The Wedge Product and Calculus PhL 8.27.15

but creation date says 8.29.15

This was an attempt to explain everything relating to wedges, but it only explains the Spivak function-based wedge world. I need to explain all of Wedge World, not just this piece.

One goal in dealing with "patches" on surfaces is to clearly define a direction for such a patch. A 2D patch has two sides. A 1D patch has two directions. A 3D patch can be left or right handed. And an nD patch spanned by $\mathbf{e_n}$ has a polarity, still called handedness, determined by the sign of det(S) where S is the matrix whose column vectors are the $\mathbf{e_n}$.

In order to incorporate "sign" into such patches, Grassmann or someone invented the idea of using antisymmetric functions and the associated wedge product. That is the first subject of this doc.

The Wedge Product

In Spivak's book, the space of k-multilinear functions of k vector variables is called $\mathcal{T}^k(V)$. Such a function is a mapping f: $V^k \rightarrow R$, hence really a "functional" which is "linear" in each of its variables. Although Sjamaar expresses the linearity property in one equation, we do it in two:

$f(\mathbf{v_1},\mathbf{v_2}, \alpha\mathbf{v_3}, \mathbf{v_4}) = \alpha\, f(\mathbf{v_1},\mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4})$ // scalar rule

$f(\mathbf{v_1},\mathbf{v_2}, \mathbf{u} + \mathbf{w}, \mathbf{v_4}) = f(\mathbf{v_1},\mathbf{v_2}, \mathbf{u}, \mathbf{v_4}) + f(\mathbf{v_1},\mathbf{v_2}, \mathbf{w}, \mathbf{v_4})$ // addition rule

Here we show linearity for the third argument, but for a multilinear function, one has this for all arguments. It is just a generalization of a regular linear functional. Often one uses $V = R^n$ but that is not necessary, V can be any vector space.

The first step in the Grassmann plan is to form a subspace within $\mathcal{T}^k(V)$ which contains only functions which are *antisymmetric* (AS) in the vectors: any swap of two vector arguments negates the function. Eventually we get to a 2D "patch" looking like dx_1dx_2 (this will not be for while), we will end up with the idea that $dx_2dx_1 = -dx_1dx_2$ which "flips" the sign of a 2D area patch. That is where the antisymmetry comes in.

That subspace of AS functions within $\mathcal{T}^k(V)$ is called $\Lambda^k(V)$ by Spivak (and others), and is called AV^k by Sjamaar. We will use the latter notation since A reminds one of "Antisymmetric" .

In algebraic operations, we always want to "multiply" elements to get new elements. For example, we can combine two vectors to get a new vector using $\mathbf{a} = \mathbf{b} \times \mathbf{c}$ and here the product **a** lives in the same space R^3 inhabited by **b** and **c**. Sometimes we do $\mathbf{a} = \mathbf{b} \otimes \mathbf{c}$ to create a new object **a** in the direct product space, which is not the same as the space of **a** or **b**. Similarly in the product $a = \mathbf{b} \bullet \mathbf{c}$ the product result a is not the same type of object as **b** and **c**.

In the Grassmann algebra, we want to combine two AS functions to create a new AS function in the direct product sense of $\mathbf{a} = \mathbf{b} \otimes \mathbf{c}$. In the regular direct product ⊗, we might have **b** in R^a and **c** in R^c and then **a** is in R^{b+c} . We would write $f(\mathbf{b},\mathbf{c}) \equiv \mathbf{b} \otimes \mathbf{c}$ with $f: R^b \otimes R^c \rightarrow R^{b+c}$.

We can define an analogous direct product ⊗ for *multilinear functions* in the following manner:

$F \equiv f \otimes g \qquad\qquad F : \mathcal{J}^m(V) \times \mathcal{J}^n(V) \rightarrow \mathcal{J}^{m+n}(V)$

f = m-multilinear function g = n-multilinear function

This is the simple rule for multiplication of two functions,

$$F(\mathbf{v}_1,\mathbf{v}_2,...\mathbf{v}_m, \mathbf{v}_{m+1}.....\mathbf{v}_{m+n}) = f(\mathbf{v}_1,\mathbf{v}_2,...\mathbf{v}_m)\, g(\mathbf{v}_{m+1}.....\mathbf{v}_{m+n}).$$

Function F has m+n vector arguments and is thus an element of $\mathcal{J}^{m+n}(V)$.

