ generated by all elements of the form $v \otimes v$, $v \in V$.

Definition. The quotient of $T(V)$ by the ideal $A(V)$ is called the exterior algebra of V , and denoted by $\bigwedge V$.

Now we define the concept of a graded algebra.

Definition. An algebra A is graded if it is the direct sum of subspaces $A = A_0 \oplus A_1 \oplus \dots$ such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. The elements of A_k are said to be homogenous of degree k .

Both the tensor algebra and exterior algebras defined thus far are graded algebras, the graded k^{th} piece in both cases being given by k -products of vectors.

3 The Clifford Algebra

The definition of a Clifford algebra is parallel to that of the exterior algebra. This time, we construct the ideal $D(V)$ generated by all elements of the form $v \otimes v - |v|^2$.

Definition. The Clifford algebra of V $C(V)$ is the quotient $T(V)/D(V)$.

The Clifford algebra is no longer graded. This is because now when one multiplies two elements together, the resulting term may drop in degree without becoming zero, by the relation $v \otimes v = |v|^2$. For example, take v itself, of degree 1. $v \otimes v$ reduces to $|v|^2$, of degree zero, rather than a homogenous term of degree 2. Still, the product of a degree i term and degree j term can still give a result of degree at most $i + j$. So the algebra is no longer graded, but it does have a weaker property, that of being a filtered algebra.

Definition. A filtration of an algebra A is a sequence of subspaces $A_0 \subseteq A_1 \subseteq \dots$ such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$.

This property is weaker than being graded in that we no longer have a direct sum decomposition, but rather a decomposition into a sequence of inclusive subspaces.

However, we can take the Clifford algebra and obtain from it a graded algebra again. We do so by defining the subspaces $gr(C(V)_p) = C(V)_p / C(V)_{p-1}$, and letting $gr(C(V)) = C(V)_0 \oplus gr(C(V)_1) \oplus gr(C(V)_2) \dots$

We end with the remarkable theorem,

Theorem 1. $gr(C(V)) \cong \bigwedge V$.