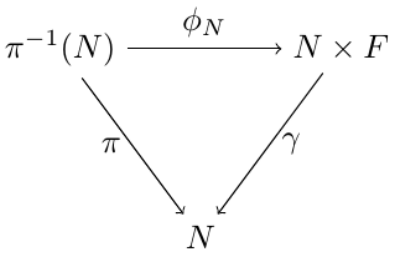
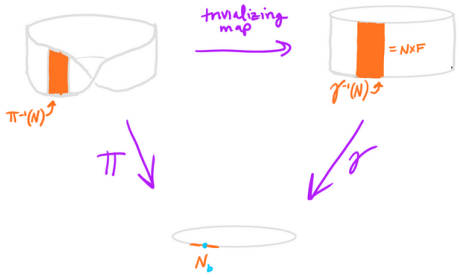
Mobius and Me PhL 9.5.15

Mobius with the cylinder forms the standard example for a fiber bundle. But there is one piece that I simply do not understand and it is this: Why under φ_i for some U_i does the fiber F have its points shuffled? Why is that necessary. If the points were not shuffled, then $\varphi_i(p,f) = (p,f)$ and then $\varphi_i = 1$ and this would also be true for φ_j and U_j and then $t_{ij} = 1$ and there is no transition group other than 1.

There must be some reason that causes $\varphi_i(p,f) = (p,f')$ with $f' \neq f$. I think the reason is that somehow this is necessary in order that the triangle picture be consistent.

For every $b \in B$ there is a neighborhood N of b s.t. the following diagram commutes.





So for the cylinder, is γ completely determined? If the cylinder points are $q = (\theta,z)$ then the only choice there would be $\gamma(\theta,z) = \theta$ for the projection function γ. No other projection is really possible. Also you would say $\gamma(\Delta\theta,z) = \Delta\theta$ where $\Delta\theta$ is a neighborhood of some point θ_1 in the base B. What do we mean then by saying $(\Delta\theta,z) = \gamma^{-1}(\Delta\theta)$? I guess this is really valid for every z in (0.1) .

So let's look at the consistency issue here:

$\pi(\Delta\theta,z) = \Delta\theta$ is the projection for Mobius for any z, where z lines along the fiber!

But if you want the real z in the R^3 space, you would find for some θ that z lies in some range (0,a) instead of (0,1) because at some θ the fiber is tipped. So if you start with some z at $\theta = 0$, you might have something like $z = z\cos(\theta/2)$ at θ of Mobius is evenly spread out. Then what is meant by $\pi(q) = p$ at some angle θ? You would think some website out of the 10^9 websites would discuss this in detail for the Mobius example. Are you supposed to put in 3D coordinates for **q** as if it were q in R^3?

Somewhere I did see some 3D equations for a Mobius. Wiki has this,

One way to represent the Möbius strip as a subset of three-dimensional Euclidean space is using the parametrization:

$$x(u,v) = \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u$$
$$y(u,v) = \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u$$
$$z(u,v) = \frac{v}{2}\sin\frac{u}{2}$$

where $0 \le u < 2\pi$ and $-1 \le v \le 1$. This creates a Möbius strip of width 1 whose center circle has radius 1, lies in the xy plane and is centered at (0, 0, 0). The parameter u runs around the strip while v moves from one edge to the other.

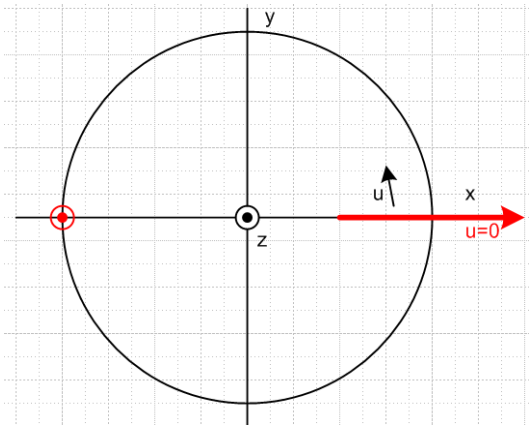
I think position on the fiber is coordinate v while θ is my angle. Now at u = θ = 0 we find

u = θ = 0 x = 1+v/2
y = 0
z = 0

So I guess this particular "fiber" runs from x = -1/2 to x = 1/2 so has length 1. So this is the way I would generally do, have the fiber be defined by f = [-1/2,+1/2] along the fiber. At u = 0 the local fiber lies in the x-plane. It will be vertical when θ = u = π (I think) let's check

θ = u = π x = -1
y = 0
z = v/2

θ = u = 2π x = 1 - v/2
y = 0
z = 0



I have drawn the two fibers, it all seems correct. So imagine this Mobius surface as a set of closely spaced fibers going around the top half, say. What then is the meaning of π(q) = p specifically for this example?

Does it mean $\pi(u,v) = u$? If so, then this map would project the center point of the fiber onto the circle. If this is it, then π is 100% specified for this example. It is just the first argument of the pair. Here v is distance along the fiber.

What are the equations for the corresponding cylinder?

x = cos(u)
y = sin(u)
z = v' = in the range [-1/2,+1/2]

OK, then γ says $\gamma(u,v') = u$ and it is also 100% defined.

The only question left is to find the function φ. For a super small neighborhood, our patches are only one fiber wide, let's say. All we really have to do is rotate a fiber on the Mobius at angle u upright. That would be something like this

(cylinder fiber) = $R_{\hat{u}}(u)$ (Mobius fiber)

I don't see any reason for this to alter the position of points along the fiber.

$\pi^{-1}(u)$ is our tilted fiber at angle u
our fiber on the cylinder is vertical.

φ: U x F $\rightarrow$ $\pi^{-1}(U)$

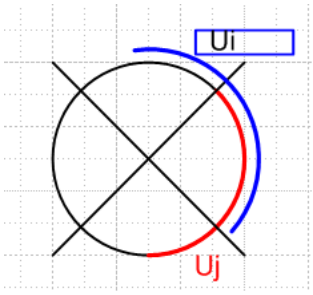
φ: u x F $\rightarrow$ $\pi^{-1}(u)$ $\varphi = R_{\hat{u}}(u)$ in some sense
up fiber tilted fiber

Conclusion: if you only consider a single fiber, you have $U_i = U_j = \delta$(1 fiber) and $t_{ij} = 1$ and you are never going to see the "issue" I am trying to uncover here. Why am I made to do this solo with a century and 10^9 people out there talking about this subject?

