

The Life and Mathematics of Karl Weierstrass

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Karl Theodor Wilhelm Weierstrass [4] was born on October 31, 1815 in Ostentfelde, Westphalia to a modest German family [5]. As a child his formal education took place at a catholic gymnasium in Paderborn, but his habit of reading copies of the *Journal of Pure and Applied Mathematics* while working as a bookkeeper provided the crucial foundational knowledge upon which his later mathematical work would build. Upon graduating from gymnasium in 1834, Weierstrass was forced by his father to enroll in the public finance and administration program at the University of Bonn, notwithstanding the many mathematics prizes that Weierstrass won while in gymnasium. After four tiresome years characterized by rebellious bouts of drinking and fencing, Weierstrass left the University of Bonn without sitting for the degree examinations. This left him in desperate need of a new career path. A family friend suggested that he enroll at the Theological and Philosophical Academy in nearby Münster, where he could sit an examination for a teaching degree within a year. Seeing no better alternative, Weierstrass did just that, and after passing the examinations in 1840, he began what became a fifteen year career as a teacher at a local gymnasium[5].

Although Karl Weierstrass thus spent the first forty years of his life without ever interacting with other professional mathematicians, his incredible mathematical ability would not go unnoticed indefinitely. While teaching at the gymnasium he found time to work on advanced mathematics, even going so far as to “publish” his results in the gymnasium’s prospectus, a document intended to entice fathers to enroll their sons in the school! This work eventually led him to publish his first genuine mathematical paper, “On the Theory of Abelian Functions,” in the *Journal of Applied Mathematics* in 1854. After publishing another paper on Abelian functions in 1856, he managed to secure a full professorship at the Industrial Institute of Berlin, and his career in academia was off and running[5].

Shortly after he accepted the offer from the Industrial Institute of Berlin, however, Weierstrass also received an offer from the prestigious University of Berlin. Although he was unable to formally occupy this position until several years later, after his agreed upon term of service with the Industrial Institute of Berlin had passed, Weierstrass effectively spent his entire academic career with the University of Berlin, starting in 1856[5]. It was here that he began interacting with the famous mathematicians Ernst Eduard Kummer (1810-1893) and Leopold Kronecker (1823-1891), with whose help he managed to make the University of Berlin the premier mathematics university in the world. Among the incredible students who were taught by these great teachers of mathematics during this period were Georg Cantor, Sofia Kovalevskaja, Lazarus Fuchs, Hermann Amandus Schwarz, Friedrich Schottky, Ferdinand Georg Frobenius, Hermann Minkowski,

Carle Runge, Ludwig Boltzmann, and Max Planck, to name only those few who remain widely known. Weierstrass personally drew up the first expertises for 28 dissertations, which was unprecedented at the University of Berlin at the time, but for Kummer, who drew up 39 over the course of his lengthier tenure[1].

With such a volume of tremendously successful students, it is natural to presume that Weierstrass and his fellow professors possessed some knowledge that gave them a distinct advantage when it came to producing exceptional mathematicians. This advantage was, in fact, the unprecedented rigor with which Weierstrass approached calculus. Rather than erecting the vast structure of calculus upon the feeble foundation of geometric intuition, Weierstrass's innovation was to develop a scheme in which carefully defining crucial concepts in terms of inequalities allowed one to approach proofs in a completely impartial and rigorous manner. In addition to establishing calculus as a rigorous branch of mathematics, Weierstrass's approach allowed mathematicians to be more careful and in depth in their analysis of the many subtleties that arose in calculus, such as the distinction between point-wise and uniform convergence of functions. Generalizing his techniques to encompass broader functional analysis led to the incredible success of Weierstrass and his students[2]. This cache of novel ideas remained the prized possession of the University of Berlin for decades due to the regularity with which Weierstrass suffered from attacks of horrible vertigo. Unable to handle much physical stress or strain, Weierstrass not only taught from a reclined chair while an advanced student wrote on the chalkboard for him, but he also found it too strenuous to publish much of his valuable work. As a result, the vast majority of the mathematical results that we attribute to Weierstrass were actually passed on through the lecture notes of his many devoted students, who continued to build upon Weierstrass's work long after his death in 1897[5],[1].

In order to better understand the significance of Weierstrass's work in reformulating real analysis, it is important to look in detail at several of his results. In order to develop calculus rigorously, it is critical that certain central concepts are given very specific and useful definitions, and this is precisely where Weierstrass began. Weierstrass was by no means the first to attempt to complete this task, but he was most assuredly the first to achieve success. Cauchy's definition of a limit, for example, is quite unsatisfactory as a mathematical definition: "When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others" [3]. Although this serves as a definition of the concept, and even hints at what the word means in a rigorous mathematical setting, it does not provide any hard and fast way of manipulating limits mathematically, and

thus falls short of the standards of a mathematical definition. Similarly intractable definitions for series convergence, continuity, and the derivative had previously been developed. Weierstrass managed to place each of these concepts, and many more besides, on a rigorous mathematical basis. Some examples illustrating his approach, as articulated by Kirkwood [6], are given below.

