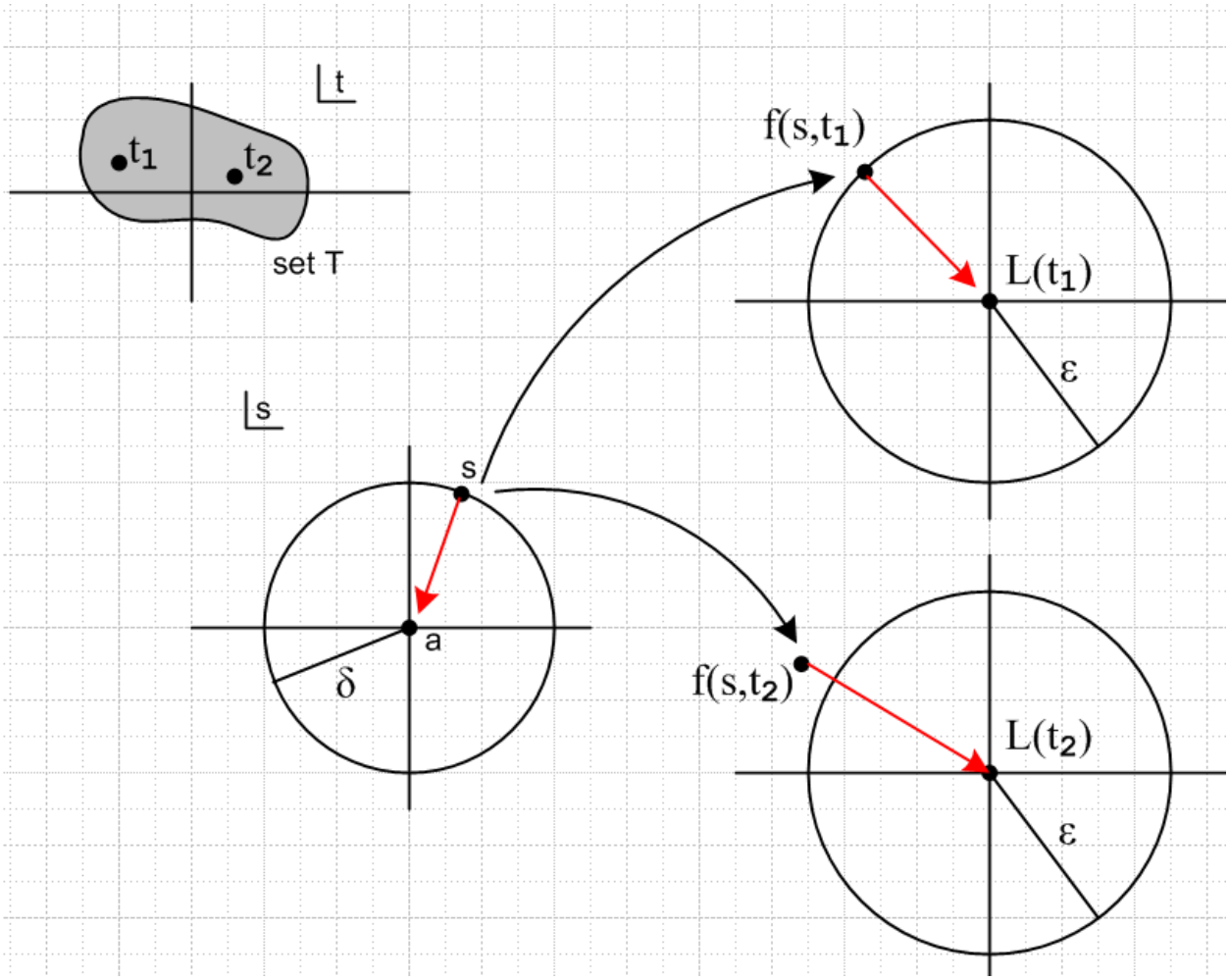
The Notion of Uniform Convergence PhL 10.2.14

This has always been a difficult concept for me. I can write down the definition, but I don't seem to really understand what it says. So let's try some pictures. Consider:



We have some function of two variables that is "of interest" called f(s,t). The second argument t lies in some well-defined set T like that shown in grey on top left.

We want to think about this idea:

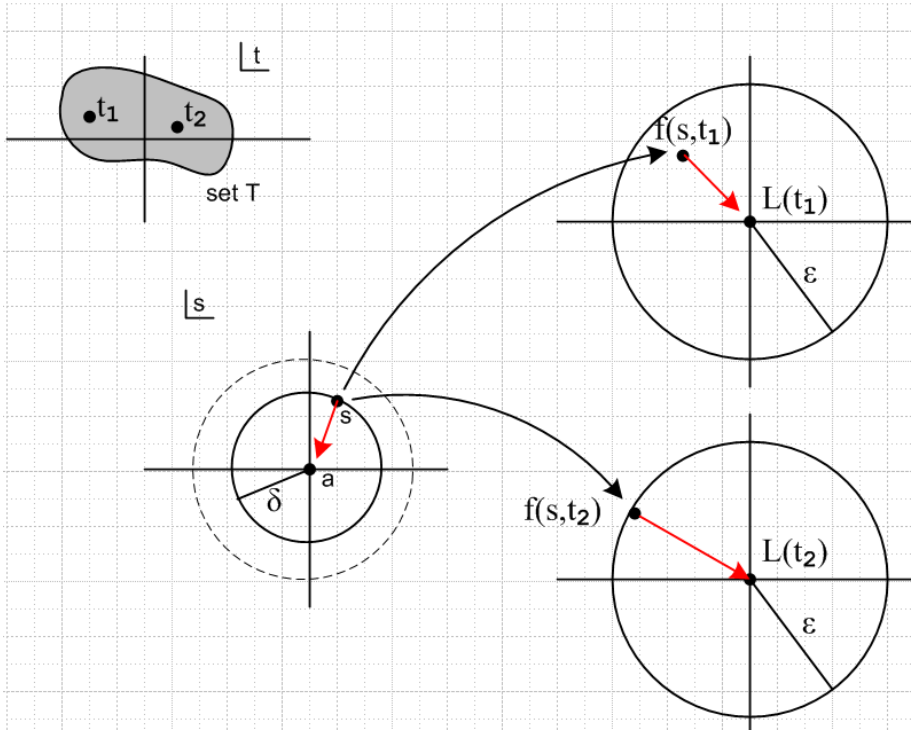
$\lim_{s \to a} f(s,t) = f(a,t) \equiv L(t)$

The parameter a is fixed, it never changes (it could be a = 0).

Suppose we pick a point t_1 in T and we pick some $\varepsilon > 0$. We then pick a certain δ and we find that all points inside the disc on the left map into the disk on the upper right. We show a sample point on the boundary which maps to the boundary. As we take $s \to a$, we find that $f(s,t_1) \to L(t)$. All is well. This just

means that at $t = t_1$, the function $f(s,t_1)$ actually converges to $f(a,t_1)$. The limits in the two spaces are shown as red arrows. This is called "pointwise convergence". For any t in T, we get this pointwise convergence.

Now suppose we pick a different value t_2 from the set T, and we draw the lower target image on the right. The disk there has exactly the same radius ε. But it happens that for t_2 our boundary point in s pace happens to land outside the circle of radius ε on the lower right. We don't want this to happen. So we can perhaps rectify the problem by making the δ radius smaller on the left, and then we have this



Now for the given ε, we find that all points in the δ disk shown map into the ε disk for both t_1 and t_2.

Suppose now there is some point t_3 in T such that the point s maps outside the ε disk on the right which surrounds the complex number $L(t_3)$. Well, then we could make the δ disk still smaller in order to force the point $f(s,t_i)$ to lie inside the ε disk for all three points t_1 and t_2 and t_3.

It may happen that as we keep considering more points in T, we have to keep making δ smaller and smaller in order to accommodate all the t_i points considered so far. It might happen that, in order to accommodate *all* the points t_i in T, the only δ that works is δ = 0!! If this is the case, then this limit

$$\lim_{s\to a} f(s,t) = f(a,t) \equiv L(t)$$

is said to converge NON-uniformly over the set T. The convergence is NOT uniform. Probably for all points in t, we will have $\lim_{s\to a} f(s,t) = f(a,t) \equiv L(t)$ occurring without any problem, so probably the convergence for any particular t is OK, and there is some δ that works for that value of t. This is called

pointwise convergence. But for a given ε, to have *uniform* convergence have to have a $\delta > 0$ value that works for ALL points in T.

So it is possible that we have a perfectly fine pointwise convergence for all points in T, but we do not have uniform convergence over T.

Often the problem is localized to a neighborhood of some point in the set T, and that point might be t = 0 as in the following example.

Example 1: Consider the case where a= 0 and $f(s,t) = \frac{1}{s+t}$. We pick some ε, and on the right we require that this be true:

$$\left| \frac{1}{s+t} - \frac{1}{0+t} \right| < \varepsilon \qquad \text{or} \qquad \left| \frac{s}{t(s+t)} \right| < \varepsilon \qquad \text{or} \qquad |s| < |t|\,|t+s|\,\varepsilon$$

Suppose set T contains the neighborhood of t = 0. We know that |t + s| is some finite value for any finite s and finite t. So we won't have |t+s| being infinite. And ε is some selected positive constant. So we see that as $t \rightarrow 0$, we will be forced to have $|s| < 0$ which basically means we need $\delta = 0$. Thus, in this example

$$\lim_{s \rightarrow 0} \frac{1}{s+t} = \frac{1}{t} \equiv L(t)$$

we find that we do have pointwise convergence for any t. For t = 0, our pointwise convergence is that the limit is ∞, which is OK, just another point. But we do not have uniform convergence for any set T which contains t = 0.

Authors often talk about uniform convergence of a sequence, whereas above we are talking only about functions. Suppose however in the above picture we defined a sequence of points like this

$$s_k \qquad k = 1,2,3.....\infty$$

and suppose this sequence converges to some value a,

$$\lim_{k \rightarrow \infty} (s_k) = a. \qquad f_{s_k}$$

We can think of the sequence as a set of points along our red arrow on the left in the drawings above. Or that arrow could be any curved path which gets from s to a. Each such path is a different sequence. Then the limit of interest is this

$$\lim_{s \rightarrow a} f(s,t) = f(a,t) \equiv L(t)$$

$$\lim_{k \rightarrow \infty} f_{s_k}(t) = f_a(t) \equiv L(t)$$

Examples 2 and 3: Go look at Case 1 and Case 2 in the "order interchange" doc.