I guess the next step is to try this with some specific finite neighborhoods U_i and U_j which overlap. Such as

U_i $u = [-\pi/2, \pi]$
U_j $u = [-\pi, \pi/2]$

$U_i \cap U_j$ $u = [-\pi/2,\pi/2]$



OK, now what is the meaning of $\pi(q) = u$? Again it is just $\pi(u,v) = u$ where u,v parameterize the Mobius surface.

What now is the map φ_i: $U_i \times F \rightarrow \pi^{-1}(U_i)$

This map has to work for the entire U_i range of u. Well, for both these ranges I think $\varphi = R_{\hat{u}}(-u)$ will do the trick. In my phys/QM doc exp calc I have this result

$$R_{\hat{n}}(\theta) = \exp(-i\theta n \bullet J) = 1 + (1-C_\theta)\, T + S_\theta[-i(n \bullet J)]$$

where $\mathbf{n} = \hat{\mathbf{n}}$ is a unit vector defining the axis of rotation and

$$T = \begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix}$$ which of course is symmetric

$$(n \bullet J) = i\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \quad \Rightarrow \quad -i(n \bullet J) = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

In our case $\hat{u}$ lies in the plane of the circle so $n_3 = 0$ and we have

$$T = \begin{pmatrix} 1-n_1^2 & -n_1n_2 & 0 \\ -n_1n_2 & 1-n_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad -i(n \bullet J) = \begin{pmatrix} 0 & 0 & n_2 \\ 0 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

so I can just add up the pieces to write out $R_{\hat{n}}(\theta)$ explicitly as a 3x3 matrix. Now

$n_1 = n_x = -\sin(u)$
$n_2 = n_y = \cos(u)$

$$R_{\hat{n}}(-u) = \exp(+iun \bullet J) = 1 + (1-C_u)\, T - S_u[-i(n \bullet J)]$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\cos(u)) \begin{pmatrix} 1-n_1^2 & -n_1 n_2 & 0 \\ -n_1 n_2 & 1-n_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \sin(u) \begin{pmatrix} 0 & 0 & n_2 \\ 0 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\cos(u)) \begin{pmatrix} \cos^2(u) & \sin(u)\cos(u) & 0 \\ \sin(u)\cos(u) & \sin^2(u) & 0 \\ 0 & 0 & 1 \end{pmatrix} - \sin(u) \begin{pmatrix} 0 & 0 & \cos(u) \\ 0 & 0 & \sin(u) \\ -\cos(u) & -\sin(u) & 0 \end{pmatrix}$$

You would then apply this matrix to the tilted fiber at u and map it into the vertical fiber in the cylinder. But whatever this mess is, it is the same for every point in either the blue or red ranges, so I not learning much here.

Even after all the above work, I am not one iota closer to solving my little mystery!

Resume Sun 9.6.15.

Obviously I need outside help on this, I have a complete internal block. So back to the web.

wiki: https://en.wikipedia.org/wiki/Fiber_bundle

Definition seems clear, makes no mention of the transitions t_{ij} or the structure group G. The specific Mobius example there makes no mention either. Later wiki does have a section on structure groups where it presents things a little differently:

Structure groups and transition functions [edit]

Fiber bundles often come with a group of symmetries which describe the matching conditions between overlapping local trivialization charts. Specifically, let *G* be a topological group which acts continuously on the fiber space *F* on the left. We lose nothing if we require *G* to act effectively on *F* so that it may be thought of as a group of homeomorphisms of *F*. A ***G*-atlas** for the bundle (E, B, π, F) is a local trivialization such that for any two overlapping charts (U_i, φ_i) and (U_j, φ_j) the function

$$\varphi_i\varphi_j^{-1}\colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

is given by

$$\varphi_i\varphi_j^{-1}(x, \xi) = (x, t_{ij}(x)\xi)$$

where $t_{ij}\colon U_i \cap U_j \to G$ is a continuous map called a **transition function**. Two *G*-atlases are equivalent if their union is also a *G*-atlas. A ***G*-bundle** is a fiber bundle with an equivalence class of *G*-atlases. The group *G* is called the **structure group** of the bundle; the analogous term in physics is gauge group.

In this notation, x must be an element of B that lies in both U_i and U_j, and ξ must be a fiber F element. So under the combination $\varphi_i\varphi_j^{-1}$ element x does not move, but you get $\xi' = t_{ij}(x)\ \xi$. I presume this equation describes some possibly non-linear map so that $[t_{ij}(x)](\xi) = \xi'$. Again, this seems to imply that when you "go over and back", the fiber elements get shuffled for some reason. That is what I don't understand. Why can't they always remain unshuffled? Why should things be different for U_i compared to U_j? Those are my questions. Wiki continues:

In the smooth category, a *G*-bundle is a smooth fiber bundle where *G* is a Lie group and the corresponding action on *F* is smooth and the transition functions are all smooth maps.

The transition functions t_{ij} satisfy the following conditions

1. $t_{ii}(x) = 1$
2. $t_{ij}(x) = t_{ji}(x)^{-1}$
3. $t_{ik}(x) = t_{ij}(x)t_{jk}(x).$

The third condition applies on triple overlaps $U_i \cap U_j \cap U_k$ and is called the **cocycle condition** (see Čech cohomology). The importance of this is that the transition functions determine the fiber bundle (if one assumes the Čech cocycle condition).

A principal *G*-bundle is a *G*-bundle where the fiber *F* is a principal homogeneous space for the left action of *G* itself (equivalently, one can specify that the action of *G* on the fiber *F* is free and transitive). In this case, it is often a matter of convenience to identify *F* with *G* and so obtain a (right) action of *G* on the principal bundle.