Definition A **sequence** of real numbers is a function from the positive integers into the real numbers, and is denoted $\{x_n\}$.

Definition We say that the sequence of real numbers $\{x_n\}$ **converges to the number** L if, for any $\varepsilon > 0$, there is a positive integer N_ε such that if n is a positive integer larger than N_ε , then $|x_n - L| < \varepsilon$. In this case we say that L is the **limit of the sequence** $\{x_n\}$, and we write $\lim\{x_n\} = L$ or $\{x_n\} \rightarrow L$. If a sequence does not converge, then it is said to **diverge**.

Definition A number x is called a **limit point**, or a **cluster point**, or an **accumulation point**, of a set of real numbers A if for any $\varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon)$ contains infinitely many points of A .

Definition Let f be a function with domain $D(f)$, and suppose that x_0 is a limit point of $D(f)$. The **limit of f as x approaches x_0 is L** if, given any $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that if $0 < |x - x_0| < \delta_\varepsilon$ and $x \in D(f)$, then $|f(x) - L| < \varepsilon$. In this case we write $\lim_{x \rightarrow x_0} f(x) = L$.

Definition Let f be a function and $x_0 \in D(f)$. Then we say that f is **continuous at x_0** if, given any $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta_\varepsilon$ and $x \in D(f)$. If f is not continuous at x_0 , then f is said to be **discontinuous at x_0** .

Definition Let f be a function with domain $D(f)$ and $A \subseteq D(f)$. We say that f is **continuous on A** if, given any $\varepsilon > 0$ and $x_0 \in A$, there is a number $\delta_{\varepsilon, x_0}$ such that if $x \in D(f)$ and $|x - x_0| < \delta_{\varepsilon, x_0}$, then $|f(x) - f(x_0)| < \varepsilon$.

Definition Let f be a function with domain $D(f)$ and $A \subseteq D(f)$. We say that f is **uniformly continuous on A** if, given any $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that if $x, y \in A$ and $|x - y| < \delta_\varepsilon$, then $|f(x) - f(y)| < \varepsilon$.

Definition Let f be a function defined on an interval (a, b) , and suppose $c \in (a, b)$. We say that f is **differentiable at c** if the limit

$$\lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h}$$

exists and is finite. If this is the case, this limit is called the **derivative of f at c** , and is denoted $f'(c)$.

This may seem an overly exhaustive list of definitions, but this list is actually only the tip of the iceberg. Every concept that mathematicians had been using and abusing since the dawn of calculus back in the late 1600's, nearly two centuries before Weierstrass, had to be rigorously defined. These definitions then had to be employed to prove, with complete and incontrovertible rigor, a multitude of calculus theorems.

Several new ideas actually came to the fore only as a result of this intense scrutiny of calculus. The definitions of continuity and uniform continuity given above, for example, are very nearly identical at first glance; however, the seemingly minor discrepancies between the two definitions are actually incredibly significant. An extension of these ideas led to the concept of uniform convergence of a series of functions, which is of central importance in functional analysis: uniformly convergent series of functions possess many properties that convergent series of functions do not, in general, possess[3].

Many important pioneering efforts in real analysis were made by Weierstrass; however, his results were not always immediately welcomed by the mathematical community as a whole. One difficulty arose as a result of opposing philosophical viewpoints regarding what constituted valid mathematics. Intuitionism only arose as a distinct school of thought in the early 1900's as a result of the efforts of Luitzen Egbertus Jan Brouwer (1881-1966), but many of its underlying ideas, such as rejecting mathematics that could not be proven constructively, and rejecting the unrestricted use of the Law of the Excluded Middle, had proponents even back in Weierstrass's day. The Law of the Excluded Middle states that any statement under consideration in a mathematical proof is either absolutely true or absolutely false: nothing is both true and false, and nothing is neither true nor false. Proofs by contradiction necessarily assume the validity of the Law of the Excluded Middle in order to reach their conclusions. Existence proofs were also called into question on the grounds that they were not demonstrable: they merely state that some quantity exists, but give no indication as to how one might

find that quantity. Weierstrass used many very subtle arguments regularly in his proofs, including some existence proofs, and thus bore the brunt of much criticism, particularly in his later years when word of many of his more unusual results began to spread. Making matters worse, by 1870 Weierstrass's colleague and close friend, Kronecker, began openly subscribing to these opposing philosophical viewpoints, particularly that of abhorring the actually infinite. In the 1880's the controversy between Kronecker and Weierstrass had become so great that there was a break between them, and Weierstrass had even considered retiring and moving to Switzerland a few weeks before his seventieth birthday [1].

