Wedge Stuff

Question 1: What is the *connection between* the Spivak world of wedge products defined in terms of multilinear functions and the Denker world of geometric blades and Clifford algebras and the geometric product?

Both worlds use the wedge product symbol and have expressions like a ^ b, but sources seem to talk only about one or the other.

1. Wiki page on exterior algebra.

The first part of the presentation concerns these objects:

exterior product = wedge product = ^
geometry
area of parallelogram
exterior algebra = Grassman algebra
k-blades
graded algebra
The definition of the exterior algebra makes sense for spaces not just of geometric vectors, but of other vector-like objects such as vector fields or functions. [hint of a connection]

Planar geometry examples : has notion of standard basis vectors e_i and column vectors.

$$\begin{aligned}\mathbf{v}\wedge\mathbf{w} &= (a\mathbf{e}_1+b\mathbf{e}_2)\wedge(c\mathbf{e}_1+d\mathbf{e}_2)\\ &= ac\mathbf{e}_1\wedge\mathbf{e}_1+ad\mathbf{e}_1\wedge\mathbf{e}_2+bc\mathbf{e}_2\wedge\mathbf{e}_1+bd\mathbf{e}_2\wedge\mathbf{e}_2\\ &= (ad-bc)\mathbf{e}_1\wedge\mathbf{e}_2\end{aligned}$$

Here the coefficient is the area of a 2-piped spanned by the vectors. I think you regard $e_1 \wedge e_2$ as a basis element in the 2-vector space, perhaps a basis 2-blade. In 2D this is all there is. You cannot evaluate $e_1 \wedge e_2$ as some number. It is perhaps a bit like $e_1 \otimes e_2$ which is a basis element in the direct product space and is not something you try to evaluate.

Next is a section on R^3 examples. Now we have these elements

e_1, e_2, e_3	$e_1 \wedge e_2,\ e_2 \wedge e_3,\ e_3 \wedge e_1$	$e_1 \wedge e_2 \wedge e_3$
vector	bivector	trivector

So there are somehow 3 + 3 + 1 = 7 basis type elements.

$$\mathbf{u}\wedge\mathbf{v} = (u_1v_2-u_2v_1)(\mathbf{e}_1\wedge\mathbf{e}_2)+(u_3v_1-u_1v_3)(\mathbf{e}_3\wedge\mathbf{e}_1)+(u_2v_3-u_3v_2)(\mathbf{e}_2\wedge\mathbf{e}_3)$$

so three of these elements appear in the general wedge of two 3-vectors. And then

$$\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} = (u_1v_2w_3+u_2v_3w_1+u_3v_1w_2-u_1v_3w_2-u_2v_1w_3-u_3v_2w_1)(\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3)$$

Notice that in both these last examples, you do have an antisymmetric element creeping in, but there are no AS functions really in the Spivak sense.

Next comes a section that I think is slowly coming into focus:

The exterior product ∧ of two elements of Λ(V) is defined by

$$\alpha \wedge \beta = \alpha \otimes \beta \pmod{I},$$

where the mod I means we do the tensor product in the usual way and then declare every element of the tensor that is in the ideal to be zero.

This says that you go ahead and compute a direct product in the usual away of two vectors α and β, but in the resulting set, you by fiat set elements of the form x ⊗ x = 0. Very strange. Let's try this in the above example:

$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$
$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$

$\mathbf{u} \otimes \mathbf{v} = [u_1\mathbf{e}_1 + u_2\mathbf{e}_2] \otimes [\, v_1\mathbf{e}_1 + v_2\mathbf{e}_2]$

= write it out.

Then if you throw out terms having $e_1 \otimes e_1$ and terms having $e_2 \otimes e_2$, you do indeed arrive at **u** ^ **v** if you replace all ⊗ with ^ . Is this true only for vectors u and v, or for any Clifs u and v ?? Try this

$u = u_{12}\ e_1 \wedge e_2 + u_{23}\ e_2 \wedge e_3 + u_{31}\ e_3 \wedge e_1$ a bivector in R^3
$v = v_1e_1 + v_2e_2 + v_3e_3$ a vector in R^3

$u \otimes v = [u_{12}\ e_1 \wedge e_2 + u_{23}\ e_2 \wedge e_3 + u_{31}\ e_3 \wedge e_1] \otimes [v_1e_1 + v_2e_2 + v_3e_3]$

Well now we have a mix of ^ and ⊗. Take all to ⊗

$u \otimes v = [u_{12}\ e_1 \otimes e_2 + u_{23}\ e_2 \otimes e_3 + u_{31}\ e_3 \otimes e_1] \otimes [v_1e_1 + v_2e_2 + v_3e_3]$

$= u_{12}v_1\ e_1 \otimes e_2 \otimes e_1 +$ other terms

So the more general rule is that you have to throw out terms where any 2 factors of the direct product are the same. So that is more than

Earlier they say x ⊗ x = 0 as shown above. So they are only talking vectors.

In this same section wiki also says

The exterior algebra $\Lambda(V)$ over a vector space V over a field K is defined as the quotient algebra of the tensor algebra by the two-sided ideal I generated by all elements of the form $x \otimes x$ such that $x \in V$.[7] Symbolically,

$$\Lambda(V) := T(V)/I.$$

Now I am supposed to be an expert in "ideals" from my Galois doc which I just looked at. Recall that an ideal I in R means you have rI = Ir = I for any r in R (a ring). The ring here is T(V) which is unfortunately not defined. But perhaps it is the direct product space V x V of which $\alpha \otimes \beta$ would be an element. So the "r" here would be any $\alpha \otimes \beta$, whereas an "i" would be any $x \otimes x$. Then check

$$r\, i = (\alpha \otimes \beta)(x \otimes x) = (\alpha x \otimes \beta x)$$

But why should this be in I ? I have fallen off the wagon. The reference for this claim is good old Birkhoff MacLane which I think I have in stock. I do have it. Where would I find such a claim in there? I don't think it is there as stated above. So I get no happiness on the above claim, but the mod statement makes some sense, throwing out "diagonal" elements somehow.

