Fiber Bundles -- Starting Out PhL 9.4.15

I knew something about this subject very long ago, it came up somewhere in my physics past. If I search my Work folder for files containing the word "fiber", I get *no hits* on this subject, so I have no "current" notes on this topic. So here I start over. Vector Bundle is another phrase. Agent Ransack shows I have no data on this subject.

1. Paul Duffell. 1

2. Catherine Ray (age 17) nickname "Rin" 10

3. Torsors by John Baez 2010. 18

4. Hainan Zhang MA thesis 19

9/8/15. OK, I spent a few days on this subject, gathered some PDF files in this folder, have notes on some of them below. A side-track was Mobius and me doc doing details of that example. I don't think I want to know any more about fiber bundles at this point, but I think I see the lay of the land. Some of these documents even mention pullbacks in the bundle context, but I did not pursue that. This topic brushes up against many other areas of math about which I know nothing!

1. Paul Duffell.

Did a first shot on this, then did mobius and me doc, then came back for a second shot at Duffell. Did not really get much further, she is NOT doing a primer here.

I found a good pdf of Paul Duffell, a 2014 PhD cosmology. It claims to be "Chapter 4" of something. Well further probing reveals:

Geometric Concepts in Physics

Paul Duffell

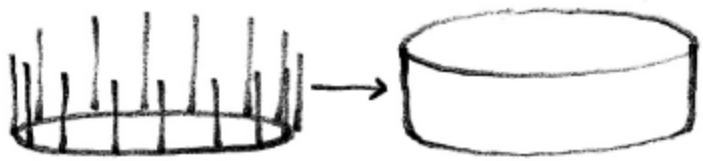
Based on lectures by Brian Greene

Illustrations by Andrew Macfarlane

I saved this as well, this has a 4-chapter TOC, but then only the first chapter. These are just informal notes he has written, so we do have an author.

OK, lets stay focused then on his Ch 4 on fiber bundles.

4.1 He uses an example of a circle as a base manifold and a line segment as "the fiber". The tensor product of these two spaces is like a cylinder



where you draw a fiber at every point of the circle, hence the "brush" analogy. He notes that if you construct a Mobius band using this idea, you have the cylinder notion locally but not globally due to the global twist. Here is his example

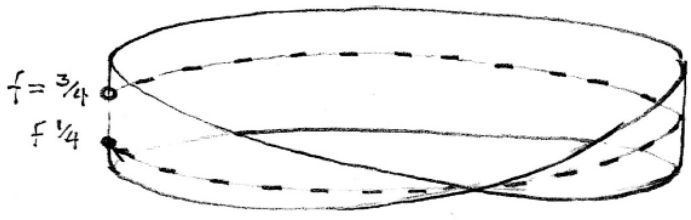


Figure 4.3 Fiber parameterization on the möbius band is not global.

You can think of the base manifold (circle) as a global thing, but the fiber then is not global. The point f = 3/4 is on the fiber before you "go around" and you find you end up with f = 1/4 due to the twist, so the value 3/4 is not "well defined". But locally you are OK. This locality is always implied when you talk about manifolds, as I now know.

He is going to build up the statement that other web pdf's start with!

In the above picture, the strip is called E, and the global base thing is M with points p. Then

π: E $\rightarrow$ M a *projection* of a point in E to a point on M $\pi(q) = p$

Many points on E project into the same point p in M, so many to 1 map. After all, the entire fiber through this point in E will map into M. He then writes

$\pi^{-1}(p) \approx F$ = all points q in E such that $\pi(q) = p$; this set is an entire fiber through p in M

to show that the pre-image of point p in the above mapping is the entire fiber F. Fine.

In manifold world one deals with a union of sets U_i and "homeomorphisms" φ_i. Sjamaar referred to these things as "embeddings". I ran into the word homeomorphism recently, where was that? In my interests I get several doc hits on this word, but newest is 2011! Well, here is from Ahlfors meta review

Topological Mapping: If f is **one-to-one** and uses up all the range (is **onto**), then you can invert any point in the range (image) and therefore $\mathbf{f^{-1}}$ **exists** and everything is nice. IF in addition both f and f^{-1} are continuous, then such a mapping is called a **topological mapping** (= **homeomorphism** -- "the isomorphism of the topological world"). Such a mapping preserves certain topological properties. Two such properties that are preserved are compactness and connectedness, according to our theorems 8 and 9. Openness is not such a property.

And here from Sja Ch 6 notes:

Comment: A **homeomorphism** is like a diffeomorphism but has a lesser requirement: h and h^{-1} have to exist and be just *continuous* in both directions, not necessarily differentiable. So the Sjamaar embedding conditions require something halfway between a diffeo and a homeo morphism.

OK, so here is Duffell's next step:

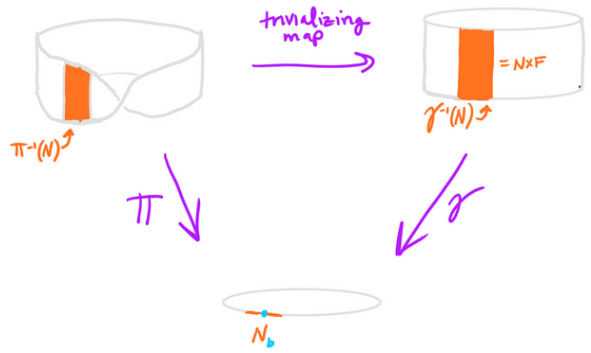
$$\Phi_i : U_i \times F \to \pi^{-1}(U_i)$$

$\Phi_i(p,f)$ = q = some point on the Mobius patch

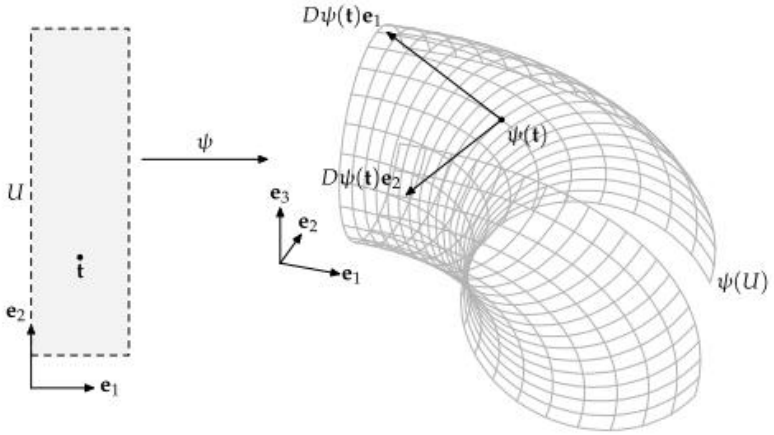
We also know that $\pi(q) = p$. Combining we get $\Phi_i(\pi(q), f) = q$ and $\pi(\Phi_i(p,f)) = p$.