Another way to write the above line is this

$$[f \otimes g](\mathbf{v}_1,\mathbf{v}_2,...\mathbf{v}_m, \mathbf{v}_{m+1}.....\mathbf{v}_{m+n}) = f(\mathbf{v}_1,\mathbf{v}_2,...\mathbf{v}_m)\, g(\mathbf{v}_{m+1}.....\mathbf{v}_{m+n})$$

where we replace the function name F with the function name [f ⊗ g].

Our task is to come up with a different but analogous mapping G such that

$G \equiv f \wedge g \qquad\qquad G : AV^m \times AV^n \rightarrow AV^{m+n}$

f = an AS m-multilinear function g = an AS n-multilinear function

Just as we wrote $F \equiv f \otimes g$ to define the ⊗ symbol for a direct product of two multilinear functions, here we write $G \equiv f \wedge g$ to define the ∧ symbol for the "wedge product" of two AS multilinear functions. This new mapping G would take two AS functions and generate a new AS function from them. Obviously the mapping F shown above does not work because if f and g are AS, F won't be AS, as for example if one were to swap arguments $\mathbf{v}_1$ and $\mathbf{v}_{m+n}$.

In order to construct the mapping G, we first define a simple operator we will call AS whose task is to "AntiSymmetrize" a function which might not be AS. So consider:

$$[AS(f)](\mathbf{v}_1,\mathbf{v}_2....\mathbf{v}_k) \equiv (1/k!) \sum_{abc..q} \varepsilon_{abc..q}\, f(\mathbf{v}_a,\mathbf{v}_b....\mathbf{v}_q) \qquad \text{// k! terms in sum}$$

where ε is the permutation tensor and we use a simple if not quite precise subscripting system. Note that

$AS: \mathcal{J}^m(V) \rightarrow AV^m \qquad \text{and} \qquad AV^m \subset \mathcal{J}^m(V)$

since AS maps general m-multilinear functions into antisymmetric m-multilinear functions.

For example,

$$[AS(f)](\mathbf{v}_1) = (1/1!)\,\Sigma_{a=1}^{1}\varepsilon_a\, f(\mathbf{v}_a) \;=\; \varepsilon_1 f(\mathbf{v}_1) \;= f(\mathbf{v}_1)$$

$$[AS(f)](\mathbf{v}_1,\mathbf{v}_2) = (1/2!)\,\Sigma_{a=1}^{2}\varepsilon_{ab}\, f(\mathbf{v}_a,\mathbf{v}_b) \;=\; (1/2)[\,\varepsilon_{12} f(\mathbf{v}_1,\mathbf{v}_2) + \varepsilon_{21} f(\mathbf{v}_2,\mathbf{v}_1)\,]\,)$$

$$= (1/2)[\,f(\mathbf{v}_1,\mathbf{v}_2) - f(\mathbf{v}_2,\mathbf{v}_1)]$$

$$[AS(f)](\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) \;=\; (1/3!)\,\Sigma_{abc}\,\varepsilon_{abc}\, f(\mathbf{v}_a,\mathbf{v}_b,\mathbf{v}_c)$$

$$= (1/3!)\,[\,\varepsilon_{123}\, f(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) + \varepsilon_{132}\, f(\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_2) \;+\; \varepsilon_{231}\, f(\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_1) + \varepsilon_{213}\, f(\mathbf{v}_2,\mathbf{v}_1,\mathbf{v}_3)$$

$$+ \varepsilon_{312}\; f(\mathbf{v}_3,\mathbf{v}_1,\mathbf{v}_2) + \varepsilon_{321}\; f(\mathbf{v}_3,\mathbf{v}_2,\mathbf{v}_1)\,]$$

$$= (1/6)\,[\,f(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) - f(\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_2) + f(\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_1) \;- f(\mathbf{v}_2,\mathbf{v}_1,\mathbf{v}_3) + f(\mathbf{v}_3,\mathbf{v}_1,\mathbf{v}_2) \;- \;f(\mathbf{v}_3,\mathbf{v}_2,\mathbf{v}_1)\,]$$

The purpose of the (1/k!) is this: if $f(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3....)$ is *already* an AS function, then AS(f) = f and there is no overall constant. (Spivak refers to operator AS as Alt and to function f as T.)

