More on the meaning of λ_i in the dual space. PhL 8.24.15

Every time I think I understand it, it just slithers away like water in my hand. It slithers away when I try to write down a coherent explanation. Here I think Plan A3 is a coherent explanation, after failing with some earlier attempts A0, A1 and A2. We shall see if it stays put.

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Overview

In the following approach (called "A") I try to find a specific set of vectors $\mathbf{v}^{(n)}$ which have coefficients $\mathbf{c}^{(n)}$ and then I define $C^{(n)}(\mathbf{v}) \equiv \mathbf{c}^{(n)\mathrm{T}}\,\mathbf{v}$ and try to show that $C^{(n)}(\mathbf{v}_j) = \delta_{n,j}$ so $\lambda_n = C^{(n)}$.

In **Plan A0** I use $\mathbf{v}^{(a)}$ where (a) is a continuum index and things done don't work out because of the fact that "a" is continuous, although I am able to come with at least some functional $C^{(a)}$.

In **Plan A1** I instead select $\mathbf{v}^{(a)} = \mathbf{v}^{(n)} = \mathbf{e}_n$ so now the selected vectors have a discrete integer index. At the same time, I choose the specific basis $\mathbf{v}_i = \mathbf{e}_i$. This approach does lead to $C^{(i)}(\mathbf{e}_j) = c^{(i)}{}_j = \delta_{i,j}$ and this explains Sjamaar as $\lambda_i = C^{(i)}$. The problem is that Sjamaar comes up with his λ_i without having to say $\mathbf{v}_i = \mathbf{e}_i$, so he is more general.

In **Plan A2** I try setting $\mathbf{v}^{(n)} = \mathbf{v}_n$ instead of $\mathbf{e}_n$, in order to *be* more general. But then $C^{(n)}(\mathbf{v}_j) = (\mathbf{v}_j)_n$ and this things don't work, since I am looking for $C^{(n)}(\mathbf{v}_j) = \delta_{j,n}$. This forces me to take $\mathbf{v}_n = \mathbf{e}_n$ which I don't want to do.

In **Plan A3** I make use of the reciprocal base vectors $\mathbf{q}_i$ where $\mathbf{q}_i \bullet \mathbf{v}_j = \delta_{i,j}$. Specifically, I suggest that maybe $C^{(i)}(\mathbf{v}) \equiv \mathbf{q}_i{}^{\mathrm{T}}\mathbf{v}$ is the correct $C^{(i)}$. This *is* the correct solution and then $C^{(n)}(\mathbf{v}_j) = \delta_{j,n}$ so I then identify $\lambda_n(\mathbf{v}) = C^{(n)}(\mathbf{v})$. In order to get this solution from the methods of Plan A1 and Plan A2 for general basis $\mathbf{v}_i$, I have to obtain the $\mathbf{v}^{(n)}$ as the solution of an obscure eigenvector problem. This solution then allows the basis $\mathbf{v}_i$ to be arbitrary, as Sjamaar says. I restate this plan in **Plan A3 Restated.**

In **Supporting Work** I show that the q_i are a basis if the v_i are a basis, and that the λ_i are a complete basis for the functional space V*.

Plan A0.

This is a generic idea where you label selected (non-basis) vectors in V this way: $\mathbf{v}^{(a)}$

1. Consider

$$\mathbf{v}^{(a)} = \Sigma_{j=1}^{n} c^{(a)}{}_j \, [\mathbf{v}_j]$$

Here the vectors $\{[\mathbf{v}_j]\}$ form a basis for vector space V, possibly non-orthogonal. The notation $[\mathbf{v}_j]$ has j being a label, not a vector index. The quantity v_i as the i^{th} component of vector $\mathbf{v}$ is not the same as the quantity $[\mathbf{v}_i]$ being one of the basis vectors of V. Don't get confused about this!

The object $\mathbf{v}^{(a)}$ is some *specific* vector in V and we have written it as a linear combination of the $\{[\mathbf{v}_j]\}$ with real coefficients $c^{(a)}{}_j$. Some other vector $\mathbf{v}^{(b)}$ in V would have some other coefficients $c^{(b)}{}_j$.

