

Interchange of limits and uniform convergence

Let $s \rightarrow a$ and $t \rightarrow b$ be two limiting processes. (For example, these could be a real variable tending to a number, or towards infinity, a natural number tending towards infinity, a partition tending towards ‘infinity’ in the sense of refinement, etc.) Let $f(s, t)$ be a function (real- or complex-valued now, though this is not necessary). We will give a general theorem justifying the interchange of the order of these two limiting processes if a certain one of them occurs uniformly, in the sense of the following definition.

Definition. Let $L(t)$ be a function of t . We say that $\lim_{s \rightarrow a} f(s, t) = L(t)$ *uniformly in t* if for every $\epsilon > 0$ there is s_0 such that for any $s \geq s_0$, and for all t , we have $|f(s, t) - L(t)| < \epsilon$.

Theorem. Suppose that

- (i) For each s , $\lim_{t \rightarrow b} f(s, t) = R(s)$.
- (ii) For each t , $\lim_{s \rightarrow a} f(s, t) = L(t)$.
- (iii) $\lim_{s \rightarrow a} (\lim_{t \rightarrow b} f(s, t)) = C$.
- (iv) $\lim_{s \rightarrow a} f(s, t) = L(t)$ uniformly in t .

Then $\lim_{t \rightarrow b} (\lim_{s \rightarrow a} f(s, t)) = C$.

Proof. Let $\epsilon > 0$. By (iv) we may choose s_0 so that if $s \geq s_0$ then for all t we have $|f(s, t) - L(t)| < \epsilon/3$. By (iii) we may choose s_1 so that if $s \geq s_1$ then $|R(s) - C| < \epsilon/3$. Choose $s_2 \geq \max\{s_0, s_1\}$. We may use (i) to choose t_0 so that if $t \geq t_0$ then $|f(s_2, t) - R(s_2)| < \epsilon/3$. Now for $t \geq t_0$ we have

$$\begin{aligned} |L(t) - C| &\leq |L(t) - f(s_2, t)| + |f(s_2, t) - R(s_2)| + |R(s_2) - C| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon. \quad \blacksquare \end{aligned}$$

The standard theorems stating that the (appropriately uniform) limit of continuous/integrable/differentiable functions is continuous/integrable/differentiable, and that the limit/integral/derivative of the limit is the limit of the limit/integral/derivative, can all be derived from the above theorem. We will use it to prove a result on double series.

Theorem. Let a_{ij} be complex numbers, for $i, j \in \{1, 2, 3, \dots\}$. Suppose that

- (i) For all i , $\sum_j |a_{ij}| < \infty$.
- (ii) $\sum_i (\sum_j |a_{ij}|) < \infty$.

Then

- (iii) For all j , $\sum_i |a_{ij}| < \infty$.
- (iv) $\sum_j (\sum_i a_{ij}) = \sum_i (\sum_j a_{ij})$.

Proof. Fix j . For any i we have that $|a_{ij}| \leq \sum_\ell |a_{i\ell}|$. Hence $\sum_i |a_{ij}|$ converges by (ii) and the comparison test. Therefore (iii) holds. In particular, $\sum_i a_{ij}$ converges (absolutely). By (ii) and the comparison test again, $\sum_i |\sum_j a_{ij}|$ converges, so that $\sum_i (\sum_j a_{ij})$ converges (absolutely). We claim that $\sum_i a_{ij}$ converges uniformly in j . Let $\epsilon > 0$. By (ii) there is i_0 such that $\sum_{i=i_0}^\infty (\sum_\ell |a_{i\ell}|) < \epsilon$. For any j , and for all $i \geq i_0$, $|a_{ij}| \leq \sum_\ell |a_{i\ell}|$. Therefore $\sum_{i=i_0}^\infty |a_{ij}| \leq \sum_{i=i_0}^\infty (\sum_\ell |a_{i\ell}|) < \epsilon$. The theorem now follows by using the previous theorem. $\blacksquare$