I had trouble with this until I read a different pdf which had this picture in which U_i is called N

For every $b \in B$ there is a neighborhood N of b s.t. the following diagram commutes.



So Duffell's map Φ_i is the inverse of the above "trivializing map". Duffell's mapping Φ_i maps from the orange patch on the cylinder (which after all is just U_i x F) to the orange patch on the Mobius band. The cylinder orange patch is like the "flat" parameter space U in this Sjamaar picture,



So Sjamaar has ψ being the mapping from the flat space to patch on fancy surface (embedding)

So Duffell has Φ_i being the mapping from the flat space to patch on fancy surface (homeomorphism)

A point in the orange flat cylinder patch is just (p,f) like ($\pi/2$, 1/4). He says,

> Since we have a map Φ_i between $\pi^{-1}(U_i)$ and $U_i \times F$, we can locally express points in E using points in $U_i \times F$. We first project the point q in E to a point p in M, using $p = \pi(q)$, then find an open neighborhood U_j of p, then there is a corresponding map Φ_j which associates q with a point in $U_j \times F$, i.e. a pair of points (p, f).

Now what exactly is this paragraph saying?? E is the Mobius, so q is on the Mobius. For a given q, I agree that we project to find its p coordinate using $p = \pi(q)$. Now why in the next sentence does he define the neighborhood of p by U_j instead of by U_i ? I guess it is just to have an alternate to U_i for what follows. Certainly p would lie in the union of U_i and U_j.

OK, suppose we come up with some U_i and U_j which both contain p of M. We then have corresponding mappings Φ_i and Φ_j. These mappings are intended for different regions U_i in the usual manifold sense, but we have to be "consistent" where those two regions U_i and U_j within M overlap. So consider:

$\Phi_i : U_i \times F \rightarrow \pi^{-1}(U_i)$
$\Phi_j : U_j \times F \rightarrow \pi^{-1}(U_f)$

$\Phi_j^{-1} : \pi^{-1}(U_j) \rightarrow U_j \times F$ // homeomorphism is 1-to-1 and onto, so invertible

$\Phi_i \circ \Phi_j^{-1} : \pi^{-1}(U_j) \rightarrow U_j \times F \rightarrow \pi^{-1}(U_f)$ for the overlap region

$\Phi_j^{-1} \circ \Phi_i : U_j \times F \rightarrow \pi^{-1}(U_f) \rightarrow U_j \times F$ for the overlap region

In a composition, the function on the right acts first. So let's write these more accurately

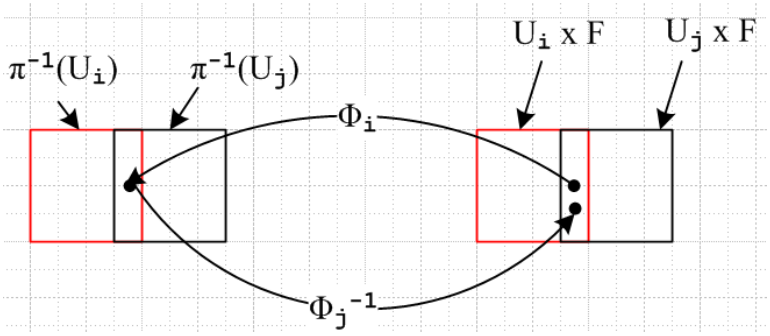
$U_i \cap U_j$

$\Phi_i \circ \Phi_j^{-1} : \pi^{-1}(U_i \cap U_j) \rightarrow \pi^{-1}(U_i \cap U_j)$

$\Phi_j^{-1} \circ \Phi_i : (U_i \cap U_j) \times F \rightarrow (U_i \cap U_j) \times F$

So here is my own little picture

Mobius Cylinder



So we could then define

$$t_{ij} \equiv \Phi_j^{-1} \text{ o } \Phi_i : (U_i \cap U_j) \text{ x } F \rightarrow (U_i \cap U_j) \text{ x } F$$

Duffel has made an error in the following line, what he means is what is shown above. I think he is used to having Φ defined in the reverse direction. I know what he means!

$$t_{ij}=\Phi_i \circ \Phi_j^{-1} : (U_i \cap U_j) \times F \rightarrow (U_i \cap U_j) \times F$$

Suppose we start with a point (p_1,f_1) as the black dot on the right. This will map to some (p_2,f_2) on the left, allowing full generality for the moment. Then we come back and land at some (p_3,f_3). But suppose we imagine that the overlap region is very small. In the limit, all three p values will be the same. So then we argue that under the t_{ij} map we have (p,f) → (p,f'). Duffell does not really make this clear.

So far I am with Duffell so let's continue. As just stated, he notes that,

$$(p,f) \rightarrow (p,f') \qquad (4.6)$$

So I agree, more informative maybe to write the mapping as

$$t_{ij}(p): F \rightarrow F \qquad (4.7)$$

and this makes complete sense to me now. He then claims that the mapping t_{ij} has these properties where only the first is not quite obvious

$$t_{ij}(p)[t_{jk}(p)]=t_{ik}(p) \qquad (closure)$$
$$t_{ii}(p)[f]=f \qquad (identity\ element)$$
$$t_{ji}(p)=t_{ij}^{-1}(p) \qquad (inverses)$$

Think now of the $t_{ij}(p)$ as elements of a set of some sort that is labeled by p. He wants to say that the set of elements { $t_{ij}(p)$ } forms a group. But what then is the product $t_{ab}\ t_{cd}$? The first line does not account

for this case. It only accounts for $t_{ab}t_{bc}$ where two indices match. The identity and inverse lines seem OK to me. Well let's try:

$$t_{ab}t_{cd} = \Phi_a^{-1} \circ \Phi_b \circ \Phi_c^{-1} \circ \Phi_d \;=?=\; \Phi_x^{-1} \circ \Phi_y = t_{xy}$$