2. We can think of a column vector $\mathbf{c}^{(a)}$ and a corresponding row vector $\mathbf{c}^{(a)T}$.

3. Consider this scalar product

$$\mathbf{c}^{(a)T} \bullet \mathbf{v}$$

where $\mathbf{v}$ is some *arbitrary* vector in $\mathbf{v}$. This dot product is a real number in R^1.

4. We might then define the following *linear functional* $C^{(1)}$ which acts on an arbitrary vector $\mathbf{v}$ in V:

$$C^{(a)}(\mathbf{v}) = \mathbf{c}^{(a)T} \mathbf{v} = \Sigma_i \, c^{(a)}{}_i \, v_i \qquad v_i = \text{component of } \mathbf{v}$$

We do not bold the functional $C^{(a)}$ because that is the name of a single scalar function. We can write

$$C^{(a)} : V \rightarrow R$$

5. One might say that the scalar linear functional $C^{(a)}$ "is represented by" a vector object $\mathbf{c}^{(a)T}$. If we want to know about the action of $C^{(a)}$, we need to have $\mathbf{c}^{(a)T}$ and then the answer is $C^{(a)}(\mathbf{v}) = \mathbf{c}^{(a)T} \mathbf{v}$.

6. The "dual space" V* of V is defined as "the space of all linear functionals on V". Therefore, we have found one element of V* : we know that the functional $C^{(a)} \in V^*$. The row vector $\mathbf{c}^{(a)T}$ is NOT an element of V*. Had we started with specific vector $\mathbf{v}^{(b)}$ we would have generated another functional named $C^{(b)}$ and we would then have $C^{(b)} \in V^*$. It certainly seems then that for each vector $\mathbf{v}^{(a)}$ in V, we obtain a scalar linear functional $C^{(a)}$. This makes you think that maybe the vector space V and the dual space V* have the same number of elements!

7. There seems to be a 1-to-1 relationship between the set of vectors $\mathbf{v}^{(a)}$ in V and the linear functionals $C^{(a)}$ in V* (both sets are uncountable). There also seems to be a 1-to-1 relationship between scalar linear functionals $C^{(a)}$ and row vectors $\mathbf{c}^{(a)T}$. So we have "three sets of things" to think about. I suppose you could say that $\mathbf{c}^{(a)T}$ is an element of a space called V^T. So then we have V, V* and V^T.

8. The following facts are true:

$$C^{(a)}(\mathbf{v}^{(a)}) = \Sigma_i\, c^{(a)}{}_i\, (v^{(a)})_i$$
$$C^{(a)}(\mathbf{v}^{(b)}) = \Sigma_i\, c^{(a)}{}_i\, (v^{(b)})_i$$
$$C^{(b)}(\mathbf{v}^{(a)}) = \Sigma_i\, c^{(b)}{}_i\, (v^{(a)})_i$$
$$C^{(b)}(\mathbf{v}^{(b)}) = \Sigma_i\, c^{(b)}{}_i\, (v^{(b)})_i$$

9. Now how do we make any connection to the object λ_i on page 84 Sjamaar? It is referred to as a "covector of V* " so that means it must be a linear functional like $C^{(a)}$. But it has an integer index i, which differs from the continuous index a. We are given this fact:

$\lambda_i(\mathbf{v}) = c_i$ where v is an arbitrary element of V.

But all this really says is that λ_i is a linear functional on V. Let's compare this with our result above,

$$C^{(a)}(\mathbf{v}) = \mathbf{c}^{(a)\mathrm{T}}\,\mathbf{v} = \Sigma_i\, c^{(a)}{}_i\, v_i$$

Then look pretty far apart, don't they!

Plan A1.

