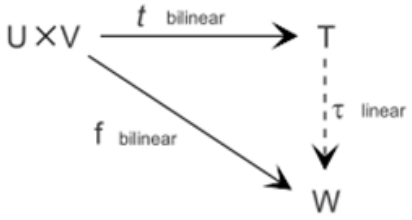
Meta Notes on Tensor Products PhL 1.19.15

First of all, I think my tensor doc makes use of the outer product, which is a special case of a tensor product. I think the term direct product there is incorrect, but people do use it. I review the tensor doc sections on such products in the raw notes.

There seem to be two major theoretical approaches to the Tensor Product.

1. The first approach starts with a free vector space, defines bilinear-like equivalence classes, and then concludes that V⊗W= F(VxW)/N where F is the free vector space defined over the Cartesian product Vx W, and N is the subspace of elements in F(VxW) which are declared equivalent to 0. This basically has the effect of imposing what I call the bilinearity relations on the ⊗ operator.

2. The second approach starts off with bilinear functions in a category triangle diagram,



Since a unique and linear τ can be constructed to satisfy the triangle (that is, to make it "commute"), the pair (T,t) is a universal pair, and then by definition the space T is the tensor product space T = U⊗V. This approach emphasizes that the tensor product is a more or less unique construct, up to isomorphism via the linear function τ. One has

$t : UxV \rightarrow U\otimes V \quad so \quad t(u,v) \rightarrow u\otimes v$

The bilinearity rules then just move from function t to the tensor product u⊗v. For example

$t(\alpha u+\beta u',v) = \alpha\, t(u,v) + \beta t(u',v)$

maps into

$(\alpha u + \beta u')\otimes v = \alpha\, u\otimes v + \beta\, u'\otimes v$

3. Both approaches arrive at the same notion of a tensor product. The main aspects are bilinearity and uniqueness of the tensor product.

4. One can generalize to a tensor product A⊗B⊗C..... . In this case, the triangle diagram has at its left vertex A x B x C.... . The bilinearity rules become multilinearity rules.

5. The **outer product** is a tensor product in which an extra ingredient is added, resulting in no conflict with the tensor product structure as outlined above. One writes for example,

$(\mathbf{A}\otimes\mathbf{B}\otimes\mathbf{C}\ \ldots)^{abc\cdots} \equiv A^a B^b C^c \ldots\ldots$

$(M\otimes\mathbf{N})_{ab,AB} \equiv M_{aA}N_{bB}$

$(M\otimes\mathbf{N})_{ab,AB,\alpha\beta} = M_{aA\alpha}N_{bB\beta}$

In all these cases, the expressions on the right have the correct multilinearity form. For example

$([\alpha M+\beta M']\otimes\mathbf{N})_{ab,AB,\alpha\beta} = [\alpha M+\beta M']_{aA\alpha}N_{bB\beta} = \alpha\, M_{aA\alpha}N_{bB\beta} + \beta\, M'_{aA\alpha}N_{bB\beta}$

6. One can write a most general element of V⊗V in this manner

$T \equiv \Sigma_{ij}T^{ij}\, e_i\otimes e_j$

but then with the extra ingredient, we can write

$T^{IJ} \equiv \Sigma_{ij}T^{ij}\,[e_i\otimes e_j]^{IJ} = \Sigma_{ij}T^{ij}\,(e_i)^I\,(e_j)^J = \Sigma_{ij}\,T^{ij}\,\delta_i{}^I\delta_j{}^J = T^{IJ}$

and so the rank-2 tensor components agree left and right end.

7. When the outer product construction has involves tensors of rank 2 or more (as shown in the last two examples above), it is called a Kronecker product, but again it is just a special version of the generic tensor product.