This antisymmetrization procedure is familiar to people who construct totally antisymmetric "Fermion wavefunctions" in quantum mechanics to come up with a multi-particle wavefunction which changes sign when any two position vectors are swapped. Notice in the last two examples above that AS(f) changes sign if any pair of vector arguments is swapped. This desired property is of course the reason for using the $\varepsilon_{abc..q}$ permutation tensor in the definition of AS(f).

Once we have this AS operator, we can now define the wedge product. It also has an overall factor K whose value we shall discuss below :

$$G = f \wedge g \;\equiv\; K\, AS(\, f \otimes g) \;.$$

If we assume that f has m vector arguments and g has n, this says

$$G(\,\mathbf{v}_1,\mathbf{v}_2,...\mathbf{v}_m,\,\mathbf{v}_{m+1}.....\mathbf{v}_{m+n}) \;=\; [f \wedge g](\,\mathbf{v}_1,\mathbf{v}_2,...\mathbf{v}_m,\,\mathbf{v}_{m+1}.....\mathbf{v}_{m+n})$$

$$\equiv\; K_{mn}\,[AS(\, f \otimes g)](\,\mathbf{v}_1,\mathbf{v}_2,...\mathbf{v}_m,\,\mathbf{v}_{m+1}.....\mathbf{v}_{m+n})$$

where the overall factor K_{mn} depends on both m and n. So the idea is that one first creates the $\otimes$ direct product of the two functions as shown above, *then* the AS operator is applied to obtain function G which is AS in any pair of arguments.

Example: Let m = 2 and n = 2 so we have the following for $G = f \wedge g$:

$$G(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4) \;= [f \wedge g](\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4) = \;K_{22}\;\; AS[(f \otimes g)(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)]$$

$$= K_{22}\;\; AS[\,F(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)] \;= K_{22}\; AS\,[\;\; f(\mathbf{v}_1,\mathbf{v}_2)\; g(\mathbf{v}_3,\,\mathbf{v}_4)\;]$$

$$= K_{22}\ (1/4!)\ \Sigma_{abcd}\ \varepsilon_{abcd}\ f(\mathbf{v}_a,\mathbf{v}_b)\ g(\mathbf{v}_c,\ \mathbf{v}_d)$$

$$= (\ K_{22}/4!)\ [f(\mathbf{v}_1,\mathbf{v}_2)\ g(\mathbf{v}_3,\ \mathbf{v}_4)\ -\ f(\mathbf{v}_2,\mathbf{v}_1)\ g(\mathbf{v}_3,\ \mathbf{v}_4) - f(\mathbf{v}_1,\mathbf{v}_3)\ g(\mathbf{v}_2,\ \mathbf{v}_4)\ + ...\ ]\ .$$

There will be 4! = 24 terms in this expansion, half with + signs and half with - signs. The sign of any term is determined by $(-1)^S$ where S is the number of pairwise argument swaps it took to get to that term from the first term $f(\mathbf{v}_1,\mathbf{v}_2)\ g(\mathbf{v}_3,\ \mathbf{v}_4)$. No rocket science here. It turns out that

$$K_{nm}\ = (m+n)!\ /\ [\ m!\ n!\ ]\ .$$

Notice that it is generally not true that $f \wedge g = -\ g \wedge f$. For example, if we were to compute $g \wedge f$ as in the above example, the first term in the result would be $g(\mathbf{v}_1,\mathbf{v}_2)\ f(\mathbf{v}_3,\ \mathbf{v}_4)\ =\ f(\mathbf{v}_3,\ \mathbf{v}_4)\ g(\mathbf{v}_1,\mathbf{v}_2)$. If we do just two swaps $3\leftrightarrow1$ and then $4\rightarrow 2$ this becomes $f(\mathbf{v}_1,\ \mathbf{v}_2)g(\mathbf{v}_3,\mathbf{v}_4)$ which is the first term of the example. Thus, for this example we in fact have $f \wedge g = +\ g \wedge f$. However, the rule $f \wedge g = -\ g \wedge f$ will always apply to the special case discussed below.

It is not hard to generalize the wedge product from 2 functions to 3 or more functions. For example, one can define

$$f \wedge g \wedge h\ \equiv\ (f \wedge g) \wedge h$$

f = an AS m-multilinear function
g = an AS n-multilinear function
h = an AS k-multilinear function .