Suppose we go back and instead of selecting all our $\mathbf{v}^{(a)}$ vectors as arbitrary vectors in V, suppose we choose them as the usual unit vectors in V. Then for example,

$$\mathbf{v}^{(1)} = (\mathbf{e}_1) = (1,0,0...0)^{\mathrm{T}}$$

Then we could at least have numeric indices on things instead of a continuous index. Tracking from the above,

$$\mathbf{v}^{(1)} = \Sigma_{j=1}^{n}\, c^{(1)}{}_j\, [\mathbf{v}_j]$$

Suppose now, at the same time, we select the basis vectors of V to also be the $\mathbf{e}_j$ so $[\mathbf{v}_j] = (\mathbf{e}_j)$. Then the above expansion becomes

$$(\mathbf{e}_1) = \Sigma_{j=1}^{n}\, c^{(1)}{}_j\, (\mathbf{e}_j)$$

More generally, we would have

$$\mathbf{v}^{(n)} = (\mathbf{e}_n) = \Sigma_{j=1}^{n}\, c^{(n)}{}_j\, (\mathbf{e}_j)$$

It is obvious looking at the above, that in this case we have

$$c^{(n)}{}_j = \delta_{n,j}$$

This is a statement about the vector $\mathbf{c}^{(n)}$ in the space V^{T}. Now consider

$$C^{(a)}(\mathbf{v}) = \mathbf{c}^{(a)T}\mathbf{v} = \Sigma_i\, c^{(a)}{}_i\, v_i$$

so

$$C^{(n)}(\mathbf{e}_j) = \mathbf{c}^{(n)T}\mathbf{v} = \Sigma_i\, c^{(n)}{}_i\, (\mathbf{e}_j)_i = c^{(n)}{}_j$$

Now we combine the last two results above to get (changing integer n to integer i)

$$C^{(i)}(\mathbf{e}_j) = c^{(i)}{}_j = \delta_{i,j}$$

This looks very much like Sjamaar's page 84 statement that

$$\lambda_i(\mathbf{v}_j) = \delta_{i,j}$$

But in order to get the above result, we needed to have $[\mathbf{v}_j]$ be the specific basis $\mathbf{e}_j$. Maybe we can redo things to avoid that necessity.

Plan A2

This time, assume a *generic* set of basis vectors $\{[\mathbf{v}_i]\}$ for V. And this time, assume that

$\mathbf{v}^{(i)} = [\mathbf{v}_i]$ = one of the basis vectors

as our selected vectors with which to define functionals. Then write

$$\mathbf{v}^{(n)} = \Sigma_{j=1}^{n}\, c^{(n)}{}_j\, [\mathbf{v}_j]$$
$$\mathbf{v}^{(n)} = \mathbf{v}_n$$

Then in this case it seems quite clear that we must have

$$c^{(n)}{}_j = \delta_{n,j}$$

just as above. What do our functionals look like now?

$$C^{(n)}(\mathbf{v}) = \mathbf{c}^{(n)T}\mathbf{v} = \Sigma_i\, c^{(n)}{}_i\, v_i$$

which we evaluate at $[\mathbf{v}_j]$

$$C^{(n)}(\mathbf{v}_j) = \Sigma_i\, c^{(n)}{}_i\, (\mathbf{v}_j)_i = \Sigma_i\, \delta_{n,i}\, (\mathbf{v}_j)_i = (\mathbf{v}_j)_n$$

Now we nothing at all about $(\mathbf{v}_j)_i$ for a general basis vector $[\mathbf{v}_j]$ so we cannot continue our development here to where we think we want to go. However, suppose instead we evaluate at the $\mathbf{e}_j$ vectors:

$$C^{(n)}(\mathbf{e}_j) = \Sigma_i\, c^{(n)}{}_i\, (\mathbf{e}_j)_i = c^{(n)}{}_j = \delta_{n,j}$$

Conclusion: In this Plan A2 we have done the following:

• defined the functionals $C^{(n)}$ based on selected vectors $\mathbf{v}^{(n)} = [\mathbf{v}_n]$, where $\{[\mathbf{v}_i]\}$ is an arbitrary basis of the space V. These vectors need not be $\mathbf{e}_n$. Specifically, we expand these $\mathbf{v}^{(n)}$ to find their coefficient vectors called $\mathbf{c}^{(n)}$, then we use $\mathbf{c}^{(n)}$ to define $C^{(n)}$ as $C^{(n)}(\mathbf{v}) = \mathbf{c}^{(n)}\mathbf{v}$.

