Find the solution of f(z)f(-z) = K where f(z) is analytic at z=0 PhL 10.11.10

Here is the answer (we assume $K \neq 0$)

Theorem: The most general solution of f(z)f(-z) = K where f(z) is analytic at z = 0 is given by:

• Start with *any odd function* of z analytic at z = 0, call that function $f_{odd}(z)$. [ok]

Note: this implies $f_{odd}(0) = 0$ [ok]

• Define the even function $f_{even}(z) \equiv \sqrt{K + [f_{odd}(z)]^2}$. [ok]

Note: since $\sqrt{K + [f_{odd}(0)]^2} = \sqrt{K}$, we are away from the SQRT branch point so $f_{even}(z)$ is analytic at z=0. [ok]

• Then either f(z) as follows will solve the equation f(z)f(-z) = K :

$f(z) = f_{odd}(z) \pm \sqrt{K + [f_{odd}(z)]^2}$ [need to prove this]

• One small example is $f_{odd}(z) = sh(Bz)$ and K=1 in which case
$f(z) = sh(Bz) \pm ch(Bz) = \pm e^{\pm Bz}$ and you can see that in either sign case, f(z)f(-z) = 1. [ok]

• If you are given f(z), you can find $f_{odd}(z)$ by doing some simple algebra. The result is

$f_{odd}(z) = [\, f(z)^2 - K^2] \,/\, [2\, f(z)]$ [pend]

In our last example, this gives $f_{odd} = [\, e^{\pm 2Bz} - 1] \,/\, [\pm 2\, e^{\pm Bz}] = \pm(1/2)[\, e^{\pm Bz} - e^{-+Bz}] = \pm sh(\pm Bz) = sh(Bz)$.

Proof of Theorem:

Decompose $f(z) = f_{even}(z) + f_{odd}(z)$

where $f_{even}(z) = [\, f(z) + f(-z)]/2$
$f_{odd}(z) = [\, f(z) - f(-z)]/2$ [ok]

$f(z)f(-z) = K$ [ok]

$[f_{even}(z) + f_{odd}(z)]\, [f_{even}(-z) + f_{odd}(-z)] = K$ [pend]

$[f_{even}(z) + f_{odd}(z)]\, [f_{even}(z) - f_{odd}(z)] = K$

$[f_{even}(z)]^2 - [f_{odd}(z)]^2 = K$

$$[f_{even}(z)]^2 = K + [f_{odd}(z)]^2$$

$$f_{even}(z) = \pm\sqrt{K + [f_{odd}(z)]^2}$$

Just run the last lines in reverse for a proof. Examples for K = 1 appear in the text below.

Question: If someone tells you that the complex function f(z) has the property f(z)f(-z) = 1, and is analytic at z = 0, then what can you say about the solution f(z) ?

An interesting question. If f(z) is analytic at z = 0, we can expand it like so

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \ \ldots \qquad [ok]$$

Our condition then becomes

$$(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \ \ldots)(a_0 - a_1 z + a_2 z^2 - a_3 z^3 + a_4 z^4 \ \ldots) = 1 \quad [ok]$$

We can then require that each coefficient of a z power match. [ok] The function on the LHS is even in z, [ok] so we know all the odd terms have manifestly vanishing coefficients, leaving only even terms. [??] Let's have Maple take care of this expansion for us:

```
> restart;
> f := (z)-> a0 + a1*z + a2*z^2 + a3*z^3 + a4*z^4 + a5*z^5 + a6*z^6 + a7*z^7 + a8*z^8;
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$$f := z \to a0 + a1\,z + a2\,z^2 + a3\,z^3 + a4\,z^4 + a5\,z^5 + a6\,z^6 + a7\,z^7 + a8\,z^8$$

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> g := f(z)*f(-z);
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$$g := (a0 + a1\,z + a2\,z^2 + a3\,z^3 + a4\,z^4 + a5\,z^5 + a6\,z^6 + a7\,z^7 + a8\,z^8)$$
$$(a0 - a1\,z + a2\,z^2 - a3\,z^3 + a4\,z^4 - a5\,z^5 + a6\,z^6 - a7\,z^7 + a8\,z^8)$$

```
> collect(g,z);
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$$a8^2 z^{16} + (2\,a6\,a8 - a7^2) z^{14} + (2\,a4\,a8 - 2\,a5\,a7 + a6^2) z^{12} + (2\,a8\,a2 - 2\,a7\,a3 + 2\,a6\,a4 - a5^2) z^{10}$$
$$+ (2\,a6\,a2 + 2\,a0\,a8 - 2\,a5\,a3 - 2\,a7\,a1 + a4^2) z^8 + (-a3^2 + 2\,a4\,a2 - 2\,a5\,a1 + 2\,a6\,a0) z^6 + (2\,a0\,a4 - 2\,a1\,a3 + a2^2) z^4$$
$$+ (2\,a0\,a2 - a1^2) z^2 + a0^2$$

The result is good through the z^8 term because adding more terms to the series $O(z^9)$ cannot affect this term. [ok] We see a pattern here.