This is analogous to det(ABC) = det([AB]C) = det(AB)det(C) = det(A)det(B)det(C). Next would be,

$$f \wedge g \wedge h \wedge i\ \equiv\ (f \wedge g \wedge h) \wedge i$$

and so on. . It turns out that the following is true (as the reader could show)

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)$$

which says that wedge operator $\wedge$ is "associative". For this reason the notation like $f \wedge g \wedge h \wedge i$ is unambiguous so one could group things in any manner, such as $f \wedge (g \wedge h) \wedge i$ or $(f \wedge g) \wedge (h \wedge i)$. It is important to keep the functions in the same order, otherwise minus signs may be incurred.

In particular, one can show that the wedge product of three antisymmetric functions is given by

$$f \wedge g \wedge h\ = K_{nmk}\ AS(\ f \otimes g \otimes h) \qquad\qquad K_{nmk}\ = (m+n+k)!\ /\ [\ m!\ n!\ k!]$$

$$G\ = f \wedge g \wedge h \qquad\qquad G:\ AV^m\ x\ AV^n x\ AV^k\ \rightarrow AV^{m+n+k}$$

Here the triple wedge product is defined in terms of the triple direct product (for functions):

$H(f,g) \equiv f \otimes g \otimes h$ $\quad H : \mathcal{T}^m(V) \times \mathcal{T}^n(V) \times \mathcal{T}^k(V) \rightarrow \mathcal{T}^{m+n+k}(V)$

f = m-multilinear function g = n-multilinear function h = k-multilinear function

One can regard this for example as $f \otimes (g \otimes h)$ where $\otimes$ is associative just as $\wedge$ is associative. The general multiple wedge product is then

$f \wedge g \wedge h \ldots = K_{nmk\ldots}\, AS(f \otimes g \otimes h \ldots)$ $\quad K_{nmk\ldots} = (m+n+k+\ldots)! / [\, m!\, n!\, k!\, \ldots.]$

Warning: In the above we had equations like $G(\mathbf{v_1},\mathbf{v_2},\mathbf{v_3},\mathbf{v_4}) = [f \wedge g](\mathbf{v_1},\mathbf{v_2},\mathbf{v_3},\mathbf{v_4})$. The function G has four vector arguments as shown where $\mathbf{v_i} \in V$. One could also write $G[f,g] = f \wedge g$, but here G has a completely different set of arguments. These arguments are not vectors in V, they are functions in AV^m and in AV^n. So then

$G(): \; V^4 \rightarrow R$

$G[]: \; \mathcal{T}^m(V) \times \mathcal{T}^n(V) \rightarrow \mathcal{T}^{m+n}(V)$

The Wedge Product Special Case: functions all of a single variable

Now in the application of the wedge product to the subject of "differential forms", one deals only with a simple subset of wedge products, namely, wedge products of functions which have *only a single argument*. In this special case, $K_{nmk\ldots} = (1+1+1+\ldots)! = N!$ where N is the number of factors in the wedge product.

Example:

$G(\mathbf{v_1},\mathbf{v_2}) = [f \wedge g](\mathbf{v_1},\mathbf{v_2}) = K_{11}\; AS[(f \otimes g)(\mathbf{v_1},\mathbf{v_2})]$ $\quad K_{11} = (2)! / [\, 1!\, 1!\,] = 2$

$= 2\, AS[\, F(\mathbf{v_1},\mathbf{v_2})] = 2\, AS[\; f(\mathbf{v_1})\, g(\mathbf{v_2})\,] = 2\,(1/2)\,[\; f(\mathbf{v_1})\, g(\mathbf{v_2}) \; - \; f(\mathbf{v_2})\, g(\mathbf{v_1})\;]$

$$= f(\mathbf{v_1})\, g(\mathbf{v_2}) \; - \; g(\mathbf{v_1})\, f(\mathbf{v_2}) \;=\; \det \begin{pmatrix} f(\mathbf{v_1}) & g(\mathbf{v_1}) \\ f(\mathbf{v_2}) & g(\mathbf{v_2}) \end{pmatrix}$$

where now the result can be written in terms of determinant. The determinant is of course the same if the rows and columns are swapped.