The last paragraph means nothing to me. But you see some references for the meaning of the t_{ij} items. The cohomology wiki page is another mystery. Has U_i forming an open cover of topo space X. That takes me too far in the "abstraction" direction. I cannot just keep extending forever 100 levels from the topic and hand, it has to be cut off at some good place. I just want to see the Mobius example! I search on

mobius "fiber bundle" group transition

First hit is wiki, second is Duffell. He says:

$$\Phi_i : U_i \times F \to \pi^{-1}(U_i)$$

Now, it is possible that we could have chosen a different U_k about p, thus a different map Φ_k associating it with a different point (p, f″) in $U_k \times F$. This is fine, but we need to understand the relationship between f and f″. In other words, to distinguish the bundle properly, we need to know about all possible choices of fiber parameterization. In the case of the cylinder, there was only one fiber parameterization, because the space was globally a product manifold. In the case of the möbius band, there are two possible parameterizations, and we can make the transformation explicit by

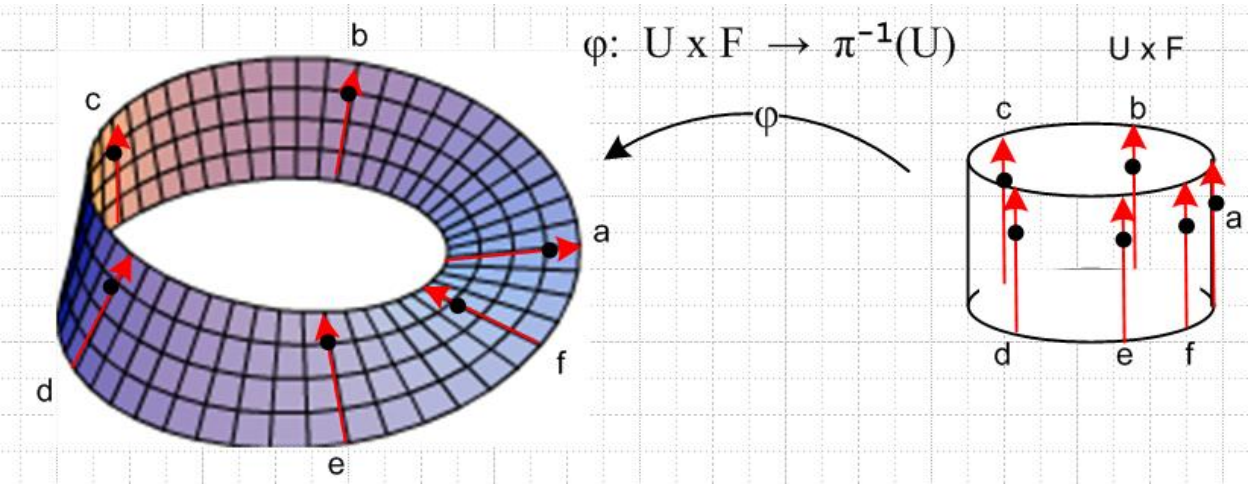
$$f' = 1 - f \qquad (4.4)$$

Neither parameterization f nor f″ works globally, but we can cover the circle with two overlapping segments, and choose one parameterization for one segment, and the opposite for the other segment.

Parse his first sentence. The object (p,f') is an argument of Φ_k where I guess $\Phi_k(p,f') = q$ in E, whereas $\Phi_i(p,f) = q$, so we have a new "parameterization k of the fiber". This is like a change of coordinate system, yes, I get that idea in general terms. So when we change from U_i to U_k, we are not JUST changing the region in B, we are also changing the coordinate system, maybe I missed that point. Suppose U_i labels fiber points in the usual manner, but suppose U_k labels them in reverse order, just to pick a possible coordinate system change on the fiber. Then φ_i does no flip, φ_k does a flip, and then t_{ij} does a flip. How many ways can you pick a coordinate system for real on [0,1] ? You could do a respeeding of course. Then in my example you end up with t_{ij} = a respeed of [0, 1]. There are an infinite number of possible respeeds, so you would think group G would be continuous. Duffell claims however that there

are only two possibly parameterizations, f' = f and f' = 1-f, so he does not include respeeds. Looking at his finally sentence. question: how would you pick U_i and U_k so f' = f on i and f' = 1-f on U_k ?

Here is a better picture showing some fibers going around:



The black dot on the a fiber marks f = +1/2. Suppose point a is θ = 0.

φ: U x F $\rightarrow$ $\pi^{-1}(U)$ $\qquad$ $\varphi(\theta, f) = q$

Then for example

$\varphi(\theta_a,\ z = 1/2) = \mathbf{q_a} = (x,y,z) = (1 + 1/2,0,0)$
$\varphi(\theta_b,\ z = 1/2) = \mathbf{q_b} = (x,y,z) =$ (some values we could figure out)
$\varphi(\theta_c,\ z = 1/2) = \mathbf{q_c}$
...
$\varphi(\theta_f,\ z = 1/2) = \mathbf{q_f}$
$\varphi(\theta_{a'},\ z = 1/2) = \mathbf{q_{a'}} = (x,y,z) = (1 - 1/2,0,0)$ $\qquad$ $\theta_{a'} = \theta_a + 2\pi$

Here I show q as points in R^3. But suppose we use parameters u and v? Well, u is the angle in restricted range $0 \le u < 2\pi$. For this range, v is measured + in the direction of the arrow. So even at point f we would say that v = +1/2. We cannot "get to" point a' in this parameterization.

OK, here is perhaps the point: In the above, if we restrict to U_i having θ=u in $0 \le \theta < 2\pi$, then the three wiki equations above for the Mobius provide a viable coordinate system (u,v), aka parameterization, for the Mobius. This particular parameterization has a *discontinuity* at θ = u = 0. Things are not continuous and smooth at that discontinuity. The single set U fails to "cover" the Mobius surface. If the parametrization had worked say for u in the range (-30 degrees, +400 degrees) [as it does for the cylinder], then this single U would "cover" the base set B.

