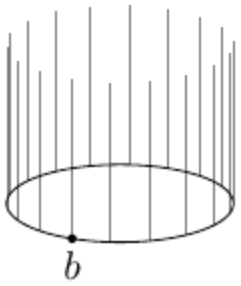
Introduction to Bundles

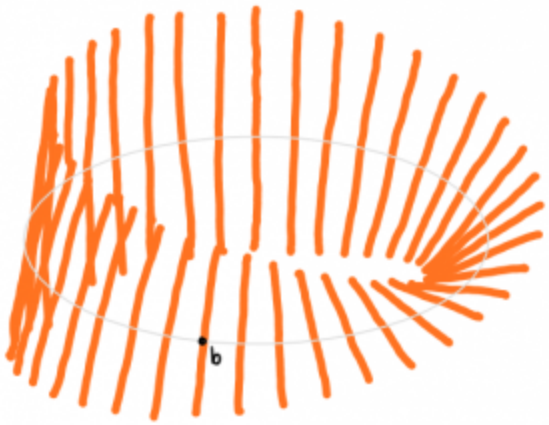
Treating spaces as fiber bundles allows us to tame twisted beasts. Most of spin geometry is phrased in the language of fiber bundles, and this post will begin to introduce that language — extremely powerful in its simplicity.

Introduction to Fiber Bundles

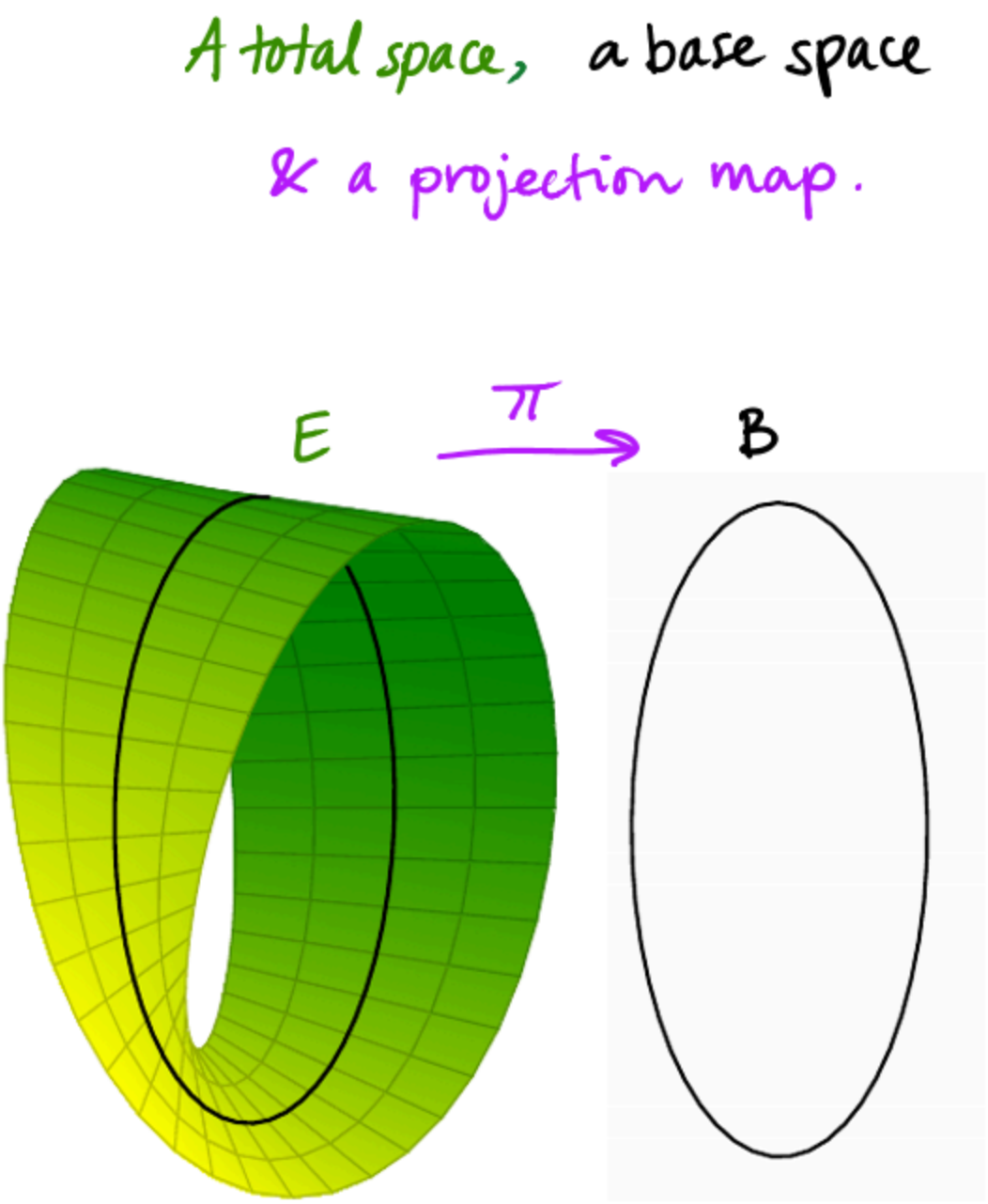
If we glue lines onto every point b in a circle (or a circle to every point of a line), we get a cylinder. In other words, a cylinder is the product space $S^1 \times [0, 1]$.