• We then find that

$C^{(n)}(\mathbf{v}_j) = (\mathbf{v}_j)_n$ which is not too helpful

$C^{(n)}(\mathbf{e}_j) = (\mathbf{e}_j)_n = \delta_{n,j}$

So in the first step we can choose either to have the $\{[\mathbf{v}_i]\}$ be arbitrary, or we could select them to be the basis $\{[\mathbf{e}_i]\}$. Regardless of what we do on that matter, the only way to get $\delta_{n,j}$ is as shown above. Rewrite this way

$C^{(i)}(\mathbf{e}_j) = \delta_{i,j}$

Now the link to Sjamaar must be this:

$\lambda_i \equiv C^{(i)}$ = a linear functional in V*

Then we have

$\lambda_i(\mathbf{e}_j) = \delta_{i,j}$

and this seems *more specific* that the claim made on page 84 A.

What Sjamaar says

So how does Sjamaar obtain what seems to be a more general result? Let us examine each of his densepack steps:

(1) $\mathbf{v} = \Sigma_j c_j(\mathbf{v})[\mathbf{v}_j]$. I have to show that the c_j are associated specifically with $\mathbf{v}$.

(2) define a covector $\lambda_i(\mathbf{v}) \equiv c_i(\mathbf{v})$. But a covector must be a linear functional, so he must be saying here that $\lambda_i = c_i$ = a linear functional. Alternatively, he is saying that $\boldsymbol{\lambda}(\mathbf{v}) = \mathbf{c}^{\mathbf{T}}(\mathbf{v})$ = a vector in $V^{\mathbf{T}}$. We have to put a label on c_i because it is specific to $\mathbf{v}$. I feel Sja notation is simply no good. Does he do something different in his new update? Yes, he has completely changed the discussion there!!! Here is his new discussion:

Now suppose that V is a vector space of finite dimension n and choose a basis $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ of V. Then every vector $\mathbf{b} \in V$ can be written in a unique way as a linear combination $\sum_j c_j \mathbf{b}_j$. Define a covector $\beta_i \in V^*$ by $\beta_i(\mathbf{b}) = c_i$. In other words, β_i is determined by the rule $\beta_i(\mathbf{b}_j) = \delta_{i,j}$, where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

is the *Kronecker delta*. We call β_i the *i-th coordinate function*.

But this is the exact same discussion where he has taken $\mathbf{v_i} \rightarrow \mathbf{b_i}$ as the basis (letter b) and $\lambda_i = \beta_i$. He now is writing $\mathbf{b} = \Sigma_j c_j(\mathbf{b})[\mathbf{b_j}]$ so again we need a **b** label on the $c_j(\mathbf{b})$.

Plan A3

Another interpretation of Sjamaar. I know that, given a basis $\{[\mathbf{v_i}]\}$ of V, I can find a unique set of dual basis vectors $\mathbf{q_i}$ which satisfy this rule

$\mathbf{q_i} \bullet \mathbf{v_j} = \delta_{i,j}$ for all i,j

The vectors $\{[\mathbf{q_i}]\}$ are really an alternate basis for V , not for V*.

Example: Think of the 3-piped in R^3 spanned by the $\mathbf{e_n}$ which form a basis. The three area vectors $\mathbf{E_n}$ form an alternate basis in R^3, the reciprocal basis.

Now how can I define a functional in V* from this?

$C^{(i)}(\mathbf{v}) \equiv \mathbf{q_i}^T \mathbf{v}$

Very good. Once the $\mathbf{v_i}$ are known, the $\mathbf{q_j}$ are completely determined. We then have at once

$C^{(i)}(\mathbf{v_i}) \equiv \mathbf{q_i}^T \mathbf{v_j} = \mathbf{q_i} \bullet \mathbf{v_j} = \delta_{i,j}$

So then we can define

$\lambda_i(\mathbf{v}) \equiv C^{(i)}(\mathbf{v}) \equiv \mathbf{q_i}^T \mathbf{v}$

and then we have our functional λ_i and it does the general result

$\lambda_i(\mathbf{v_j}) = \mathbf{q_i}^T \mathbf{v_j} = \delta_{i,j}$

just as he says!