Once again, here are the three Mobius equations for $0 \le u < 2\pi$:

x(u,v) = [1 + (v/2)cos(u/2)] cos(u)
y(u,v) = [1 + (v/2)cos(u/2)] sin(u)
z(u,v) = (v/2)sin(u/2)] OK for $0 \le u < 2\pi$ and $-1 \le v \le 1$

How would we write a different param for say $-\pi \le w < \pi$? Try taking w = u-π. Then

x'(w,v) = [-1 + (v/2)sin(w/2)] cos(w)
y'(w,v) = [-1 + (v/2)sin(w/2)] sin(w)
z'(w,v) = (v/2)cos(w/2)] OK for $-\pi \le w < \pi$ and $-1 \le v \le 1$

where I have made use of these facts

cos u/2 = cos([w+π]/2 = cos[w/2 + π/2] = -sin(w/2)
sin u/2 = sin[w+π]/2 =sin[w/2 + π/2] = + cos(w/2)

cos u = cos(w+π) = -cosw
sinu = cos(w+π) = -sinw

Here is a Maple plot of the first coordinate system for u = 0 to 2π-0.5:

```
restart; with(plots):
x := (1 + (v/2)*cos(u/2))*cos(u);
```

$$x := \left(1 + \frac{1}{2} v \cos\left(\frac{1}{2} u\right)\right) \cos(u)$$

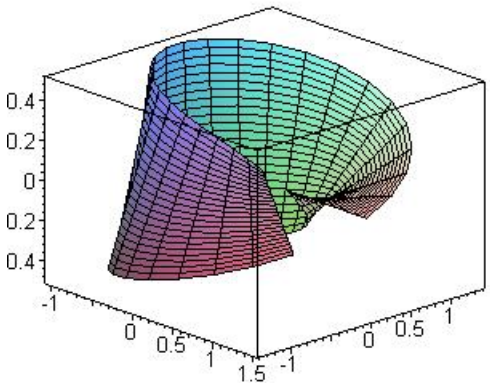
```
y := (1 + (v/2)*cos(u/2))*sin(u);
```

$$y := \left(1 + \frac{1}{2} v \cos\left(\frac{1}{2} u\right)\right) \sin(u)$$

```
z := (v/2)*sin(u/2);
```

$$z := \frac{1}{2} v \sin\left(\frac{1}{2} u\right)$$

```
plot3d([x,y,z],u=0..2*Pi-.5,v=-1..1);
```



Here you see the starting horizontal segment "a" which goes around and ends up trying to be flipped at θ = 2π, but I stop it a little short. So the seam here is at the u = 0/2π location. Here is plot using the second parameterization

```
restart: with(plots):
xp := (-1 + (v/2)*sin(w/2))*cos(w);
```

$$xp := \left(-1 + \frac{1}{2} v \sin\left(\frac{1}{2} w\right)\right) \cos(w)$$

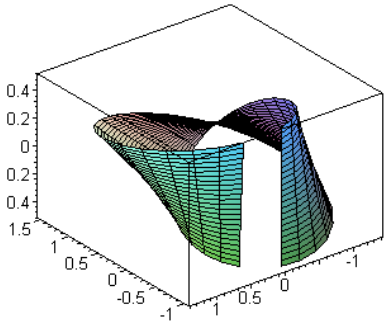
```
yp := (-1 + (v/2)*sin(w/2))*sin(w);
```

$$yp := \left(-1 + \frac{1}{2} v \sin\left(\frac{1}{2} w\right)\right) \sin(w)$$

```
zp := (v/2)*cos(w/2);
```

$$zp := \frac{1}{2} v \cos\left(\frac{1}{2} w\right)$$

```
plot3d([xp,yp,zp],w=0..2*Pi-.5,v=-1..1);
```



and now the seam is at the vertical part of the strip, which was u = π in the previous param which is now w = 0.

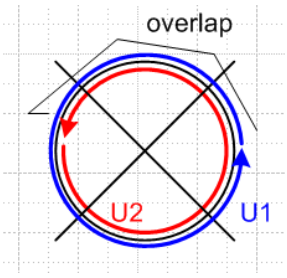
Fact: There exists no single parameterization of the Mobius strip which works for an open set of u which includes the entire range 0,2π.

Fact: We can *cover* the entire range 0,2π with these *two* different parametrizations:

x(u,v) = [1 + (v/2)cos(u/2)] cos(u)
y(u,v) = [1 + (v/2)cos(u/2)] sin(u)
z(u,v) = (v/2)sin(u/2)] $0 \le u < 2\pi$ and $-1 \le v \le 1$ U_1

x'(u,v) = [-1 + (v/2)sin(u/2)] cos(u)
y'(u,v) = [-1 + (v/2)sin(u/2)] sin(u)
z'(u,v) = (v/2)cos(u/2)] $-\pi \le u < \pi$ and $-1 \le v \le 1$ U_2

Notice that $U_1 \cap U_2 = 0 \le u < \pi$



Question: For these two different coordinate systems, what are the two φ_i mappings?

$\varphi_1(u, v) = (x,y,z) = (x(u,v),y(u,v),z(u,v))$ $\qquad 0 \le u < 2\pi$

$= ([1 + (v/2)\cos(u/2)]\cos(u), [1 + (v/2)\cos(u/2)]\sin(u), (v/2)\sin(u/2))$

$\varphi_2(u, v) = (x',y',z') = (x'(u,v),y'(u,v),z'(u,v))$ $\qquad -\pi \le u < \pi$

$= ([-1 + (v/2)\sin(u/2)]\cos(u), [-1 + (v/2)\sin(u/2)]\sin(u), (v/2)\cos(u/2))$

So I am going with the theory that in φ: U x F $\rightarrow$ $\pi^{-1}(U)$, the right side represents a point in R^3 which is a point on surface E. Yes, these are two *different* mappings.

Question: What is $\varphi_1^{-1}\varphi_2$?