In this new scenario, one has more generally,

$f_i(\mathbf{v})$ = 1-multilinear functions $f_i(\mathbf{v})$ $\quad f_i: V \rightarrow R$, so f_i is a normal "functional"

$F[f_1,f_2 \ldots f_N] \equiv f_1 \otimes f_2 \otimes \ldots . \otimes f_N$ $\quad F: \; \mathcal{T}^{n1}(V) \times \rightarrow R$

$$F(\mathbf{v}_1,\mathbf{v}_2\ldots.\mathbf{v}_N) = [f_1 \otimes f_2 \otimes \ldots.\otimes f_N](\mathbf{v}_1,\mathbf{v}_2\ldots.\mathbf{v}_N) = f_1(\mathbf{v}_1)\, f_2(\mathbf{v}_2)\ldots.f_N(\mathbf{v}_N)$$

$$G(f_1,f_2\ldots f_N) \equiv [f_1 \wedge f_2 \wedge \ldots.\wedge f_N]$$

$$G : AV^1(V) \text{ x } AV^1(V)\ldots.. \rightarrow AV^N(V)$$

$$G(f_1,f_2\ldots f_N) = K_{111\ldots}\, AS(\ f_1 \otimes f_2 \otimes \ldots.\otimes\ f_N) = N!\ AS(\ f_1 \otimes f_2 \otimes \ldots.\otimes f_N) = N!\ AS(F)$$

Now using the above definition of AS we find

$$[AS(F)](\mathbf{v}_1,\mathbf{v}_2\ldots.\mathbf{v}_k) \equiv (1/k!)\,\Sigma_{abc..q}\,\varepsilon_{abc..q}\,F(\mathbf{v}_a,\mathbf{v}_b\ldots.\mathbf{v}_q)$$

$$= (1/N!)\,\Sigma_{abc..q}\,\varepsilon_{abc..q}\ f_1(\mathbf{v}_a)\, f_2(\mathbf{v}_b)\ldots.f_N(\mathbf{v}_q)$$

If we were do define a matrix M such that $M_{ij} = f_i(\mathbf{v}_j)$, then the above would say

$$[AS(F)](\mathbf{v}_1,\mathbf{v}_2\ldots.\mathbf{v}_k) = (1/N!)\,\Sigma_{abc..q}\,\varepsilon_{abc..q}\,M_{1a}M_{2b}M_{3c}\ldots.M_{Nq}$$

$$= (1/N!)\ \det[M_{ij}] = (1/N!)\det[\ f_i(\mathbf{v}_j)]$$

Then we find that

$$G(f_1,f_2\ldots f_N) \equiv f_1 \wedge f_2 \wedge \ldots.\wedge\ f_N = N!\ AS(F) = N!\ \ (1/N!)\det[\ f_i(\mathbf{v}_j)] = \det[\ f_i(\mathbf{v}_j)]$$

so we end up with this very simple result

$$f_1 \wedge f_2 \wedge \ldots.\wedge f_N = \det[\ f_i(\mathbf{v}_j)]$$

Again, this result is only valid for wedge products of functions of a single argument.

Looking at the final result, it is clear that swapping any two subscripts causes a minus sign. In the above line if we swap to get $f_2 \wedge f_1 \wedge \ldots.\wedge f_N$, this swaps rows 1 and 2 of the determinant and thus creates a minus sign. So

$$f_2 \wedge f_1 \wedge \ldots.\wedge f_N = -\ \ f_1 \wedge f_2 \wedge \ldots.\wedge f_N$$

More generally

$$f_a \wedge f_b \wedge \ldots f_k \ldots. f_m\ \ldots.\wedge f_q = -\ \ f_a \wedge f_b \wedge \ldots f_m \ldots. f_k\ \ \ldots.\wedge f_q$$

One can write using the ε symbol

$$f_a \wedge f_b \wedge \ldots. \wedge f_q = \varepsilon_{abc..q}\ f_a \wedge f_b \wedge \ldots.\wedge f_q$$

This is clear form the ε tensor form of the result, or from the determinant result since such a swap switches two rows of the determinant's matrix, as is seen in the simple example above.

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I am not really happy with the above attempt, but it at least has the right pieces.