If we glue lines onto every point of a circle, progressively twisting each individual line, we get a Mobius strip.

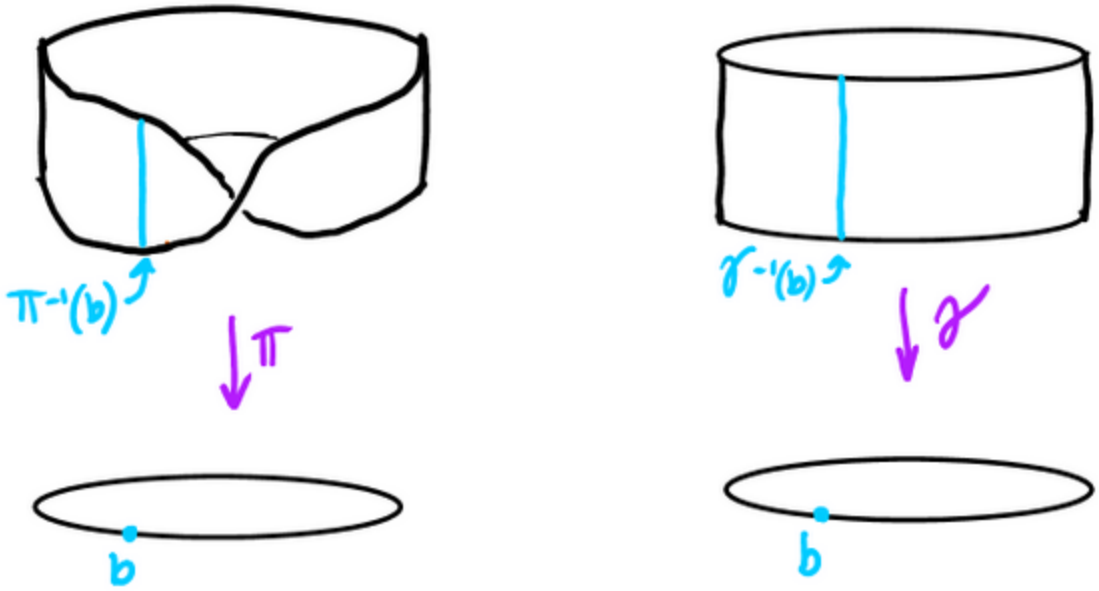


A fiber bundle with fiber F consists of: 2 topological spaces, and a projection map which projects the total space onto its base space.