Well, this is just fine if you define

$$\Phi_y \equiv \Phi_c^{-1} \circ \Phi_d$$

$$\Phi_x^{-1} = \Phi_a^{-1} \circ \Phi_b \qquad \Rightarrow \qquad \Phi_x \equiv \Phi_b^{-1} \circ \Phi_a$$

and since all inverses exist, I guess this is all OK. So OK, the t_{ij} form a group. There is a group element for each choice U_i and U_j which contain p, so this will be a continuous group. A very strange group until I see an example!

$$t_{ij}: U_i \cap U_j \to G$$

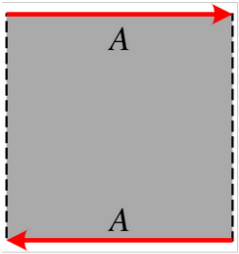
The t_{ij} are the "transition functions" and G is the "structure group". Here are some examples:

into the structure group G which acts smoothly on the fiber F. The transition functions characterize the fiber bundle. In the case of the cylinder, the structure group is just the trivial group of one element. In the möbius band, the structure group is the group of two elements, Z_2, given by $\{1, x\}$, where $x^2 = 1$. In other words, we only have two parameterizations, and thus only one transition function other than the identity, which is its own inverse:

$$f'' = 1 - (1 - f) = f \qquad (4.10)$$

If you had the cylinder on the left and on the right, the round trip would map f → f and you have only one group element, a trivial group. For the Mobius what happens? First, wiki says

Topologically, the Möbius strip can be defined as the square $[0, 1] \times [0, 1]$ with its top and bottom sides identified by the relation $(x, 0) \sim (1 - x, 1)$ for $0 \le x \le 1$, as in the diagram on the right.



To turn a rectangle into a Möbius strip, join the edges labelled *A* so that the directions of the arrows match.

But I want some nice parameters. For cylinder I would use θ in 0 to 2π and z in 0,1 so a point on the cylinder would be given by (p,f) = (θ,z).

I do not see why the structure group G is equal to {1,x} where $x^2 = 1$ for a Mobius strip. The claim is that G actually determines the surface or the fiber bundle. I need another source on that!

In more detail, the claim is that G tells you how the neighborhoods U_i and their maps are "sewn together". Buck talks about this same thing on page 345, but not really in this same context. Here is a mysterious Duffell picture

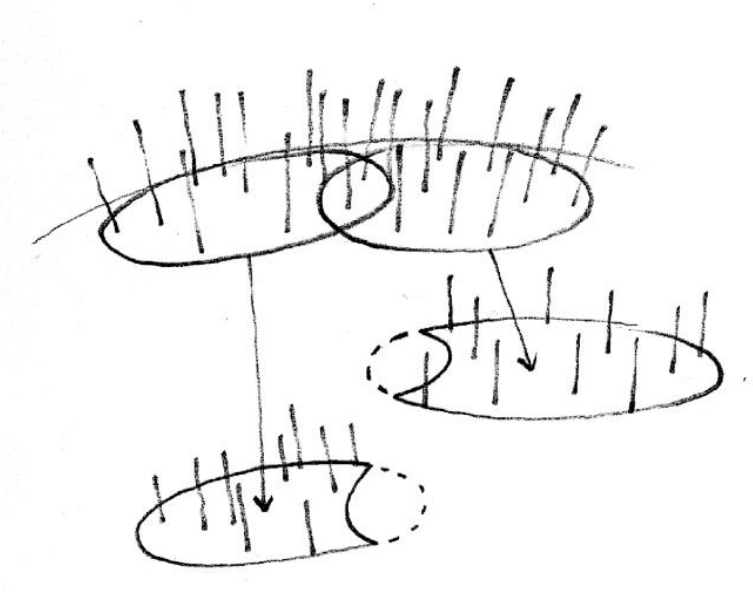


Figure 4.4 The transition functions specify precisely how the fibers are sewn together.

Maybe the idea is that if you enforce your rules on any intersecting regions, that determines things for the entire regions like U_i . So at this point I am not really with it on these transition functions and that structure group. So pause at this point.

Resuming now after writing "Mobius and Me.doc" where I apply most of the above stuff to the Mobius example of the fiber bundle. I was able even to obtain Z_2 as the structure group. But the above picture is not clear because the transition functions only act on a region of intersection, but the arrows in the above picture seem not related to the intersection region. I will let that ride.

4.4 Special types of bundles.

The **Trivial Bundle** is what I expect it to be. A simple direct product B x F with G = 1.

Vector Bundle. In our example we had F = [0,1] which is NOT a vector space. If you require that F be a vector space that is basically R^k then you are talking "**vector bundle**". Recall that we could view those transition functions t_{ij} as maps t_{ij}: F → F. We now add a *requirement* that such functions are all linear

on F (linear on the vector space F). In our example we had t_{12}: f = 1-f which is not linear because $t_{12}(f)$ = 1-f does not give $t_{12}(\alpha f) = \alpha\ t_{12}(f)$, just for example. So if all the t_{ij} act linearly on the vector space = fiber F, with some group G, then we have a true "vector bundle". But some Duffell statements make no sense to me at this point. For example:

> "the t_{ij} have a k-dimensional representation on the fibers."
>
> "the t_{ij} are subgroups of GL_kR "

I don't even know what that group is! OK for now. // It is GL(k,R), linear transformations.

Example Next, imagine the base is a 2D surface manifold so we can think about something. It has points on it which we still call p. At any point p, I know there is a **tangent space** T_pM whose little vectors are tangent to the surface. So this T_pM is a vector space, and we have a different vector space at every point p, so this sure *looks like* an example of a vector bundle. The whole thing E=TM is the tangent bundle associated with M. For any small neighborhood we can regard the F space axis as our coordinate system which we associate then with some φ_{Ni} function.

Example: What about a vector field **F**(**x**)? At each point in space we have a vector. But axes are usually the same at each point. But having a direction at each point seems a little like SO(3) element for each point so F = G and maybe this is a principle bundle as defined below.

Now how do you write a tangent vector as some kind of derivative thing? I often write

> if d**r** lies on a surface f = K, then df = ∇f • d**r** and if you stay on the surface ∇f • d**r** = df = 0. So this says $\partial_i f\ dx^i = 0$. If d**r** = **v** ds where v is a tangent vector, then $\nabla f \bullet \mathbf{v} = 0$. We could write this as $\mathbf{v} \bullet \nabla f = 0$ which says $v^i \partial_i f = 0$.