In order to better understand the specific arguments and ideas to which Kronecker and others objected, consider the Bolzano-Weierstrass Theorem. Given the validity of a less significant theorem, whose validity we shall assume, this famous theorem can be proven as shown below [6].

Lemma Let $A_n = [a_n, b_n]$ be a sequence of intervals such that $A_n \supseteq A_{n+1}$ for $n = 1, 2, 3, \dots$. Suppose that $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$. Then there is a real number p for which

$$\bigcap_{n=1}^{\infty} A_n = \{p\}.$$

That is, there is exactly one point common to every A_n .

The Bolzano-Weierstrass Theorem Every bounded infinite set of real numbers has at least one limit point.

Proof Let A be a bounded set of real numbers. Since A is bounded, there is a positive number M such that $A \subseteq [-M, M]$. Divide $[-M, M]$ into two closed intervals of equal length, $[-M, 0]$ and $[0, M]$. At least one of these intervals contains an infinite number of points of A . Choose one of the intervals that contains an infinite number of points of A , and call it A_1 . Notice that the length of A_1 is $M = \frac{2M}{2}$.

Divide A_1 into two closed intervals of equal length. Choose one of the subintervals that contains infinitely many points of A , and call it A_2 . The length of A_2 is $\frac{2M}{2^2}$.

Continue this process inductively so that for each positive integer k , A_k is a closed interval of length $\frac{M}{2^{k-1}}$, and A_k contains infinitely many points of A . Notice

that $A_n \supseteq A_{n+1}$, and if $A_n = [a_n, b_n]$, then

$$\lim_{n \rightarrow \infty} (b_n - a_n) = \lim_{n \rightarrow \infty} \frac{M}{2^{n-1}} = 0.$$

Thus, by the above lemma,

$$\bigcap_{n=1}^{\infty} A_n = \{p\}.$$

We now claim that this point p is a limit point of A .

Let $\varepsilon > 0$ be given. We shall show that $(p - \varepsilon, p + \varepsilon)$ contains infinitely many points of A by showing that $(p - \varepsilon, p + \varepsilon) \supseteq A_N$ for some positive integer N .

Choose N so that $\frac{M}{2^{N-1}} < \varepsilon$. Now $p \in A_N$, since $p \in A_n$ for every n , and the total length of the interval A_N is $\frac{M}{2^{N-1}}$. Thus we must have that

$$(p - \varepsilon, p + \varepsilon) \supseteq [p - \frac{M}{2^{N-1}}, p + \frac{M}{2^{N-1}}] \supseteq A_N.$$

Since each A_n contains infinitely many points of A , we see from this that $(p - \varepsilon, p + \varepsilon)$ does indeed contain infinitely many points of A . We conclude that p is indeed a limit point of A , by the definition of limit point given above. **Q.E.D.**

Note that the Bolzano-Weierstrass Theorem is an existence theorem: it provides no way of determining a limit point p , and it gives no indication as to how many such limit points there are in such an infinite set of real numbers. It does not offer any method for determining whether a given set is infinite, either, which would be required in order to actually carry out the inductive procedure used in the proof. When viewed in this light, it is understandable why so many mathematicians had objections to many of Weierstrass's results; however, the value of these results was also unquestionable. Struggling to come to terms with such difficulties was what brought about the significant philosophy of mathematics movement that defined the early twentieth century.

Weierstrass proved many theorems to which his contemporaries had philosophical objections. Among these theorems, however, one in particular stands apart from the rest. Making use of several of his previous results, Weierstrass defined a class of functions that were everywhere continuous, but nowhere differentiable. This result was so counterintuitive that it led most mathematicians to recoil at the remarkable result. Henri Poincare called Weierstrass's example

“an outrage against common sense,” while Charles Hermite declared “I turn away with fright and horror from this lamentable evil of functions that do not have derivatives” [3]. As an example of the remarkable results that can follow from the careful and tireless application of rigorous analysis, this theorem is a true gem. Although a complete proof is beyond the scope of this essay, even a partial proof is quite illuminating [3]. Several other theorems are needed for the proof, and the truth of these theorems will here be taken on faith.

Lemma If $\{f_k\}$ is a sequence of continuous functions converging uniformly to f on $[a, b]$, then f itself is continuous.

Weierstrass M-Test If a sequence $\{f_k\}$ of functions defined on a common domain has the property that, for each k , there exists a positive number M_k so that $|f_k(x)| \leq M_k$ for all x in the domain and if the infinite series $\sum_{k=1}^{\infty} M_k$ converges, then the series of functions $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly.