Well, I need to figure out this;

$(u,v) = \varphi_1^{-1}(x,y,z)$

I see that y/x = tan(u) which says $u = \tan^{-1}(y/x)$ so we know u. How to get v? We know

$z = (v/2)\sin(u/2)$

Now in doc "maple and half angle" I have shown that sin(u/2) is given by

$$f := \frac{1}{2}\sqrt{2}\sqrt{\frac{\sqrt{1+\frac{y^2}{x^2}}+1}{\sqrt{1+\frac{y^2}{x^2}}}}$$

as long as $0 \le u \le \pi/2$, in which case all radicals are positive. So then

$$z = (1/\sqrt{2}\,)\,(v/2)\sqrt{\frac{1+\sqrt{1+x^2/y^2}}{\sqrt{1+x^2/y^2}}}$$

and therefore our solution is this:

$$(u,v) = \varphi_1^{-1}(x,y,z)$$

$$u = \tan^{-1}(y/x) \;\; = u(x,y,z)$$

$$v = 2\sqrt{2}\; z \sqrt{\frac{\sqrt{1+x^2/y^2}}{1+\sqrt{1+x^2/y^2}}} \;\; = v(x,y,z)$$

Meanwhile we know that

$$(x,y,z) = \varphi_2(u,v)$$

or

$$x(u,v) = [-1 + (v/2)\sin(u/2)]\cos(u)$$
$$y(u,v) = [-1 + (v/2)\sin(u/2)]\sin(u)$$
$$z(u,v) = (v/2)\cos(u/2)] \qquad -\pi \le u < \pi \;\text{ and }\; -1 \le v \le 1 \qquad U_2$$

So we could insert the above expressions for u and v into this to get

$$\varphi_1^{-1}\varphi_2 \;\; = \text{big mess}$$

My main point is that it can be done (the two distinct params) and there is some answer (for t_{12}) and the result will be t_{12} and we will not end up with $t_{12} = 1$!

Do I know anything about $(t_{12})^2$?

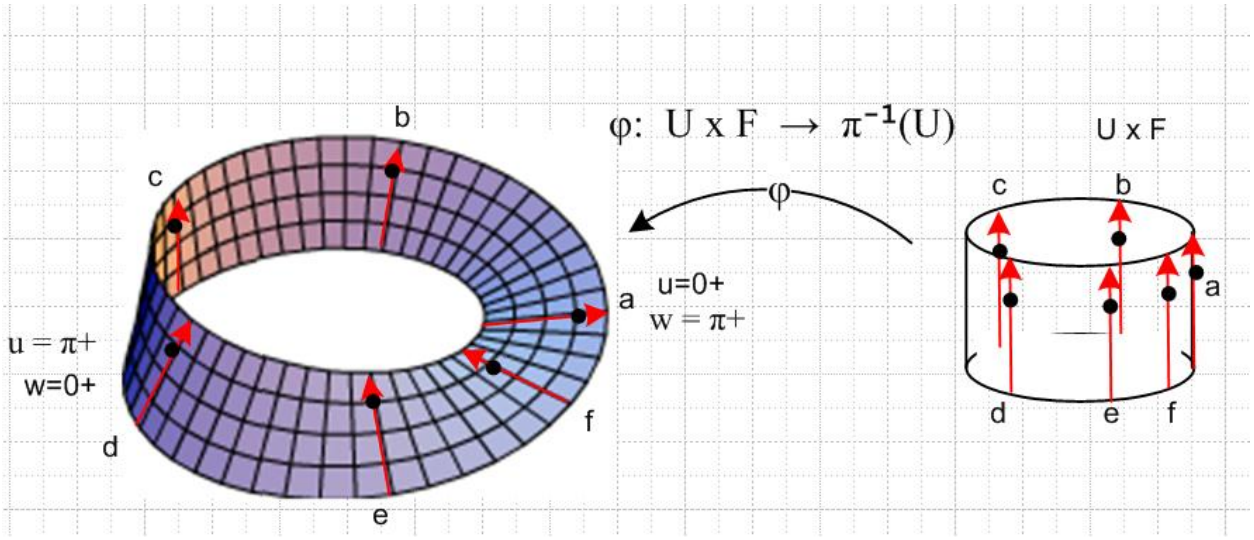
$$(t_{12})^2 = t_{12}\, t_{12} = \;\varphi_1^{-1}\varphi_2\, \varphi_1^{-1}\varphi_2 \;\; = \text{not known.}$$

Task: With our two explicit parametrizations above, let's try to trace a single point in terms of our pictures above. Here are those two params, and the overlap region is $0 \le u < \pi$

$$x(u,v) = [1 + (v/2)\cos(u/2)]\cos(u)$$
$$y(u,v) = [1 + (v/2)\cos(u/2)]\sin(u)$$
$$z(u,v) = (v/2)\sin(u/2)] \qquad 0 \le u < 2\pi \;\text{ and }\; -1 \le v \le 1 \qquad U_1$$

$$x'(u,v) = [-1 + (v/2)\sin(u/2)]\cos(u)$$
$$y'(u,v) = [-1 + (v/2)\sin(u/2)]\sin(u)$$
$$z'(u,v) = (v/2)\cos(u/2)] \qquad -\pi \le u < \pi \;\text{ and }\; -1 \le v \le 1 \qquad U_2$$

Consider again our mapping:



We are cartographers and we have two different "maps" of the surface of the Mobius. Map 1 uses coordinates (u,v) where u = 0+ is at point a on the Mobius (the flat) and the cylinder. On the Mobius, this a arrow points to the right, because that is how we lay out the grid in that area. The arrow is at u = 0+ and we define v = 0 at inner edge of the Mobius. This is the way our φ_1 explicitly works.

Meanwhile, Map 2 uses coordinates (w,g) where w = 0+ is near d on the left (the cliff), and the arrow points up here (and the matching arrow on the cylinder at d also points up). Again, that is how Map 2 explicitly works. So in the Map 2 coordinate system, where is the black dot on arrow a? The answer is w = π+ and g = -1/2 ! In Map 2, the arrow at point a is like that at point f, it points to the left. So the black dot on a left-pointing arrow a is at v = -1/2.

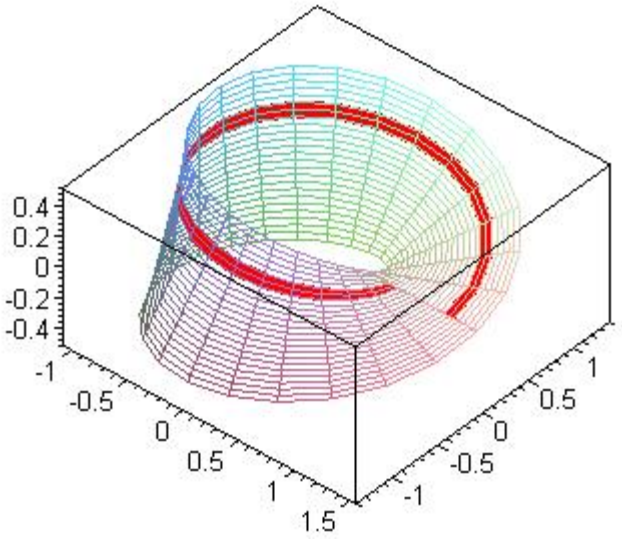
More generally suppose we start with point (u=0+, f) and move to the left and this is then some point on arrow a. When we express this location in φ_2 map coordinates, it is at (w=π+, 1-f). There is that 1-f that Duffell keeps talking about.