But this does not seem too helpful. Sjamaar had $T_xM = (D\psi)$ and the columns of this thing were the tangent vectors, so $\xi_i = (D\psi)_{ni}$ for n = 1...k. But Duffell has some new notation, here it is

In the neighborhood U_A we use coordinates $\{x^i\}$ and

$$V_p \in TM = V^i \frac{\partial}{\partial x^i}\Big|_p \qquad (4.15)$$

and in the neighborhood U_B we use coordinates $\{y^j\}$ and

$$V_p \in TM = \tilde{V}^j \frac{\partial}{\partial y^j}\Big|_p \qquad (4.16)$$

I have no clue what this notation means. The derivatives look like operators evaluated at point p. I guess the first equation is saying $V_p = \mathbf{V} \bullet \nabla|_p$?? This entire Duffell section has fallen apart for me, it means nothing at all. Mention of parallel transport.

Principle Bundle (F = G). Remember that F = fiber and G = structure group, usually some Lie group, though I have seen no examples. I guess you can regard the parameter of a Lie group as a manifold, so

you could then choose G = F! That certainly seems strange, but it is a key thing to do, he says. This bundle then is fully defined by a base manifold B and some Lie group.

Possible example: in tensor doc I have the R matrix $R_{ij}(\mathbf{x})$ as a rotation defined at each point in R^N. Perhaps one could interpret this as B = R^N and F = G = SO(3). G is not the trivial group, so this cannot be a simple R^N x SO(3) direct product space. This would be a principle bundle example. But it also seems to be an example of the frame bundle defined below. All a bit vague.

Frame Bundle. Here (at point p) the fiber F is taken to be "the set of all possible frames $\{\mathbf{e}_i)$ that could span the tangent space at point p." That is certainly a large space even for R^2. I would say one frame for exact element of SO(n). Elements of F are now sets of 2-vectors, say, so F does not seem to be any simple kind of vector space.

Example of a Lie group. I guess GL_kR is the same as GL(k,R) in Duffell notation. Here it is.

- The 2×2 real invertible matrices form a group under multiplication, denoted by GL(2, **R**) or by $GL_2(\mathbf{R})$:

$$\mathrm{GL}(2,\mathbf{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = ad - bc \neq 0 \right\}.$$

I suppose you could identify each "frame" with some element of GL(k,R), perhaps the column vectors of same are those axes. So perhaps $\mathbf{e}'_j = M_{ij}\mathbf{e}_j$ where M is in GL_kR and in this way you can generate all sets of axis. So in some sense you now have F = GL(k,R) if the tangent space has dimension k. Duffell eventually makes that point.

Associated Vector Bundle. This is too fancy for me right now. None of the sentences mean anything.

4.5 More elegant description of Fiber Bindles.

He argues here that *basically* the bundle is defined by the **projection map** π, it is this map that is often identified with the "fiber bundle". He then talks about the idea that two bundles with π_1 and π_2 might in some sense be "**equivalent**".

Section. Seems that any $\pi^{-1}(N)$ would be a section, but maybe you are supposed to also pick a point in the fiber F. So if you had a version of $\pi^{-1}(p)$ that picked out a specific f in the fiber F, that thing would be a real function. and I suppose you could do it for all p. Perhaps this is like my thin red band in my Mobius pictures, where I more or less pick out only f = +1/2 (range around).

Fact: Think again of the tangent space vectors $\mathbf{e}_i(p)$. As you move on M, these vectors change, and I am very used to thinking of this in terms of the **affine connection** Γ !

$$(\partial_j \mathbf{q}_n) = \Gamma^k{}_{jn}\,\mathbf{q}_k \qquad \Rightarrow \qquad (d\mathbf{q}_n)^i = (\partial_j \mathbf{q}_n)dx^a = \Gamma^k{}_{jn}\,(\mathbf{q}_k)^i\,dx^a \quad . \qquad \text{(F.1.4)}$$

The coefficients $\Gamma^k{}_{jn}$ are known as the **affine connection**. They measure how the tangent basis vectors change in ξ-space as a function of **x** in x-space. One regards $\Gamma^k{}_{jn}(\mathbf{x})$ as an object associated with x-space,

So I should not be surprised to find that for a vector bundle, where the fiber is the tangent space, that the **affine connection** should enter the picture. How do the tangent vectors change as you change p ? Duffell quotes some of my messy basis change for Γ stuff. He is then off for the rest of his paper talking about fancy curvature on fiber bundles and all that good stuff. I am not ready to go there right now, but it is on the general relativity agenda. I still want to see a simple example of a fiber bundle where the structure group is a Lie group.

2. Catherine Ray (age 17) nickname "Rin"

This is a very unusual person, and this fiber bundle thing is just one of many items on her website. She did college at George Mason U, graduating at age 16. She is on a Thiel fellowship:

The Thiel Fellowship is unlike anything you've ever experienced. The Fellowship brings together some of the world's most creative and motivated young people, and helps them bring their most ambitious projects to life. Thiel Fellows are given a grant of $100,000 to focus on their work, their research, and their self-education while outside of university. Fellows are mentored by our community of visionary thinkers, investors, scientists, and entrepreneurs, who provide guidance and business connections that can't be replicated in any classroom.

The Thiel Fellowship is open to applicants under 23 years old who do not have a degree and agree to not be enrolled in college during the program.

She has done many things already in her sort existence. She has many PDF like objects on her site, but they are all web pages. Including Clifford things, and little detail topics of the kind I often do.

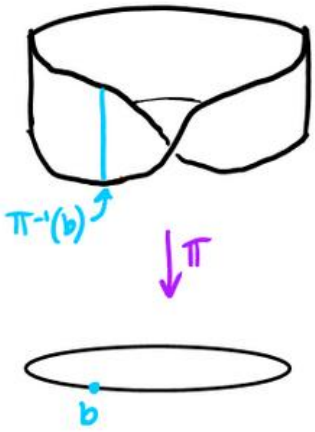
"In autumn 2014, I began to teach myself algebraic topology full-time."

Good pictures, a web page and not a pdf.