Question: where does that area business come from?

$$G(\mathbf{v}_1,\mathbf{v}_2) = [f \wedge g](\mathbf{v}_1,\mathbf{v}_2) = K_{11}\ AS[(f \otimes g)(\mathbf{v}_1,\mathbf{v}_2)] \qquad K_{11} = (2)! / [\ 1!\ 1!\ ] = 2$$

$$= 2\ AS[\ F(\mathbf{v}_1,\mathbf{v}_2)] = 2\ AS\ [\ f(\mathbf{v}_1)\ g(\mathbf{v}_2)\ ] = 2\ (1/2)\ [\ f(\mathbf{v}_1)\ g(\mathbf{v}_2)\ -\ f(\mathbf{v}_2)\ g(\mathbf{v}_1)\ ]$$

$$= f(\mathbf{v}_1)\ g(\mathbf{v}_2)\ -\ g(\mathbf{v}_1)\ f(\mathbf{v}_2) = \det\begin{pmatrix} f(\mathbf{v}_1) & g(\mathbf{v}_1) \\ f(\mathbf{v}_2) & g(\mathbf{v}_2) \end{pmatrix}$$

or

$$[f \wedge g](\mathbf{v}_1,\mathbf{v}_2) = f(\mathbf{v}_1)\ g(\mathbf{v}_2)\ -\ g(\mathbf{v}_1)\ f(\mathbf{v}_2)$$

I don't see any connection between the above line and what some have said about 2-vectors and area. Here is from wiki on exterior products

Areas in the plane [edit]

The Cartesian plane $\mathbf{R}^2$ is a vector space equipped with a basis consisting of a pair of unit vectors

$$\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}.$$

Suppose that

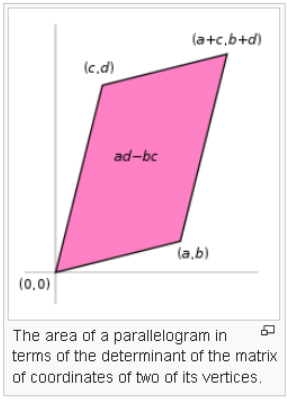
$$\mathbf{v} = \begin{bmatrix}a\\b\end{bmatrix} = a\mathbf{e}_1 + b\mathbf{e}_2, \quad \mathbf{w} = \begin{bmatrix}c\\d\end{bmatrix} = c\mathbf{e}_1 + d\mathbf{e}_2$$

are a pair of given vectors in $\mathbf{R}^2$, written in components. There is a unique parallelogram having **v** and **w** as two of its sides. The *area* of this parallelogram is given by the standard determinant formula:

$$\text{Area} = \left|\det\begin{bmatrix}\mathbf{v} & \mathbf{w}\end{bmatrix}\right| = \left|\det\begin{bmatrix}a & c\\ b & d\end{bmatrix}\right| = |ad - bc|\,.$$

Consider now the exterior product of **v** and **w**:

$$\begin{aligned}\mathbf{v} \wedge \mathbf{w} &= (a\mathbf{e}_1 + b\mathbf{e}_2) \wedge (c\mathbf{e}_1 + d\mathbf{e}_2)\\ &= ac\mathbf{e}_1 \wedge \mathbf{e}_1 + ad\mathbf{e}_1 \wedge \mathbf{e}_2 + bc\mathbf{e}_2 \wedge \mathbf{e}_1 + bd\mathbf{e}_2 \wedge \mathbf{e}_2\\ &= (ad - bc)\mathbf{e}_1 \wedge \mathbf{e}_2\end{aligned}$$



The area of a parallelogram in terms of the determinant of the matrix of coordinates of two of its vertices.

Whereas I am doing wedge products of functions in $AV^n(V)$, *they* are doing them on vectors in R^2. In my world I found for example that

$$[f \wedge g](\mathbf{v}_1,\mathbf{v}_2) = \det\begin{pmatrix} f(\mathbf{v}_1) & g(\mathbf{v}_1) \\ f(\mathbf{v}_2) & g(\mathbf{v}_2) \end{pmatrix}$$

How would I replace functions by vectors? It has to be a functional.

$f_{\mathbf{x}}(\mathbf{a}) = a_{\mathbf{x}}$

I see that I only have a tiny piece of this Grassmann puzzle.

Note added 9.9.15. I reread all of the above. The Spivak-based "wedge products in terms of functions" main part still seems pretty good. And then the concluding observation that this seems to have nothing whatsoever to do with the geometric pictures also seems good. I presumably went on to establish the connection between these two "wedge worlds", but right now I have forgotten it. This led I think to a study of Clifford World.