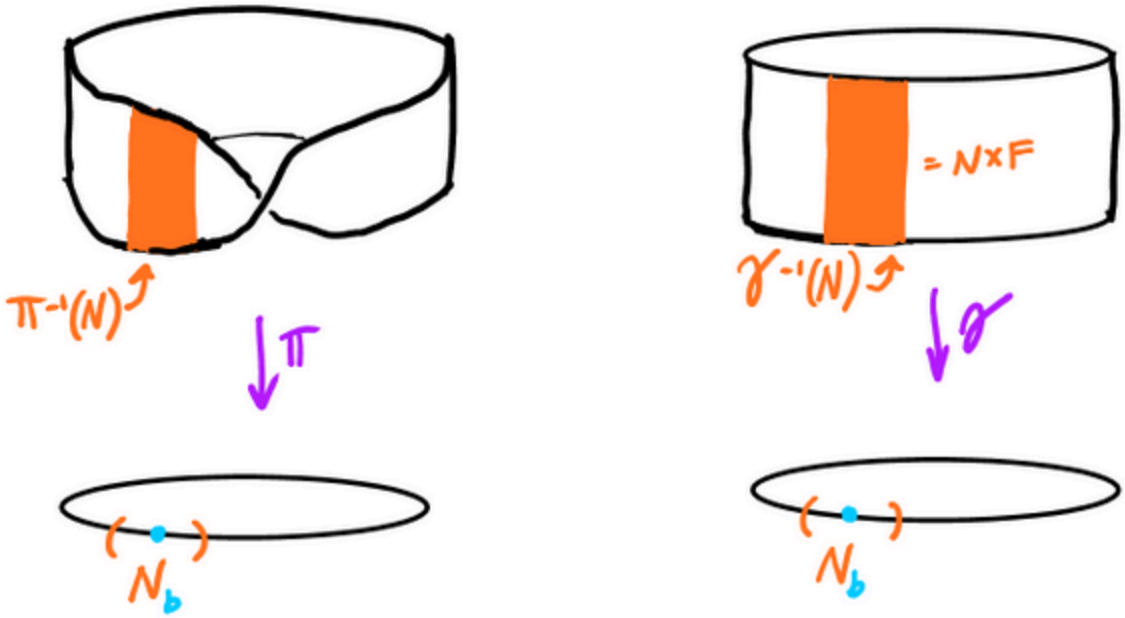


In our example, E := Mobius strip, B := base circle, where the fiber F := $[0, 1]$.

If you flip the arrow around, π^{-1} , the inverse image of the projection map, maps every b in the basespace to its corresponding fiber π^{-1} (b) in the total space.



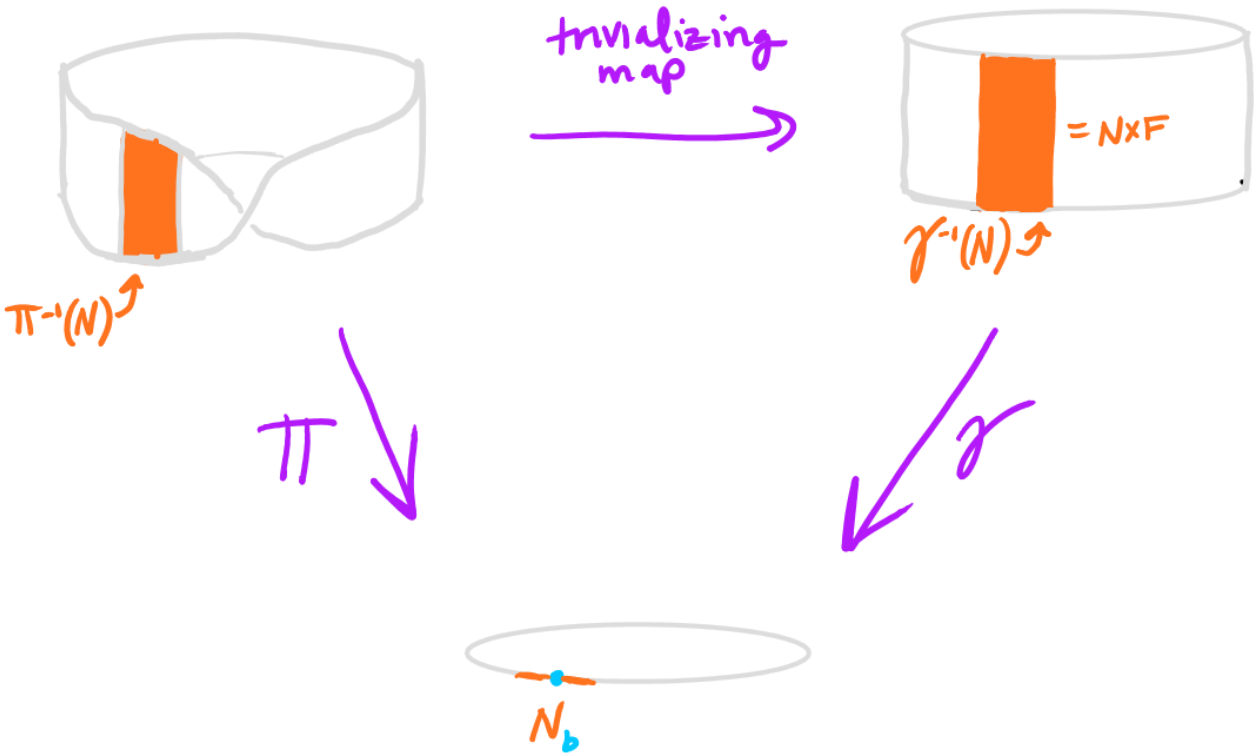
Similarly, the $\pi^{-1}(N)$ maps every point in the neighborhood N of b to their corresponding fibers $\pi^{-1}(N)$ in the total space.