Note that the $C^{(i)}$ defined in this Plan A3 are not the same as those defined by me in Plan A2. For Plan A2 I had $C^{(n)}(\mathbf{v}) = \mathbf{c}^{(n)T} \mathbf{v}$ where $\mathbf{c}^{(n)} = \mathbf{e_n}$. In Plan A3 I have $\mathbf{c}^{(n)} = \mathbf{q_n}$.

I will now show that Plan A3 is NOT the same as Plan A2 where you just select $\mathbf{v}^{(i)} = [\mathbf{q}_i]$:

Redo those Plan A steps again: We now try this *third* choice for the reference vectors

$$\mathbf{v}^{(i)} = [\mathbf{q}_i]$$

$$\mathbf{v}^{(n)} = \Sigma_{j=1}^{n} c^{(n)}{}_j \, [\mathbf{v}_j]$$
$$\mathbf{v}^{(n)} = \mathbf{q}_n$$

Then comparing we have

$$\mathbf{q}_n = \Sigma_{j=1}^{n} c^{(n)}{}_j \, [\mathbf{v}_j] \qquad \text{// this does not say } \mathbf{q}_n = \mathbf{c}_n$$

Then for one thing we can do this:

$$\mathbf{q}_n \bullet \mathbf{v}_i = \mathbf{q}_n \bullet \left(\Sigma_{j=1}^{n} c^{(n)}{}_j \, [\mathbf{v}_j] \right) = \Sigma_{j=1}^{n} c^{(n)}{}_j \, \mathbf{q}_n \bullet \mathbf{v}_j = c^{(n)}{}_i = \delta_{n,i}$$

What then is our functional?

$$C^{(n)}(\mathbf{v}) \equiv \mathbf{c}^{(n)\mathrm{T}} \mathbf{v} = \Sigma_i (c^{(n)})_i \, v_i = v_n$$

But this is the wrong functional to choose. If you choose this, you get

$$C^{(n)}(\mathbf{v}_i) = (\mathbf{v}_i)_n \neq \delta_{i,n}$$

So again my method does not produce the λ_i of Sjamaar.

So what vectors might I select for the $\mathbf{v}^{(n)}$ to make Plan A2 agree with Plan A3? Consider:

$$\mathbf{v}^{(n)} = \Sigma_{j=1}^{N} c^{(n)}{}_j \, [\mathbf{v}_j]$$

$$\mathbf{q}_i \bullet \mathbf{v}^{(n)} = c^{(n)}{}_i$$

$$\mathbf{v}^{(n)} = \Sigma_{j=1}^{N} (\mathbf{q}_j \bullet \mathbf{v}^{(n)}) \, [\mathbf{v}_j]$$

This looks like a messy system of N equations in N unknowns that I would have to solve for $\mathbf{v}^{(n)}$. I just showed on scratch that you end up with an equation of the form $\mathbf{v}^{(n)} = A\mathbf{v}^{(n)}$ where A is a matrix which is a function of the $\mathbf{q}_i$ and $\mathbf{v}_i$ vectors. This then is an eigenvalue problem. Leave it there!

Sjamaar's discussion is still no good in my book! Here is my replacement:

Plan A3 Restated

Let $\{[\mathbf{v}_i]\}$ be some arbitrary basis for V. Find the vector set $\{[\mathbf{q}_i]\}$ called "the dual basis" for which the following is true:

$$\mathbf{q}_i^T \mathbf{v}_j = \delta_{i,j}$$

I know from tensor doc that the solution set $\{[\mathbf{q}_i]\}$ exists and is unique. In general the $\{[\mathbf{q}_i]\}$ form a basis for V. Define a linear functional λ_i as follows:

$$\lambda_i(\mathbf{v}) \equiv \mathbf{q}_i^T \mathbf{v}$$

Then it follows that

$$\lambda_i(\mathbf{v}_j) = \mathbf{q}_i^T \mathbf{v}_j = \delta_{i,j}$$

just as Sjamaar states in page 84 A.