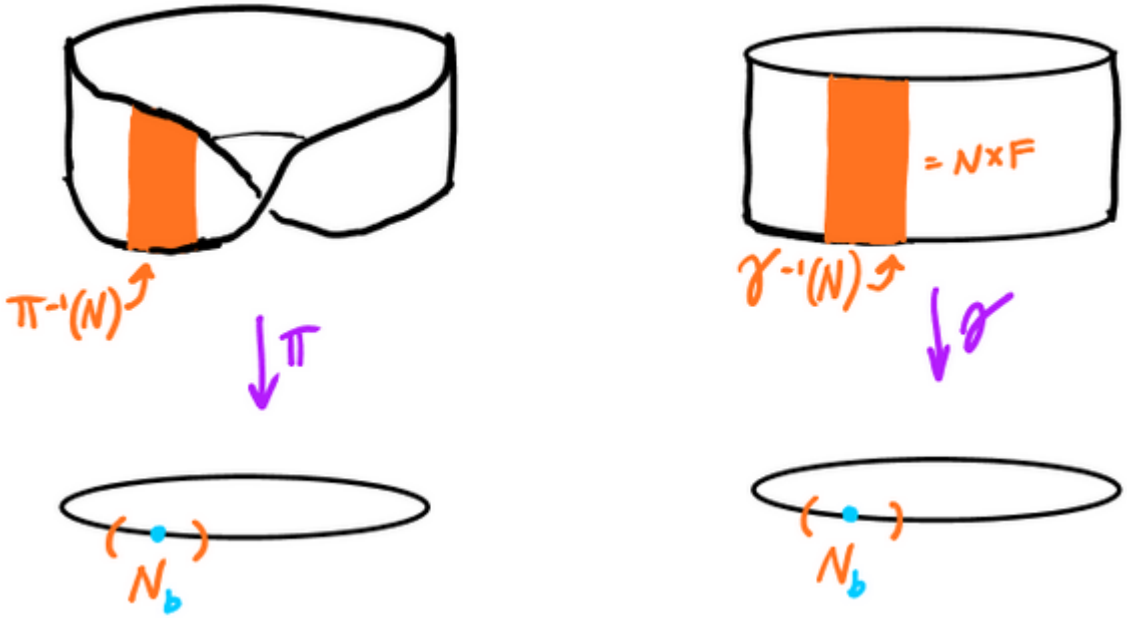
http://rin.io/intro-to-bundles/

I stored this locally as htm file and support folder in the usual manner.

First, we have the definition of π which is the same as Duffell:

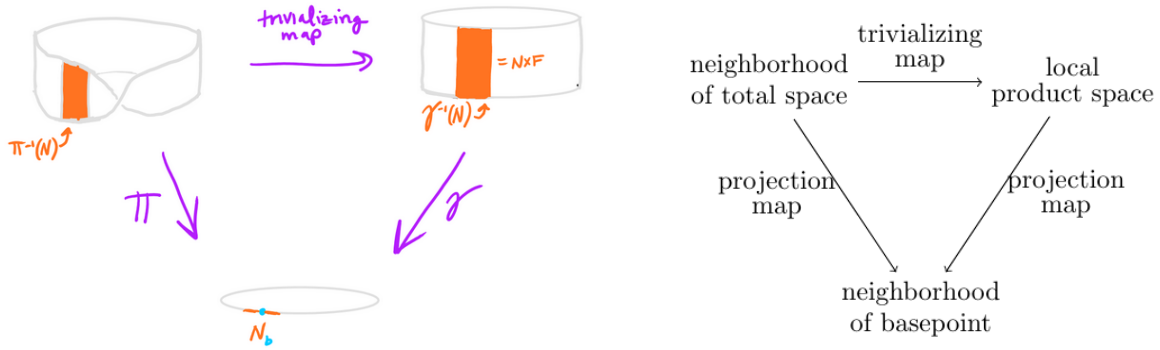


Now we do the neighborhood thing in the base manifold (circle) and it back-maps into a handful of fibers as shown on the left for Mobius, and on the right for cylinder. The maps are given different names because we have different total surfaces.



"We can locally treat the Mobius strip as a plane, in the same way that we can locally treat a cylinder as a plane. This property allows us to vastly simplify calculations; it allows us to locally treat twisted spaces like their non-twisted counterparts." [I see differential forms in here somewhere]

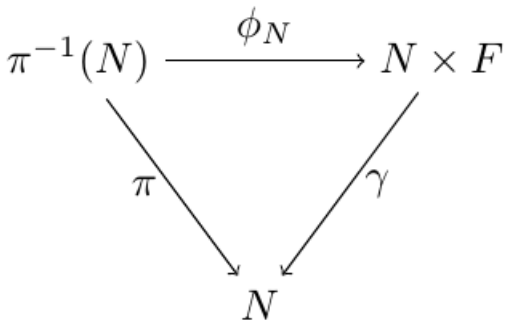
For every $b \in B$ there is a neighborhood N of b s.t. the following diagram commutes.



(1)

Note: π is a "surjection", it is "onto", its image includes *all* of the base B/

What does she mean "**commutes**"? I guess you can to the π route, or the map + γ route and end up with the same thing, which is neighborhood N_b. OK, sounds possible at least. Words on the right. The "total space" is our manifold of interest, the Mobius. The product space is that cylinder. Here is a third picture of the above triangle:



(2)

You could start with N, then go to top right with π^{-1} to get the orange small bundle on the top left of Fig 1. From that patch you go the top right patch with some assumed function φ_N . The patch on the top right is exactly N x F (so this object now appears!). Then γ takes that down to the same N.

Note these facts:
1. The spread of fibers top left is a patch of surface E = Mobius = our surface of interest.
2. N on the bottom lies in the base manifold B (circle in Mobius example)
3. F = the fiber thing, I guess there is only one F , a little segment of the real axis 0 to 1, say.
4. the map π: E→B covers all of the base thing, so is a "surjection" (onto)

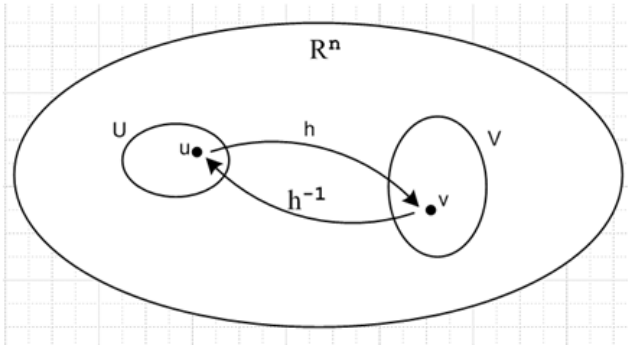
Here are his words for Fig 2 (they are presented above Fig 2, so his commute statement refers to Fig 2 :

A fiber bundle (with structure group G and fiber F) over B is a smooth surjection $\pi : E \to B$ together with a local triviality condition: every $b \in B$ has a neighborhood N and a diffeomorphism $\phi : \pi^{-1}(N) \to N \times F$ s.t. the following commutes:

We don't know what G is yet. The left to right mapping at the top is indeed φ: π^{-1}(N) → N x F which is a map from one patch to the other that only has to be reasonable locally since N is "small". They want φ to be a **diffeomorphism** which I know about so derivatives are well defined in all directions. Recall:

If U and V are open sets in $\mathbf{R}^n$, a differentiable function $h: U \to V$ with a differentiable inverse $h^{-1}: V \to U$ will be called a diffeomorphism. ("Differentiable" henceforth means "C^∞".)

