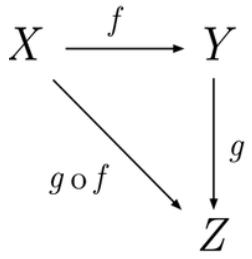
Category Theorem PhL 9.8.15

Just want to take a very quick look at this room in the House of Math. The wiki page is pretty good and here are a few notes on that page. I of course omit most topics addressed.

"Category theory[1] formalizes mathematical structure and its concepts in terms of a collection of **objects** and of **arrows** (also called **morphisms**). A category has two basic properties: the ability to **compose** the arrows associatively and the existence of an **identity arrow** for each object. Category theory can be used to formalize concepts of other high-level abstractions such as sets, rings, and groups."

Here is a typical picture,



Here X,Y,Z are **objects** and f,g and f o g are **arrows (morphisms)** and o indicates "**arrow composition**". The arrows are called morphisms, but this is not the same sense of morphism used elsewhere, it is a special word used in category theory.

If you were to regard X,Y,Z as themselves categories (objects = categories), then the arrows are called **functors**, a term I see here and there. So this all makes sense:

"Diagram chasing is a visual method of arguing with abstract "arrows" joined in diagrams. Functors are represented by arrows between categories, subject to specific defining commutativity conditions. Functors can define (construct) categorical diagrams and sequences (viz. Mitchell, 1965). A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second."

Here is a formal presentation of the category idea:

Categories [edit]

A *category* C consists of the following three mathematical entities:

- A class ob(C), whose elements are called *objects*;
- A class hom(C), whose elements are called morphisms or maps or *arrows*. Each morphism ***f*** has a *source object* ***a*** and *target object* ***b***.
  The expression $f : a \to b$, would be verbally stated as "*f* is a morphism from *a* to *b*".
  The expression **hom(*a*, *b*)** — alternatively expressed as $\mathbf{hom}_C(a, b)$, **mor(*a*, *b*)**, or **C(*a*, *b*)** — denotes the *hom-class* of all morphisms from *a* to *b*.
- A binary operation ∘, called *composition of morphisms*, such that for any three objects *a*, *b*, and *c*, we have $\hom(b, c) \times \hom(a, b) \to \hom(a, c)$. The composition of $f : a \to b$ and $g : b \to c$ is written as $g \circ f$ or gf,[4] governed by two axioms:
  - Associativity: If $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
  - Identity: For every object *x*, there exists a morphism $1_x : x \to x$ called the *identity morphism* for *x*, such that for every morphism $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.
    From the axioms, it can be proved that there is exactly one identity morphism for every object. Some authors deviate from the definition just given by identifying each object with its identity morphism.

I would say that an arrow is a "mapping" but generalized from its usual use between functions or between say vector spaces. It is between two objects in the category space. I suppose hom(C) is the set of mappings which can link objects in the category C whose objects are ob(C). Not sure what the letters "hom" signify.

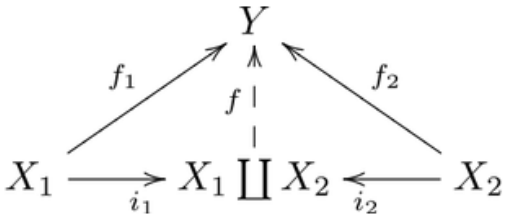
"In 1942–45, Samuel Eilenberg and **Saunders Mac Lane** [book Birkhoff (my 1967 teacher) & Mac Lane] introduced categories, functors, and natural transformations as part of their work in topology, especially algebraic topology. Their work was an important part of the transition from intuitive and **geometric homology** to **axiomatic homology** theory. Eilenberg and Mac Lane later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories. [Category theory is not included in the Birkhoff Mac Lane book.]

Stanislaw Ulam, and some writing on his behalf, have claimed that related ideas were current in the late 1930s in **Poland**. Eilenberg was Polish, and studied mathematics in Poland in the 1930s. Category theory is also, in some sense, a continuation of the work of **Emmy Noether** (**one of Mac Lane's teachers**) in formalizing abstract processes; Noether realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Eilenberg and Mac Lane proposed an axiomatic formalization of the relation between structures and the processes preserving them".

Example of what you see in category theory:

Definition [edit]

Let C be a category and let X_1 and X_2 be objects in that category. An object is called the coproduct of these two objects, written $X_1 \coprod X_2$ or $X_1 \oplus X_2$ or sometimes simply $X_1 + X_2$, if there exist morphisms $i_1 : X_1 \to X_1 \coprod X_2$ and $i_2 : X_2 \to X_1 \coprod X_2$ satisfying a universal property: for any object Y and morphisms $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, there exists a unique morphism $f : X_1 \coprod X_2 \to Y$ such that $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$. That is, the following diagram commutes:



The unique arrow f making this diagram commute may be denoted $f_1 \coprod f_2$ or $f_1 \oplus f_2$ or $f_1 + f_2$ or $[f_1, f_2]$. The morphisms i_1 and i_2 are called *canonical injections*, although they need not be injections nor even monic.

So here they are defining the notion of a **coproduct** in category theory which is written using an upside-down capital π symbol. I finally located this in the Arial Unicode MS font which is a very large font! So $X_1 \amalg X_2$ is an object which makes the above picture work! I think this is in effect some kind of SUM operator as the other two symbols suggest.

Here is a context in which this coproduct animal appears

Definition [edit]

Let $\{X_i : i \in I\}$ be a family of topological spaces indexed by I. Let

$$X = \coprod_i X_i$$

be the disjoint union of the underlying sets. For each i in I, let

$$\varphi_i : X_i \to X$$

be the **canonical injection** (defined by $\varphi_i(x) = (x, i)$). The **disjoint union topology** on X is defined as the finest topology on X for which the canonical injections $\{\varphi_i\}$ are continuous.

Explicitly, the disjoint union topology can be described as follows. A subset U of X is open in X if and only if its preimage $\varphi_i^{-1}(U)$ is open in X_i for each $i \in I$.

Yet another formulation is that a subset V of X is open relative to X iff its intersection with X_j is open relative to X_j for each i.

So in the fiber bundle world, when you write $\pi^{-1}(U)$ as the preimage of the projection π, maybe you can think of things this way

$$\pi^{-1}(U) = \amalg_i \pi^{-1}(U_i)$$

so maybe you are summing over the fibers of the bundle in this way. In fact, my html fiber doc says this

$$E = \coprod_{b \in B} E_b = \coprod_{b \in B} \pi^{-1}(b)$$

so you are summing over all elements b of the base manifold B, and yes, this is a sum of fibers to obtain the continuous surface E.