Now take that flipped arrow a and using φ_2, map that back to the cylinder, and it ends up exactly as the up arrow at point a, but our original point that was at (0,f) in cyl space is now at (0,1-f).

I think I am onto something here, pause for a while then test it.

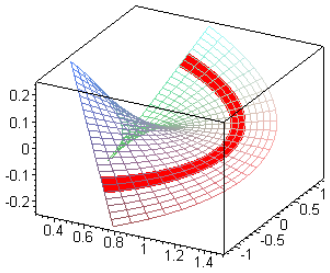
Question: What happens if we take U_1 and U_2 to be small overlapping regions? In that case, we might take φ_1 to be the same mapping system for both regions, and then $\varphi_1 = \varphi_2$ and $t_{12} = 1$. So for THIS pair of U_1 and U_2 we have $t_{12} = 1$.

Question: Why can't we get an open cover of the Mobius with such small regions all using φ_1 ? OK, let's try this using φ_1 everywhere. In the following plot, the wireframe is a map for u in (0,2π) and v in (-1,1). The red portion is for u in the same (0,2π) but v in (.4,.6):



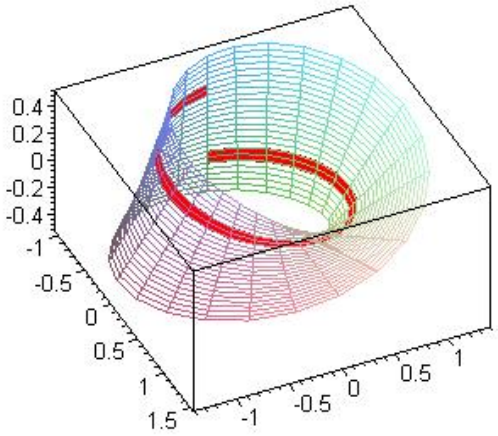
The angle u = 0/2π is toward the lower right where one sees the discontinuity in the red band. Now we can use this arrangement for lots of overlapping U_i regions (all using φ_1) where none of these regions straddles the u = 0 boundary. All of these overlapping regions are "fine" and $t_{ij} = +1$ for any pair of them since we are using the same φ_1.

But we need our regions to be an open *cover* for the Mobius surface, and that means to complete the above picture, we have to add a region U_k which straddles the u = 0 boundary. If we do this with the same function φ_1, here is what we get (using φ_1 for negative u which is out of its range; in this map we use φ_1 and let u be in (-1,1) radians:



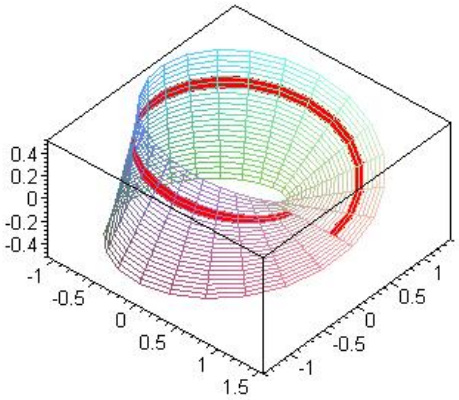
The problem of course is that the local red band for this U_k region does not match the global red band shown in the previous plot for the u < 0 portion of the plot. So we then have a contradiction in the region where U_k overlaps the region to its left. Thus, we cannot obtain a full cover using little regions U_i all having φ_1 as the coordinate system.

Were we to repeat the above using φ_2, we end up with a similar problem, which this figure shows,

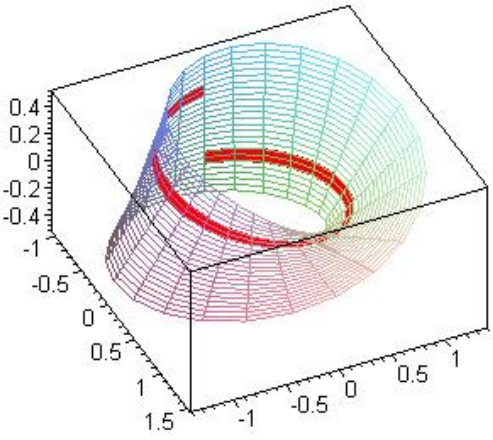


The problem formerly "in the plane" has now moved to the vertical cliff.

So if we use our large covers U_1 with φ_1 and U_2 with φ_2, how does this solve the problem? Here are those two covers:



φ_1 region



φ_2 region

Here are the key facts

(1) within each region, everything is smooth (although the red band is not the same)

(2) each region shows that if you go around the horn, you get f → 1-f

(3) the two regions cover the entire Mobius with plenty of overlap between them

(4) for both of these maps, the corresponding band on the cylinder is the same, and is on upper half of cylinder. We have for example $\varphi_1 : U \times F \rightarrow \pi^{-1}(U)$ where this upper band of cylinder in U x F maps into $\pi^{-1}(U)$ which is the red region on the left above.