Here is an interesting idea:

$$0 = (x+y) \wedge (x+y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y = x \wedge y + y \wedge x$$

So by just assuming the left =, you obtain the antisym rule for all x and y in your "space". Does this then apply for x and y being any clifs? Seems not true for example if x = scalar and y = vector. In fact if s is a scalar we have $s \wedge s = s^2 \neq 0$. I think wiki stuff is only for vectors.

The kth **exterior power** of V, denoted $\Lambda^k(V)$, is the vector subspace of $\Lambda(V)$ spanned by elements of the form

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k, \quad x_i \in V, i = 1, 2, \ldots, k.$$

Notice the use of $\Lambda^k(V)$ which is the Spivak notation in the parallel function world. This space has the alternating property, but we have no functions yet.

Question: Is there an isomorphism between $x_1 \wedge x_2 \wedge \ldots\ldots \wedge x_k$ and $f(x_1, x_2 \ldots\ldots x_k)$? On the left we have the wedge product of k vectors, on the right a function of k vectors. Maybe isomorphism only for vector clifs?

Basis and dimension [edit]

If the dimension of V is n and $\{e_1, \ldots, e_n\}$ is a basis of V, then the set

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

The above is just fine, there are n items wedged here.

There is then more which I skip, then we come to a Duality section.

Alternating operators [edit]

Given two vector spaces V and X, an **alternating operator** from V^k to X is a multilinear map

$$f : V^k \to X$$

such that whenever $v_1,\ldots,v_k$ are linearly dependent vectors in V, then

$$f(v_1, \ldots, v_k) = 0$$

The map

$$w : V^k \to \Lambda^k(V)$$

Suddenly we have multilinear functions appearing here! We then get the Spivak wedge. But no connection is made to the earlier geometry stuff. Is this just another implementation of the higher level construct ^ ? The page then peters out for me, and my question is not answered.

1. Wolfram page on exterior algebra.

exterior algebra = alternating algebra = Grassmann algebra

space = space of all possible "forms"

This offering does not help me at all, too fancy.

Chapter 3 PDF of Someone

Here is an interesting statement,

Definition 17 *Given $u_1, \ldots, u_p \in V$ the exterior product $u_1 \wedge u_2 \wedge \ldots \wedge u_p \in \Lambda^p V$ is the linear map to F which, on an alternating multilinear form M takes the value*

$$(u_1 \wedge u_2 \wedge \ldots \wedge u_p)(M) = M(u_1, u_2, \ldots, u_p).$$

The multilinear function M of V space vectors appear as function arguments on the right. But on the left we have an abstract wedge of all these vectors. That seems new to me.

Proposition 16 *The exterior product $u_1 \wedge u_2 \wedge \ldots \wedge u_p$ of p vectors $u_i \in V$ vanishes if and only if the vectors are linearly dependent.*

Earlier they state this simpler case

Definition 14 *Given $u, v \in V$ the exterior product $u \wedge v \in \Lambda^2 V$ is the linear map to F which, on an alternating bilinear form B, takes the value*

$$(u \wedge v)(B) = B(u, v).$$

Alternating bilinear just means 2-multilinear. Same thing as above. The wedge product is an operator which operates on the function B, or M above.

Sophomore Document. Only does wedge of vectors, and states that

2. Fundamental Facts about the Wedge Product

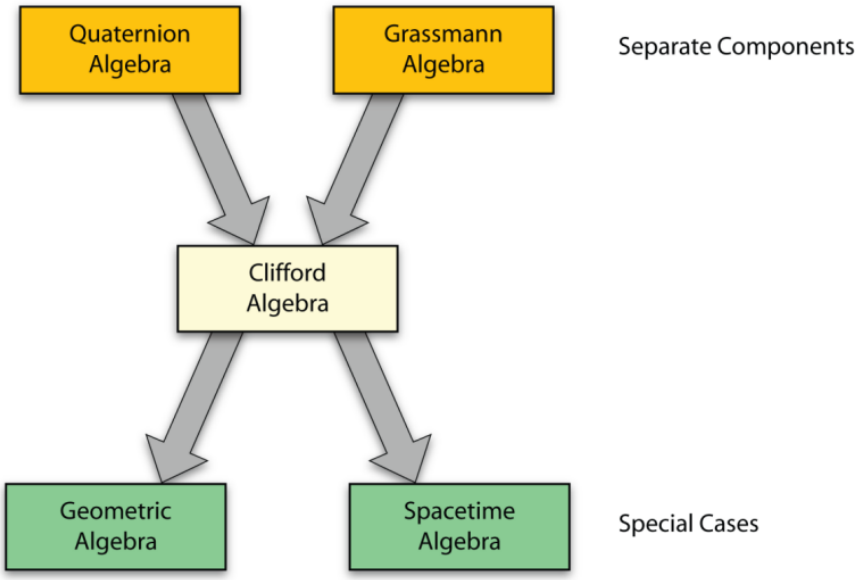
Theorem 1. For $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in \mathbb{R}^n$,

$$\|\underline{v}_1 \wedge \underline{v}_2 \wedge \cdots \wedge \underline{v}_k\| = \mathrm{vol}_k \mathcal{P}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k)$$

where vol_k is understood traditionally as volume of the $(k-1)$ dimensional base times the altitude.

A Presentation pdf

History



Signing off for 8.25.15.