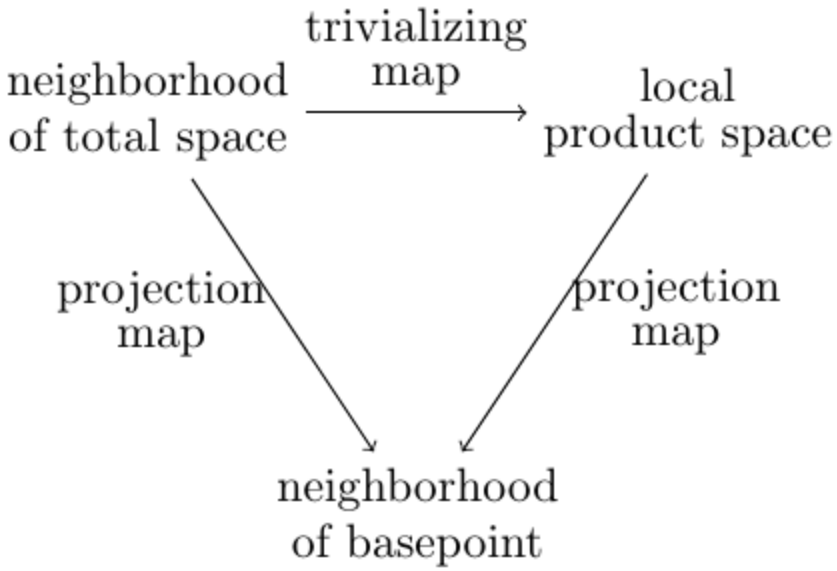


We can locally treat the Mobius strip as a plane, in the same way that we can locally treat a cylinder as a plane.

This property allows us to vastly simplify calculations; it allows us to locally treat twisted spaces like their non-twisted counterparts. We can generalize this property as follows:

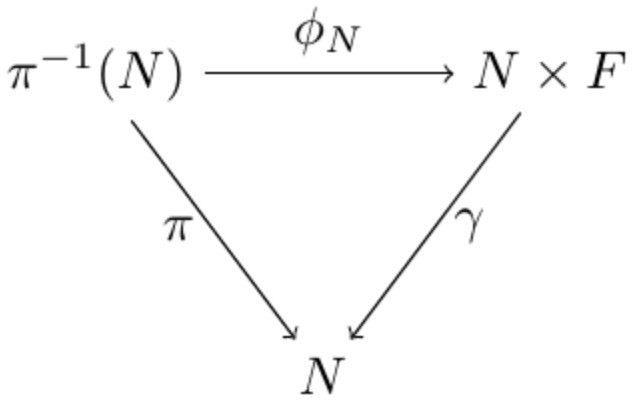
For every $b \in B$ there is a neighborhood N of b s.t. the following diagram commutes.