Here is a picture to go with the above definition



Here are some interesting ways to write the total surface E as a sum over its fibers. The first line below defines E_b as the fiber "over" the point b in the base, then the second line sums up all the fibers.

$$E_b \equiv \pi^{-1}(b)$$

$$E = \coprod_{b \in B} E_b = \coprod_{b \in B} \pi^{-1}(b)$$

The ∐ symbol is a form of summation used in "category theory", see my little doc on that subject. It is called the coproduct, but is often written as ⊕ or +, so it is more like a sum than a product IMHO.

Now we come to the next mystery notation:

A Mobius strip $:= [0,1] \times [0,1]/\sim$, where the equivalence relation is $(0,t) \sim (1,1-t)$.

Small digression. First, a generic "equivalence relation" us written a ~ b meaning that in some sense a is equivalent to b, though perhaps not exactly a = b. Given some ~ definition, you could define an "equivalence class" of a according to [a] = {b ϵ X | a ~ b}. But there is more to this as Wolfram shows:

Equivalence Relation

An equivalence relation on a set X is a subset of $X \times X$, i.e., a collection R of ordered pairs of elements of X, satisfying certain properties. Write "$x\ R\ y$" to mean (x, y) is an element of R, and we say "x is related to y," then the properties are

1. Reflexive: $a\ R\ a$ for all $a \in X$,

2. Symmetric: $a\ R\ b$ implies $b\ R\ a$ for all $a, b \in X$

3. Transitive: $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in X$,

where these three properties are completely independent. Other notations are often used to indicate a relation, e.g., $a \equiv b$ or $a \sim b$.

So given a set X and an equivalence relation ~, you can list off all pairs which are equivalent. Since not all pairs are going in general to be equivalent, the "equivalence relation on a set X" is in fact a subset of the direct product set X x X represent by a list of the equivalent pairs (x_1, x_2). The three properties shown are assumed to be valid as part of the ER definition I think.

Now from wiki we have this:

Quotient set [edit]

Main article: Quotient set

The set of all possible equivalence classes of X by ~, denoted $X/\sim := \{[x] \mid x \in X\}$, is the **quotient set** of X by ~. If X is a topological space, there is a natural way of transforming $X/\sim$ into a topological space; see quotient space for the details.

So fine, X/~ is the set of all equivalence classes of elements in X. If we are not interested in the details within the classes, we can focus just on the classes themselves, and X / ~ is the set of classes [x].

Now look back at the statement

A Mobius strip $:= [0,1] \times [0,1]/\sim$, where the equivalence relation is $(0,t) \sim (1,1-t)$.

Here X is the set of all points (u,v) that make up the direct product shown. We know that the ~ thing says that the edges are connected in this flipped over manner. Those two edges at the glue-together point are basically the same edge, but the fiber is flipped. So I sort of see his notation, but not fully.

If two elements of X are equivalent, here we would say $(u_1,v_1) \sim (u_2,v_2)$ and then the equivalence class would include ((u_1,v_1), (u_2,v_2)) in the sense of (x_1, x_2). Obviously the set of (u,v) points is doubly infinite in the unit square. Most of these points have no equivalence connection to other points. The only points which have equivalence are those of the form ($(0,v_1)$, $(1,v_2)$) and for these points, we have these equivalence pairs : ($(0,v_1)$, $(1,1-v_1)$). Thus we have only one equivalence class and this is it.

Author then talks a bit about groups which I am OK on. You have group elements acting on elements in a space, such as $\mathbf{x}' = R_z(g)\ \mathbf{x}$ as a 2D rotation, g = group element, G = group, R = matrix representation. Fine. Now write this abstractly as x' = gx . Now,

> These properties may look familiar to the categorically inclined. Every group G is a category with a single object whose morphisms are the elements of G.
>
> An "action" of G on an object in the category C is simply a functor $G \to C$.
>
> - If C is a group, this action is a group homomorphism.
> - If C is Vect, this action is a linear representation of G.

The category C is a set of only one object which is group G. Each G has its own category C.
You could regard the elements of group G as themselves forming a category, but we are the higher level where the group itself is the category. Recall that in this scenario an arrow is called a functor.
In a diagram for C the only meaningful arrow connection is this

$$G \xrightarrow{g} G$$

where the functor is labeled by some element of the group G. Well obviously I don't really have the right interpretation, but at least I know what they are generally talking about, I know a little about the language now.

Torsors.

Suppose you are given x and x', and you want to find g. One says that that solution g is the "**torsor**" for x and x', a word I have never heard before! It sort of torques one point into another in my mind. You can symbolically write g = x'/x . Well, g is the element of G which makes x' = gx be true, and you can call that a "torsor" if you want. It seems to me that set of torsors = group G.

Now in our Mobius example, you can perhaps view the fibers as being "twisted" as you go around the base circle, and you might think of this twist as a rotation about the circle axis or something like that. These rotations form a group, and I think that is our group of interest.

A principal-bundle is a bundle whose fibers are torsors.

Duffell says that a **principle bundle** has F = G, so here torsor is another name for group.

So, pick any two fibers from the Mobius strip. One is a rotation of the other by some G element g, so the two fibers are therefore "torsors" and we might write f' = gf, but this is a little hazy since we need to define these two fibers f and f' better. Fibers are the "orbit of G" ?

OK. I am fine with this language. You can think of the set of Mobius fibers as each being represented by a group element g which I think of as a rotation about the central line $R_z(g)$ or some such. But then we have the conflict that here we have a continuous G associated with Mobius, but we know on the other hand that the structure group is Z_2 which is discrete.

