New Section for Tensor Doc? PhL 9.4.15

As I read the opening section of the book by Spivak, I was confused by his definition of a tensor of a certain rank. Their discussion seemed to have no connection to tensor doc which is supposed to be about tensor algebra among other things.

Tensor doc talks about $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ as its general non-linear transformation. And it talks about a contravariant vector transforming as $V'_i = R_{ij}V_j$ under the R matrix which is the linearization of F. This becomes a vector field $V'_i(\mathbf{x}') = R_{ij}(\mathbf{x})V_j(\mathbf{x})$. A rank 2 tensor field goes as $M'_{ij}(\mathbf{x}') = R_{ii'}(\mathbf{x})\, R_{jj'}(\mathbf{x})\, M_{i'j'}(\mathbf{x})$ and so on. A viable rank-2 tensor can be made from the outer product of two vectors like this

$$M_{ij}(x) = A_i(x)B_j(x)$$

But objects with subscripts like M_{ij} don't appear in Spivak! His discussion concerns functions with multiple vector arguments, and here is a typical Spivak statement

$$S \otimes T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = S(v_1, \ldots, v_k) \cdot T(v_{k+1}, \ldots, v_{k+l}).$$

where here S, T, and S⊗T are three functions mapping V^k, V^ℓ and $V^{k+\ell}$ to R, the reals. There is no transformation at this point. But then soon we get this statement:

> One important construction, familiar for the case of dual spaces, can also be made for tensors. If $f: V \to W$ is a linear transformation, a linear transformation $f^*: \mathfrak{T}^k(W) \to \mathfrak{T}^k(V)$ is defined by
>
> $$f^*T(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k))$$
>
> for $T \in \mathfrak{T}^k(W)$ and $v_1, \ldots, v_k \in V$. It is easy to verify that $f^*(S \otimes T) = f^*S \otimes f^*T$.

So here he is talking about a transformation on one of his "k-tensors". And it is in fact a linear transformation. What does tensor doc have to say about linear transformations? Not much, just obvious things, there is a tiny section 2.8 on the topic.

Plan A. Now if we restrict tensor doc to F being a linear transformation, then $\mathbf{x}' = F\mathbf{x}$ for any vector $\mathbf{x}$. Suppose we have a set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$..... $\mathbf{v}_k$. Then we have $\mathbf{v}'_1 = F\mathbf{v}_1$ and so on. Then for a vector field we could say that $V'_i(\mathbf{v}'_1) = F_{ij}V_i(\mathbf{v}_1)$ under transformation $\mathbf{v}'_1 = F\mathbf{v}_1$. But this does not seem relevant. Spivak is talking about functions which are functions of more than one vector, whereas tensor doc's tensors are always functions of just one vector, which is space.

Plan B. Suppose we start in the Spivak world with the above transformation

$$f^*T(\mathbf{v}_1,\mathbf{v}_2 \ldots \mathbf{v}_k) = T(\mathbf{f}(\mathbf{v}_1), \mathbf{f}(\mathbf{v}_2), \ldots .\mathbf{f}(\mathbf{v}_k))$$

where we have a *linear* mapping f: $R^k \rightarrow R^k$ shall we say. For fun, rewrite with v→ x

$$f^*T(\mathbf{x}_1,\mathbf{x}_2...\mathbf{x}_k) = T(\mathbf{f}(\mathbf{x}_1), \mathbf{f}(\mathbf{x}_2), ...\mathbf{f}(\mathbf{x}_k))$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are different general points in R^n. Suppose next that $\mathbf{f}(\mathbf{x}) = R\mathbf{x}$ where R is a matrix defining the linear transformation f. then

$$f^*T(\mathbf{x}_1,\mathbf{x}_2...\mathbf{x}_k) = T(R\mathbf{x}_1, R\mathbf{x}_2, ...R\mathbf{x}_k) \ .$$

This somehow reminds me of Tinkham. If we write $\mathbf{x}_i' = R\mathbf{x}_i$ (linear!), this reads

$$f^*T(\mathbf{x}_1,\mathbf{x}_2...\mathbf{x}_k) = T(\mathbf{x}'_1, \mathbf{x}'_2, ...\mathbf{x}'_k)$$

Now suppose this can be inverted to say

$$[(f^{-1})^*T] (\mathbf{x}'_1, \mathbf{x}'_2, ...\mathbf{x}'_k) = T(\mathbf{x}_1,\mathbf{x}_2...\mathbf{x}_k)$$

and then suppose we define a new function $T' \equiv [(f^{-1})^*T]$. Then we would have

$$T'(\mathbf{x}'_1, \mathbf{x}'_2, ...\mathbf{x}'_k) = T(\mathbf{x}_1,\mathbf{x}_2...\mathbf{x}_k)$$

This strangely looks like a "scalar field of k vector variables". I am trying to get it to look like a tensor field. If I put in unit vectors, it says

$$T'(\mathbf{e}'_a, \mathbf{e}'_b, ...\mathbf{e}'_q) = T(\mathbf{e}_a,\mathbf{e}_b...\mathbf{e}_q)$$

which one could perhaps write as

$$T'_{ab..q} = T_{ab..q}$$

but again, this does not seem to useful. Trying to find a path. Trying to find some connection between the tensors of tensor doc, and the tensors of Spivak. Are these unrelated meanings of "tensor"?

What does **wiki** say about "tensor" ? It has some disambig, but I don't like it much. Let's try another source.

Mathworld starts off with the tensor doc definition of a tensor, but at the end ends with a short description of things like "tensor space" and "vector bundle" and "tangent bundle" and "dual space V*". and "pullback" and "Jacobian". No connection is made between the two discussions.

I think this question of the connection between tensors of tensor doc and tensors of Spivak (or of Benn Tucker) is at the bottom of my general confusion about a lot of things involving wedge products and differential forms and the space $\Lambda^k(V)$ and so on. I am the boy who knew too much.

One construct which *perhaps* plays a role is the fiber bundle. Since this is on my "list" of holes, maybe I will digress right now and try to learn something about fiber bundles. I think this is for sure something one needs to understand for general relativity, including parallel transport. So push the stack one level.

Note added 9.9.15 I went off for about 5 days and learned a little about fiber bundles, and now I am done with that and I return here. Offhand, I don't see how this resolves my confusion stated above about the two kinds of tensors: tensor doc tensors M_{ij}, and Spivak tensors which are functions $M(\mathbf{v_1},\mathbf{v_2})$. Another unresolved issue is the connection between wedge products of the Spivak type involving functions, and wedge products involving little vectors in R^3. I think this is Clifford related, but I have to go review that topic because I have forgotten it all!