Formally:

A fiber bundle (with structure group G and fiber F) over B is a smooth surjection $\pi : E \to B$ together with a local triviality condition: every $b \in B$ has a neighborhood N and a diffeomorphism $\phi : \pi^{-1}(N) \to N \times F$ s.t. the following commutes:



Another notation commonly used to represent the fiber in E over b is E_b

$E_b \equiv \pi^{-1}(b)$

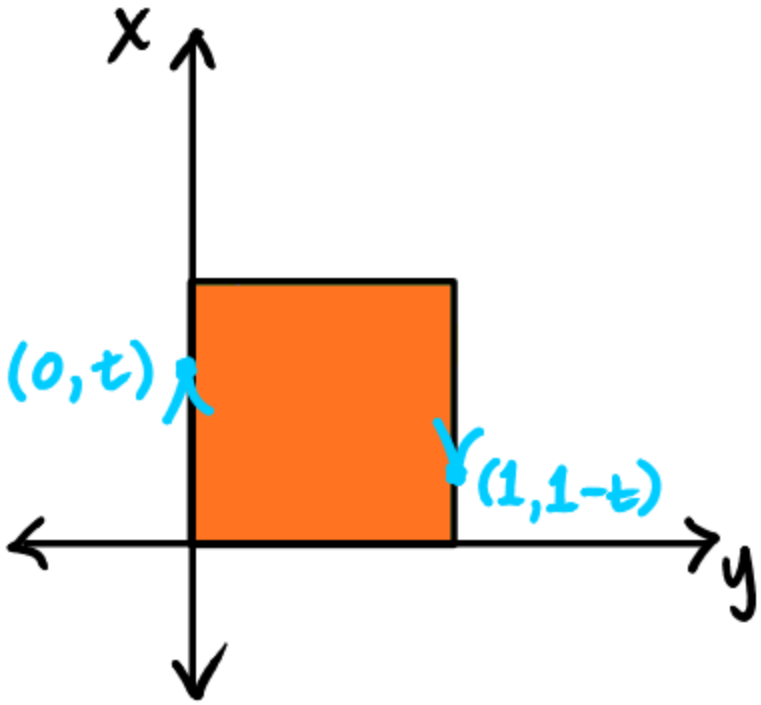
$E = \coprod_{b \in B} E_b = \coprod_{b \in B} \pi^{-1}(b)$

As an aside: How can we formally construct a twisted space?

A Mobius strip := $[0, 1] \times [0, 1] / \sim$, where the equivalence relation is $(0, t) \sim (1, 1 - t)$.

Basically, this equivalence relation gives us gluing instructions.

We must twist the plane an odd number of times s.t. $(0, t)$ are the same as $(1, 1 - t)$.



Get Your Group On: Actions and Torsors

If you are unfamiliar with smooth groups and representations, I recommend reading Studying Symmetryfor context before venturing onward.

What does it mean for $G \curvearrowright X$ (a group G to "act" on X)?

Suppose we write the group operation as multiplication and the identity element as 1.

A G-action on X takes any $g \in G$ and any $x \in X$, and returns $gx \in X$. In other words:

$$G \times X \to X$$
$$(g, x) \mapsto gx$$



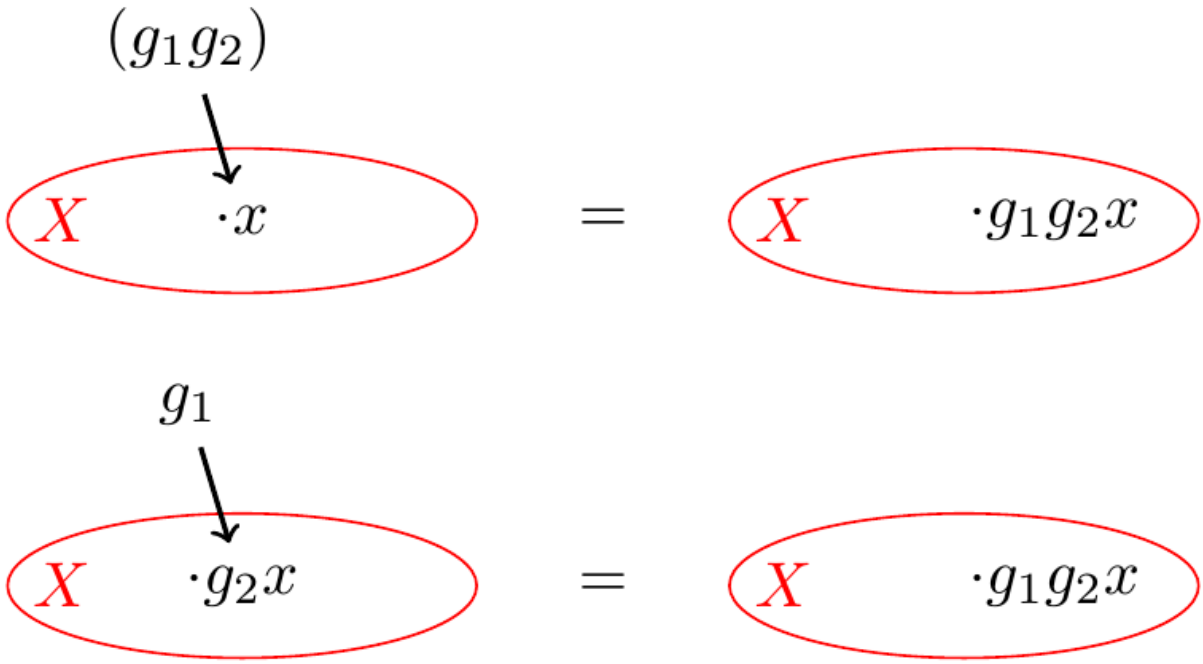
For it to be a G-action, we demand that it obeys:

$$1x = x$$



and

$$(g_1 g_2)x = g_1(g_2 x)$$



These properties may look familiar to the categorically inclined. Every group G is a category with a single object whose morphisms are the elements of G.

An "action" of G on an object in the category C is simply a functor $G \to C$.

- If C is a group, this action is a group homomorphism.
- If C is Vect, this action is a linear representation of G.

A G-torsor is a special type of G-action which satisfies the following: for any 2 elements x_1, x_2 in our G-torsor, $\exists! g \in G$ that satisfies $gx_1 = x_2$.

This means that for any two elements of our torsor, we can talk about their "ratio" x_2/x_1, which defines the unique element g which satisfies the above equation.

Principal-Bundles

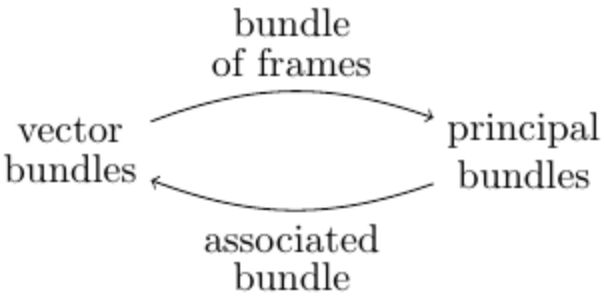
A principal-bundle is a bundle whose fibers are torsors.

A principal G-bundle over a base space B is essentially a bundle of "affine G-spaces" over B.

To be more precise, it is a fiber bundle $E \xrightarrow{\pi} B$ together with a continuous right action of G on $E \xrightarrow{\pi} B$ which preserves the fibers and acts freely and transitively on them. In other words, our fibers are the orbits of G.

Let G be a Lie group, and F be our typical fiber. We have a faithful smooth group action:
$\rho : G \times F \to F$, which we can curry into
$\rho : G \to (F \to F)$ and rewrite as
$\rho : G \to \text{Aut}(F)$

In the most general case, $\text{Aut}(F) = \text{Diff}(F)$ (the group of diffeomorphisms), but we will shortly be working with vector bundles, for which F is a vector space and $\text{Auto}(F) = GL(F)$.



Let $(GL(E))_b$ be the set of orthonormal frames of the vector space E_b. Note that the set of all orthonormal frames is a right $\text{O}(n)$ -torsor.

Given a vector bundle $E \xrightarrow{\pi} B$ with a vector space F as the fiber, we can construct a it's principal bundle $GL(E) \xrightarrow{\Pi} B$ by mapping each fiber E_b to the bundle of orthonormal frames over that fiber $GL(E_b)$.

A Taste of What's to Come

Fiber bundles are themselves merely the culmination of the centuries-long struggle to come fully and properly to grips with the idea of a multiple-valued function.

One of the purposes of cohomology is to specify how the typical fibers and the base may be combined to make a variety of fiber bundles (a way to classify and distinguish the different possibilities) — these are the so-called characteristic classes (such as Chern classes).

Sources

Spin Manifolds
Torsors Made Easy
Spin Geometry, Appendix A

Author: Rin Ray
https://rin.io/about/