Question: Exactly what is the overlap region $U_1 \cap U_2$ for these two maps? Consider our two maps using the same variables u and v:

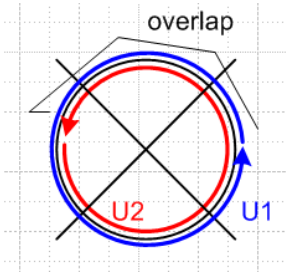
$x(u,v) = [1 + (v/2)\cos(u/2)]\cos(u)$
$y(u,v) = [1 + (v/2)\cos(u/2)]\sin(u)$
$z(u,v) = (v/2)\sin(u/2)]$ $\quad 0 \le u < 2\pi$ and $-1 \le v \le 1$ $\quad U_1$

x'(u,v) = [-1 + (v/2)sin(u/2)] cos(u)
y'(u,v) = [-1 + (v/2)sin(u/2)] sin(u)
z'(u,v) = (v/2)cos(u/2)] $-\pi \le u < \pi$ and $-1 \le v \le 1$ U_2

When expressed in the same variables, the overlap region is ONLY this

$U_1 \cap U_2 = [\, 0 \le u \le \pi \,] \times [-1 \le v \le 1]$

as indicated by this picture from above



The overlap region is the first half of the blue U_1 region and the second half of the red U_2 region.

What is supposed to happen in an overlap region? We have

$$\varphi_i\varphi_j^{-1}: (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

so in our case this is:

$\varphi_1\varphi_2^{-1} : (U_1 \cap U_2) \times F \to (U_1 \cap U_2) \times F$

This does not say $\varphi_1\varphi_2^{-1} = 1$. It says that the map $\varphi_1\varphi_2^{-1}$ maps from "the overlap region" to "the overlap region". Now if we take a piece of the overlap region (the right of the two figures above), for that piece of the overlap we have (u,f) → (u,1-f) so we do NOT have $\varphi_1\varphi_2^{-1} = 1$.

Now I quote another equation

$$\varphi_i\varphi_j^{-1}(x,\xi) = (x, t_{ij}(x)\xi)$$

which I translate to say

$\varphi_1\varphi_2^{-1}$ (u,f) = (u, t_{12}(u)f)

where f = v = a point in the fiber, for u in an overlap region. We then have for any u in our overlap region,

$t_{12}(u)f = 1\text{-}f$ for u anywhere in $0 \le u < \pi$ (that is to say, u lies in $U_1 \cap U_2$)

Now where is the "group"? Suppose we have

$g_1 f = f$
$g_2 f = 1\text{-}f$

Then it seems clear that

$g_1 g_2 = g_2 = g_2 g_1$
$g_1 g_1 = 1 = g_1$
$g_2 g_2 = g_1 = 1$
$g_1^{-1} = g_1 = 1$
$g_2^{-1} = g_2$: $g_2^{-1} g_2 f = g_2^{-1}(1\text{-}f) = f$
: $g_2 \; g_2 f = g_2 \; (1\text{-}f) = [\, 1 - (1\text{-}f)] = f$

So abstractly, if we had g_1 and g_2 defined as above, we have a little group of two elements. You could call the two elements $\{ 1, g_2\}$ and it is true that $g_2^2 = 1$. Since all elements of the group can be obtained by doing powers of g_2, this is a cyclic group according to Galois doc. The cyclic group of order 2 has a name, it is called Z_2.

Now back to the Mobius. How is this Z_2 related to Mobius? It is not obvious to me at this point. We have

$t_{12}(u)f = 1\text{-}f$ $t_{12}(u) = g_2$

This does not really fit with what I have read, so go read again! Wiki says this

$$\varphi_i \varphi_j^{-1}(x, \xi) = (x, t_{ij}(x)\xi) \qquad (*)$$

so it is wrong to say that $t_{ij} = \varphi_i \varphi_j^{-1}$, though that is exactly what Duffell says:

$$t_{ij} = \Phi_i \circ \Phi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

The wiki notation makes if more clear that t_{ij} acts on the fiber coordinate which they call ξ and which I call v or f. The better statement I think is the following, which Duffell to his credit does say

$$t_{ij}(p) : F \to F$$

Well, you could say that $t_{ij} = \varphi_i \varphi^{-1}$ if you ignore the first argument but keep in mind that it must be in the intersection region. Wiki would say (and it is implied by the quote * above)

$t_{ij}(x)$: $F \to F$ // this agrees with idea $t_{12}(u)f = 1\text{-}f$

Now let's look at the next thing wiki says:

1. $t_{ii}(x) = 1$
2. $t_{ij}(x) = t_{ji}(x)^{-1}$
3. $t_{ik}(x) = t_{ij}(x)t_{jk}(x).$

where x = some fixed value of u. Now I always have $t_{ii}(x) = 1$, since $\varphi_i^{-1}\varphi_i = 1$. And item 2 also seems clear as well, I could prove if from the wiki definition of t_{ij}.

The problem is the third item above! It only applies to a triple overlap region.

Experiment: OK break U_1 into $U_3 + U_4$ within the overlap region. Then we have U_2, U_3 and U_4. But U_3 and U_4 have the same coordinates φ_1. So any $t_{34}(u) = 1$ for u in the triple overlap. Then we get

$$t_{24} = t_{23}t_{34} = t_{23}$$

and we don't learn much. Any product is just a single t_{ij} in this experiment.

Here are names of some small groups

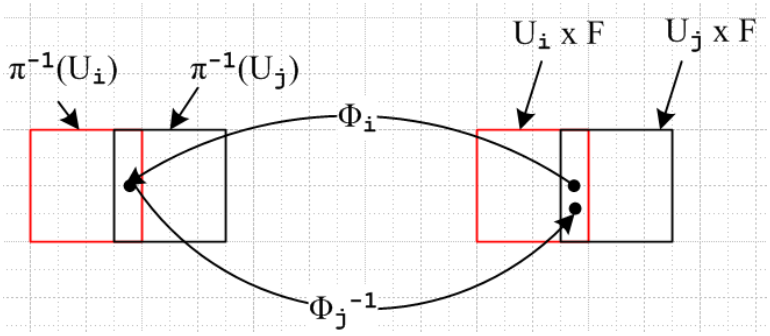
List of all abelian groups up to order 30

Order	G_o^i	Group	Subgroups	Cycle graph	Properties
1 [1]	G_1^1	Z_1[2] = S_1 = A_2	–		Trivial group. Cyclic, alternating group, symmetric group. Elementary.
2 [3]	G_2^1	Z_2[4] = S_2 = Dih_1	–		simple, the smallest non-trivial group. Symmetric group. Cyclic. Elementary.
3 [5]	G_3^1	Z_3[6] = A_3	–		simple. Alternating group. Cyclic. Elementary.