If we *happen* to choose $\mathbf{v}_i = \mathbf{e}_i$, then we will find that $\mathbf{q}_i = \mathbf{e}_i$ as well.

Comment: My presentation works specifically for R^n where I know how to compute the q_i if I am given the v_i starting basis. But suppose things are more abstract for space $V \neq R^n$. Then maybe you can still define the λ_i completely by stating the action of λ_i on all the basis vectors $\mathbf{v}_i$. So the definition you make is that $\lambda_i(\mathbf{v}_j) = \delta_{i,j}$. This I think is the way Sjamaar states things. However, it might not be obvious that you really can impose that definition and obtain a set of λ_i that work. For R^n I know it works, and R^n is generally where Sjamaar wants to work.

Supporting Work

(1) show that the $\mathbf{q}_i$ are a basis in some space. That space must be V, so $\mathbf{q}_i^T$ basis for V^T.

(2) in tensor doc I seem to have omitted a proof that if the $\mathbf{b}_i$ are complete, so are the $\mathbf{B}_i$. Do that proof and think about maybe adding it to tensor doc.

(3) Show λ_i is complete basis of the dual functional space.

(1) Assume that the $\mathbf{b}_i$ are a basis. Then we can write

$$\mathbf{B}_m = w'_{mn} \mathbf{b}_n$$

Want to show that one can write any vector $\mathbf{v}$ in V this way

$$\mathbf{v} = \Sigma_n \alpha_n \mathbf{B}_n$$

so the α_n must exist for any $\mathbf{v}$. Try this constructive derivation: try,

$$\alpha_n = (\mathbf{b}_n \bullet \mathbf{v}).$$

Then we would have

$$\mathbf{v} = \Sigma_n \ (\mathbf{b}_n \bullet \mathbf{v})\, \mathbf{B}_n$$

For this equation to be valid, the components of both sides on the complete $\mathbf{b}_m$ basis must be equal. If they are equal, then we have proven what we want. So

$$\mathbf{b}_m \bullet \mathbf{v} \ =?= [\Sigma_n \ (\mathbf{b}_n \bullet \mathbf{v})\, \mathbf{B}_n] \bullet \mathbf{b}_m \ = \ \Sigma_n \ (\mathbf{b}_n \bullet \mathbf{v})\, \delta_{nm} = \ \mathbf{b}_m \bullet \mathbf{v}$$

But this is obviously true, so the dual basis is a complete basis for V.

Conclusion:
(1) $\mathbf{q}_i$ are a complete basis for V just as $\mathbf{v}_i$ are a complete basis for V.
(2) $\mathbf{q}_i^T$ are a complete basis for V^T
(3) If you think of V^T as the "dual space" whose elements are row vectors, then
you can say that $\mathbf{q}_i^T$ form a basis in the dual space.

However, above I am thinking of the dual space as a space of functionals, not vectors, and I have

$$\lambda_i(\mathbf{v}) = \mathbf{q}_i^T \mathbf{v}$$

as the functional λ_i.

(3) Proof that the functionals λ_i are complete in V*

Must show that for arbitrary λ in V* you can write $\lambda = \Sigma_i c_i \lambda_i$. If you can find the c_i that work, then you are done with this proof. Now consider

$$\lambda(\mathbf{v}) = \Sigma_i c_i \lambda_i(\mathbf{v}) = \Sigma_i c_i \ (\mathbf{q}_i^T \mathbf{v})$$

$$\lambda(\mathbf{v}_j) = \Sigma_i c_i \ (\mathbf{q}_i^T \mathbf{v}_j) = c_j$$

So if you can write $\lambda = \Sigma_i c_i \lambda_i$, then the coefficients must be $c_i = \lambda(\mathbf{v}_j)$. We have no issue with whether λ exists or not, it is just a question of whether we can expand λ as shown. Thus, we can for sure compute the coefficients $c_i \equiv \lambda(\mathbf{v}_j)$. But then these coefficients do the job! Sja shows also unique solution.

So I think with this use of the dual basis $\mathbf{q}_i^T$ of V^T, I understand that the λ_i form a complete basis for the dual space of functionals V* !