Let G be a Lie group, and F be our typical fiber. We have a faithful smooth group action:

$\rho : G \times F \to F$, which we can curry into

$\rho : G \to (F \to F)$ and rewrite as

$\rho : G \to \mathrm{Aut}(F)$

So the group G here is the fiber → fiber rotation of the Mobius, it is not the Z_2 thing. I am OK with the above list of three mappings, especially the middle one.

The first line just says for a given g and a given f on E, you get f' = gf which is another fiber on E. I can see that in the example, f can be any fiber on E, and g can be any element of G, and then the output f' will be another fiber on E. Curry into here means to bash or massage into something else. The second line says that you first pick a g in G, then you can consider all the F→F mappings under that G, maybe all fibers that are 10 degrees of rotation apart. This F → F maps the figure into itself so is an automorphism. So then for each element of g, you get each f in E mapping into some f' in E, so the whole surface E is mapped into itself.

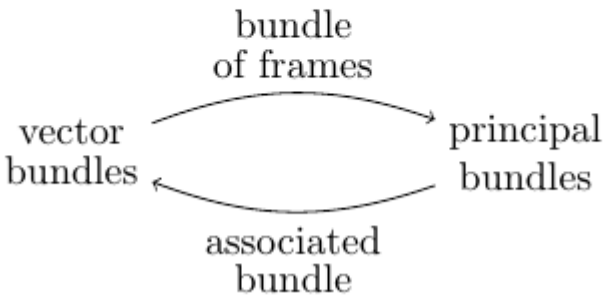
Next, consider this fine statement:

Let $(GL(E))_b$ be the set of orthonormal frames of the vector space E_b. Note that the set of all orthonormal frames is a right $\mathrm{O}(n)$-torsor.

Well this is in the Duffell language of a Frame Bundle. For point b in base B we have all these possible frames. In each frame, the basis vectors are the columns of the matrix GL(n,R; b) where k is the dimension of the vector space attached at b. As I noted in the Duffel presentation, this set of frames seems very similar to SO(n). Maybe E = $\{\mathbf{e_1},\mathbf{e_2}....\mathbf{e_n}\}$ in his notation.

Next we have this mysterious picture after the text,

In the most general case, $\mathbf{Aut}(F) = \mathbf{Diff}(F)$ (the group of diffeomorphisms), but we will shortly be working with vector bundles, for which F is a vector space and $\mathbf{Auto}(F) = GL(F)$.



The idea G: F→F is fine, and that is Aut(F), and you could require smoothness on everything so these are then diffeomorphisms from F to F.

Now in the picture: I know that a vector bundle has a vector space at each point b. Such a space has a set of possible "frames", yes. And those frames can be connected with G = SO(3). And F = G gives you a principle bundle, so somehow that is the path across the top of this picture. Somehow that strange "associated bundle" goes the other direction.

OK, lets try out his next claim

Given a vector bundle $E \xrightarrow{\pi} B$ with a vector space F as the fiber, we can construct a it's principal bundle $GL(E) \xrightarrow{\Pi} B$ by mapping each fiber E_b to the bundle of orthonormal frames over that fiber $GL(E_b)$.

I think this is the top path in the figure expressed in words. It certainly is strange.

She closes out his little web offering with this statement:

Fiber bundles are themselves merely the culmination of the centuries-long struggle to come fully and properly to grips with the idea of a multiple-valued function.

That function I think is $\pi^{-1}(b) = F$ where the single point b maps to multiple points F.

She mentions cohomology. But first, what is homology?

"Homology is a rigorous mathematical method for defining and categorizing **holes** in a shape. "

1. In algebra, which is a broad division of mathematics, **abstract algebra** (occasionally called modern algebra) is the study of **algebraic structures**. Algebraic structures include **groups**, **rings**, **fields**, **modules**, **vector spaces**, **lattices**, and algebra over a field. The term abstract algebra was coined in the early 20th century to distinguish this area of study from the other parts of algebra.

2. **Algebraic topology** is a branch of mathematics that uses tools from abstract algebra (see item 1 above) to study topological spaces. The basic goal is to find algebraic invariants that **classify** topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence.

3. In mathematics, **homotopy groups** are used in algebraic topology to classify topological spaces. The first and simplest homotopy group is the **fundamental group**, which records information about **loops** in a space. Intuitively, homotopy groups record information about the basic shape, or **holes**, of a topological space.

4. **Cohomology** is some kind of extension of homology theory (they lose me here)

This then is the end of this nice web page/

3. Torsors by John Baez 2010.

It is nice to avoid having to select an absolute origin or absolute reference frame orientation. Then you don't have to worry about such things when you move smoothly along a manifold and watch your tangent vectors rotate, for example. Torsor World has no origins or specific absolute frame alignment, it has only differences.

Anything that is defined only as a difference (energy of a particle in a field, voltage difference, wavefunction phase) is an example of a torsor. Since abs zero exists, T is not a torsor.

Math examples: indefinite integral has that additive constant, so a torsor.

Dots in a plane are torsors (forget origin, does not matter, Δ of two points is a vector)

" An affine space is a torsor for a vector space".

"Here's a famous example: the set of orthonormal frames at some point of a n-dimensional Riemannian manifold is not the group O(n), but it's an O(n)-torsor. You can take any frame and rotate it by an element of O(n); you can take two frames and work out their "difference", which is an element of O(n) - but the frames don't form a group. We can *pretend* the frames are the group O(n) - but only after we arbitrarily choose one frame and decree it to be the identity. Then every other frame is a rotated version of this one, so we can pretend it *is* a rotation! "

****

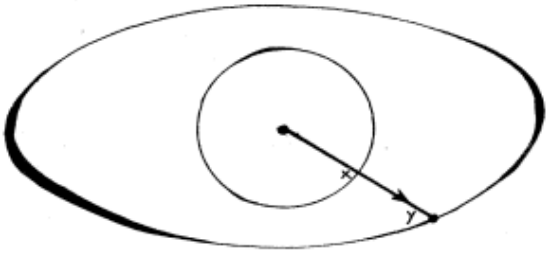
Notice the NOTATION used here to indicate two surfaces are homeomorphic:

Definition of homeomorphism.
We now come to the central idea that distinguishes topology from any other form of geometry. Two surfaces X and Y are said to be ***homeomorphic*** if there is a one-to-one continuous function between them. We often write this as

$$X \cong Y \text{ or } X \overset{\cong}{\to} Y.$$

We give some examples of homeomorphisms.