Pathological Function Theorem If $a \geq 3$ is an odd integer and if $b \in (0, 1)$ such that $ab > 1 + \frac{3\pi}{2}$, then the function $f(x) = \sum_{k=0}^{\infty} f_k(x) = \sum_{k=0}^{\infty} b^k \cos(\pi a^k x)$ is everywhere continuous and nowhere differentiable.

Beginning and Sketch of Proof First we must demonstrate that $f(x)$ is continuous. If we can show that the sequence of partial sums $\{S_K(x)\} = \{\sum_{k=0}^K f_k(x)\}$ converges uniformly to $f(x)$, then we can invoke our lemma to conclude that $f(x)$ is continuous.

To this end, we notice that

$$|f_k(x)| = |b^k| |\cos(\pi a^k x)| \leq |b^k| = b^k.$$

Thus for $M_k = b^k$, we see that, since $0 < b < 1$, we have

$$\sum_{k=0}^{\infty} |f_k(x)| \leq \sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} b^k = \frac{1}{1-b}.$$

We can now invoke the Weierstrass M-Test to conclude that $f(x) = \sum_{k=0}^{\infty} f_k(x)$ converges uniformly. As any finite sum of cosines is a continuous function, we can now also invoke our lemma on the sequence of partial sums $\{S_K(x)\}$ to conclude that $f(x)$ is continuous.

We have demonstrated that Weierstrass's pathological function is everywhere continuous, but we still need to prove that it is nowhere differentiable. This is by far the more demanding task, and only the general flow of the proof will be communicated.

Weierstrass's method is as follows. He picks any arbitrary real number r and goes about showing that the derivative does not exist at this arbitrary point. By demonstrating that his pathological function has no derivative at this arbitrary point, he is really demonstrating that his pathological function is nowhere differentiable, since the specific point was arbitrary.

In order to complete this task, he defines an h_m such that $\frac{1}{2a^m} \leq h_m < \frac{3}{2a^m}$ so that $\lim_{m \rightarrow \infty} h_m = 0$. He then uses this h_m in the definition of the derivative, as given above, so that

$$f'(r) = \lim_{m \rightarrow \infty} \frac{f(r + h_m) - f(r)}{h_m} = \lim_{m \rightarrow \infty} \frac{\sum_{k=0}^{\infty} b^k \cos(\pi a^k(r + h_m)) - \sum_{k=0}^{\infty} b^k \cos(\pi a^k r)}{h_m}.$$

In order to better manipulate the terms involved, he considers a fixed m , only later to reintroduce the limit. This leaves, after breaking up the summation into two distinct parts,

$$\sum_{k=0}^{m-1} \frac{b^k \cos(\pi a^k(r + h_m)) - b^k \cos(\pi a^k r)}{h_m} + \sum_{k=m}^{\infty} \frac{b^k \cos(\pi a^k(r + h_m)) - b^k \cos(\pi a^k r)}{h_m}.$$

Let the first sum be S_1 and the second sum be S_2 . The bulk of the proof then demonstrates that $|S_1| < \frac{\pi(ab)^m}{ab-1}$ and that $|S_2| > \frac{2(ab)^k}{3}$. Going back to the expression for $f'(r)$, you then isolate S_2 on one side of the equation and take the absolute value of both sides. After applying the triangle inequality and using the two inequalities for S_1 and S_2 , the final result is that

$$|f'(r)| > \lim_{m \rightarrow \infty} (ab)^m \left[\frac{2}{3} - \frac{\pi}{ab-1} \right].$$

The constant term on the right side is positive precisely when $ab > 1 + \frac{3\pi}{2}$, as specified in the statement of the theorem. Thus, for the specified conditions, the absolute value of the derivative of f at r must be greater than ∞ , if it exists. This would be a contradiction, and we are thus led to conclude that $f'(r)$ does not exist. Thus, as described earlier, Weierstrass's pathological function is indeed nowhere differentiable. **“Q.E.D.”**

Karl Weierstrass was clearly a master of real analysis, and is indeed the founding father of the entire modern discipline; however, his mathematical brilliance spans far beyond the narrow confines of real analysis. In addition to reformulating calculus as a rigorous discipline, Weierstrass made many significant contributions to several fields of mathematics. Along with Augustin Louis Cauchy (1789-1857) and Bernhard Riemann (1826-1866), Weierstrass is also considered one of the founding fathers of complex analysis. Differential geometry, the calculus of variations, the theory of elliptic functions, and especially a generalization of this latter work to the theory of Abelian functions, of which elliptic functions are a subset, also occupied Weierstrass throughout his career. Weierstrass was a brilliant mathematician and made significant contributions to all of these fields, particularly with regards to his restructuring of the theory of elliptic functions based upon his $\wp$ -function; however, due to the ubiquity of his approach to real analysis in modern mathematics, his name will forever be associated first and foremost with that discipline and the “dreaded” ε - δ definitions that he introduced[1].

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