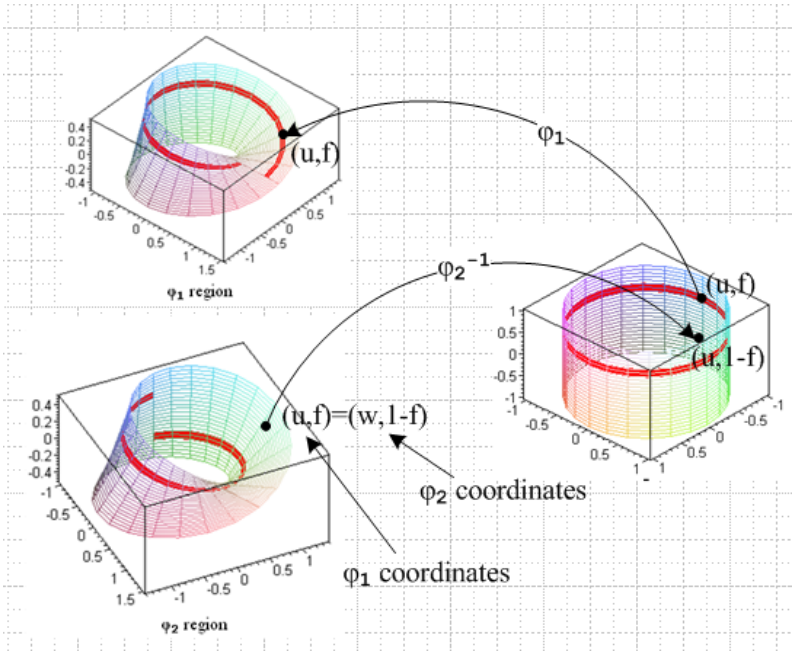
So I would say that the group $\{1, t_{12}(u)\}$ is Z_2 for u being any point in the overlap region.

Wiki is not clear that the structure group depends on x. Also, my two groups are discrete, so I have no example of a case where the structure group would be continuous or Lie group.

By the way, consider an earlier picture:



Maybe I can clarify the above picture with my 3D pictures. In the following picture, for a particular patch in the overlap region, we "go over" with φ_1 from cylinder to Mobius, and then we" come back" with φ_2^{-1} and we end up at a different place (namely 1-f) in the cylinder. So this is the situation where $t_{12}f = 1-f$ and you have the group Z_2: That is to say, $t_{11}\, f = f$ and $t_{12}\, f = 1-f$ so in Z_2 we have $t_{11} = 1$ and we have $t_{12} = g$ and we have $g^2 = 1$, so the group is $G = \{t_{11}= 1, t_{12}= g\}$ where $g^2= 1$. I guess it is really a mapping onto the group Z_2 because we really have $G = \{t_{11}= t_{22} = 1, t_{12}= t_{21} = g\}$ and $Z_2 = \{1,g\}$.



Conclusion: I think finally I understand how the fiber bundle theory applies to the Mobius strip. In this example we have

E = surface of the Mobius strip (surface has dimension 2 in R^3)
B = circle with coordinate u (wiki calls x) (circle has dimension 1 in R^3)
F = [-1,1] of reals, with coordinate f (or v) (fiber has dimension 1 in R^3)
$G = Z_2$ $\quad$ $G = \{t_{11}= 1, t_{12}= g;\ g^2 = 1\ \}$
$t_{12}(u)$ = only transition function for my two large overlapping U_1 and U_2

You can think of the Mobius surface as consisting of a "bundle of fibers" which tip as you go around and as you go all the way around, they flip completely upside down.

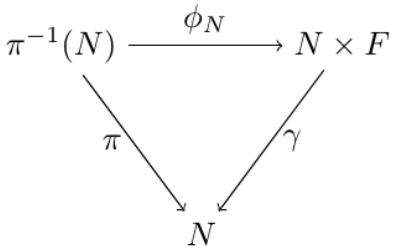
Question: What do you call a Mobius strip which has, instead of 1/2 turn, n/2 turns n = 2,3,4... ?

Well, people talk here and there about such things. Consider the case of the full twist. I was unable to produce the right equation. But I now see that if I make a cylinder of paper and then put in a full twist, this is the "Mobius with a full twist". This would have group Z_0 and is topologically exactly the same as the cylinder. So a Mobius with n full turns is a cylinder, and a Mobius with a n + 1/2 turns is the same as a Mobius strip (with just a half turn).

The whole claim is that you can sort of classify things as shown above.

I think I can now continue in my various readings on fiber bundles, having *to some extent* surmounted the obstacle of the meaning of that t_{ij} structure group. At least I see the meaning for the Mobius strip.

Question: So how does the Mobius example fulfill this generic fiber bundle picture? The NxF top right is a portion of the cylinder, while $\pi^{-1}(N)$ is a portion of the Mobius strip, N is the matching portion of the base circle B.



The picture is valid for any neighborhood N on the circle B. In particular:

$N_1 = [0,2\pi)$ for u v ϵ F

φ_{N1}^{-1} : $N_1 \times F \rightarrow \pi^{-1}(N_1)$ $\varphi_{N1}^{-1}(u,v) = \mathbf{r}$ = (x,y,z) where

x(u,v) = [1 + (v/2)cos(u/2)] cos(u)
y(u,v) = [1 + (v/2)cos(u/2)] sin(u)
z(u,v) = (v/2)sin(u/2)] OK for $0 \leq u < 2\pi$ and $-1 \leq v \leq 1$

π is the *projection* of the Mobius strip section down to the circle range N_1
$\pi^{-1}(N_1)$ is the large portion of the Mobius surface whose projection if N_1.
The function π is many-to-one and thus has no inverse, so π^{-1} indicates a region, not an inverse function.

The large region $N_1 \times F$ lies on the cylinder projects via γ to N_1 on the circle.

We have a different neighborhood N_2 which is $[-\pi,\pi]$ for u and we just replace N1 by N2 in the above, and we change the three equations accordingly. Our two neighborhoods form an open cover for the circle and their inverse projections form an open cover for the Mobius strip. The transformation t_{12} takes v to 1-v and the structure group is $G = Z_2$ as shown above.

The φ_i functions are called "local trivializations" for some reason unknown to me.