Example (i) A sphere is homeomorphic to an ellipsoid by radial projection $x \to y$.



4. Hainan Zhang MA thesis

This doc is a master's thesis done in 2014 at Kansas State (BS there in 2012). It seems to be a little voyage through this region of the House of Math, and is written therefore from a young and innocent perspective. He is not the boy who knew too much. It is quite readable. He is on Researchgate, but that is a medical person of the same name.

Chapter 1: Introduction to Fiber Bundles

He comments that the base B "carries the quotient topology determined by π", as if E/F = B in some sense. He does well on the transition function business. His notation is that h_α and h_β are two trivializations or homeomorphisms and then $\overline{\psi}_{\alpha\beta} \equiv h_\alpha^{-1} \circ h_\beta$. All his equations are good:

$$h_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha) \qquad (1.2)$$

$$h_\beta : U_\beta \times F \to \pi^{-1}(U_\beta) \qquad (1.3)$$

$$h_\alpha^{-1} \circ h_\beta : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F \qquad (1.4)$$

$$h_\alpha^{-1} \circ h_\beta(x, v) := \overline{\psi}_{\alpha\beta}(x, v) = (x, \psi_{\alpha\beta}(x)(v)) \qquad (1.5)$$

$$\psi_{\alpha\beta}(x) : F \to F \qquad (1.7)$$

He then shows that

Lemma 1. $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$.

which other authors just assume

1.1 Trivial Bundles. The usual thing, but then suddenly he tosses out a phrase

contractible CW-complex

as if any reader would know what this means. So I wander off to wiki:

Roughly speaking, a *CW complex* is made of basic building blocks called *cells*. The precise definition prescribes how the cells may be topologically *glued together*. The *C* stands for "closure-finite", and the *W* for "weak topology".

Wiki examples are helpful:

- The *standard CW structure* on the real numbers has 0-skeleton the integers $\mathbb{Z}$ and as 1-cells the intervals $\{[n, n+1] : n \in \mathbb{Z}\}$. Similarly, the standard CW structure on $\mathbb{R}^n$ has cubical cells that are products of the 0 and 1-cells from $\mathbb{R}$. This is the standard *cubic lattice* cell structure on $\mathbb{R}^n$.

So each unit interval is a cell, and the CW complex is the set of all such cells as shown. I think contractible means no holes to block you, Buck had something on this.

So Hainan claims that if B is one of these contractible CW complexes, then the only fiber bundle you can have over it is the trivial one.

1.2 Vector Bundles.

1.3 Principle Bundles.

Chapter 2. An Example of a Bundle of Frames

He talks about the matrix groups SO(2) and SO(3) which can act on frames.

An interesting example is $(\pi, E, B, F) = (\pi, SO_3, S^2, SO_2)$ where S^2 means the sphere.

$E = SO_3$	think ψ,θ,φ
$F = SO_2$	think ψ
$B = S^2$	think θ,φ surface

He then sort of studies this example in some detail in this chapter.

Chapter 3: Trivialization and Section

This entire chapter is this:

Lemma 3. *A principal bundle is trivial if and only if it admits a global section.*

The rest of his thesis ties in lots of famous phrases:

topological group
G-bundle
homotopy lifting property
Serre-fibration
paracompact
Hopf-fibration
cohains and cocycles
skeleton
Stiefel classes
Stiefel-Whitney classes
cohomology
Chern class
Cech cohomology
simplicial complex
sheafs

He ends with a bibliography.

So interesting. Here is a guy who probably know a lot about this field as an undergraduate at Kansas State, perhaps he was a student of Algebraic Topology ("pure methematics"). Then he did 2 years to get his MS and this thesis is the result. I think his name is common, like Tom Jones, and he was probably a Chinese visiting student and maybe he disappeared back home. But Linked In says he is now a student at Colombia, so he is still here.

5. No-author Chapter 2 on Bundles

This is an all-text document, no pictures, no identification of the author. It has lots of words, but still it is short on graphic examples. It does address a large hit parade of topics. I will do a few notes.

2.1 Vector Bundles.

Example 2.1 of a vector bundle is the "tangent bundle" TM.

$$TM = \bigcup_{x \in M} T_x M$$

We think of the spaces T_xM varying smoothly with x, not violently. For $M = B = S^2$, each T_xM is a plane, and we can imagine these glued to the sphere and varying smoothly, since the surface varies smoothly. The fiber is $F = T_xM$, the base is B = M, the total space E = TM is the sum or union or whatever of all

these T_xM spaces. Generally think of T_xM of dimension n so there are n axes. But then S^2 is 2D, so I guess E then has dimension 2n. We have $\pi : TM \rightarrow M$.

You could take M as the base, and R^m as your fiber, and construct space M x R^m and arrange to have π: $MxR^m \rightarrow M$ and this is a direct product space like the cylinder.

Imagine x in M, and (x,f(x)) in E, aas in this direct product example. A section maps x in M into a point in the fiber. For hairbrush, if your map takes you do the fibers a half inch over the surface, then that is a cross section of the hairs of the hairbrush, that is where the name section comes from. For the vector bundle example, the section is a map from x in M to a specific vector in each vector space, and that sounds to me like a vector field **V**(x) over x in M. He confirms this!

OK, I have started underlining a bit. The dual space of T_xM is called T_x^*M as is the space of linear functionals on T_xM, and this is called the cotangent space

Tensor bundles. This is very weird. At each point x you specific $[T_xM]^k$ k times in cross product, and you throw in also $[T_x^*M]^\ell$ ℓ times, and call the whole mess $T^k_\ell M$.

$$\left(\bigotimes_{j=1}^{\ell} T_x^*M\right) \otimes \left(\bigotimes_{j=1}^{k} T_xM\right)$$ at each x !!

AS I said, this is very strange.

which consists of multilinear maps

$$\underbrace{T_xM \times \ldots \times T_xM}_{\ell} \times \underbrace{T_x^*M \times \ldots \times T_x^*M}_{k} \rightarrow \mathbb{R}.$$

I am lost here! Is this all one multivariable functional? But some are * and some are not.

OK. Uncle! I have had enough. Spivak is in his short bibliography.