- $a_0^2 = 1$ so at least we know a_0 is a phase, and we just keep it as a_0.
- a_1 is not determined, it is a free constant.
- $a_2 = a_1^2/(2a_0)$ so a_2 is determined.
- a_3 is a free constant.
- $a_4 = [2\,a_1 a_3 - a_2^2]/(2a_0)$ so a_4 is determined.
- a_5 is a free constant

- a_6 = ... so a_6 is determined
- a_7 is a free constant
- a_8 = ... so a_8 is determined.

What is all this telling us? [not sure] Suppose we break f(z) into its even and odd parts:

$$f(z) = f_{even}(z) + f_{odd}(z)$$

where

$$f_{even}(z) = \Sigma_{n=0,2,4}^{\infty} a_n z^n$$

$$f_{odd}(z) = \Sigma_{n=1,3,5}^{\infty} a_n z^n \qquad \text{[ok]}$$

We find that $f_{odd}(z)$ can be *anything you like*, and this then forces a certain $f_{even}(z)$. Let's go back to our original equation then which says $f(z)f(-z) = 1$:

$$[f_{even}(z) + f_{odd}(z)]\,[f_{even}(-z) + f_{odd}(-z)] = 1 \qquad \text{[ok]}$$

$$[f_{even}(z) + f_{odd}(z)]\,[f_{even}(z) - f_{odd}(z)] = 1 \qquad \text{[ok]}$$

$$[f_{even}(z)]^2 - [f_{odd}(z)]^2 = 1 \qquad \text{[ok]}$$

$$[f_{even}(z)]^2 = 1 + [f_{odd}(z)]^2 \qquad \text{[ok]}$$

$$f_{even}(z) = \sqrt{1 + [f_{odd}(z)]^2} \qquad \text{[ok]}$$

So here we see our claim more clearly. Select any f_{odd} you want, then f_{even} is given by the above expression.

Example 1:

$$f_{odd}(z) = z$$

$$f_{even}(z) = \sqrt{1+z^2} \qquad \text{[ok]}$$

$$f(z) = \sqrt{1+z^2} + z \qquad \text{[ok]}$$

$$f(-z) = \sqrt{1+z^2} - z \qquad \text{[ok]}$$

$$f(z)f(-z) = [\sqrt{1+z^2} + z][\sqrt{1+z^2} - z] = (1+z^2) - z^2 = 1 \qquad \text{[ok]}$$

Well I'll be a monkey's uncle! [why?]

Example 2:

$f_{odd}(z) = A\sin(Bz)$

$f_{even}(z) = \sqrt{1+A^2\sin^2(Bz)}$ [ok]

$f(z) = \sqrt{1+A^2\sin^2(Bz)} + A\sin(Bz)$

$f(-z) = \sqrt{1+A^2\sin^2(Bz)} - A\sin(Bz)$

$f(z)f(-z) = 1$ OK

Example 3:

$f_{odd}(z) = i\sin(Bz)$

$f_{even}(z) = \sqrt{1-\sin^2(Bz)} = \cos(Bz)$

$f(z) = \cos(Bz) + i\sin(Bz) = e^{iBz}$

$f(z)f(-z) = 1$ OK

Example 4:

$f_{odd}(z) = \mathrm{sh}(Bz)$

$f_{even}(z) = \sqrt{1+\mathrm{sh}^2(Bz)} = \mathrm{ch}(Bz)$

$f(z) = \mathrm{ch}(Bz) + \mathrm{sh}(Bz) = e^{Bz}$

$f(z)f(-z) = 1$ OK

So when I started this little problem, I thought surely e^{Bz} was the only possible solution where B is some complex number, and then $f(z)f(-z) = e^{Bz}\, e^{-Bz} = 1$. I was wrong!

Summary:

Theorem: The most general solution of $f(z)f(-z) = 1$ where $f(z)$ is analytic at $z = 0$ is given by

- Start with any odd function of z analytic at z = 0, call that function $f_{odd}(z)$.

Note: this implies $f_{odd}(0) = 0$

- Define the even function $f_{even}(z) \equiv \sqrt{1 + [f_{odd}(z)]^2}$.

Note: since $\sqrt{1 + [f_{odd}(0)]^2}$ = 1, we are away from the SQRT branch point so $f_{even}(z)$ is analytic at z=0.

- Then f(z) as follows will solve the equation f(z)f(-z) = 1 :

  $f(z) = f_{odd}(z) + \sqrt{1 + [f_{odd}(z)]^2}$

- One small example is the case $f_{odd}(z) = sh(Bz)$ in which case $f(z) = e^{Bz}$.

Proof

Decompose $f(z) = f_{even}(z) + f_{odd}(z)$

where $f_{even}(z) = [\, f(z) + f(-z)]/2$
$f_{odd}(z) = [\, f(z) - f(-z)]/2$

$f(z)f(-z) = 1$

$[f_{even}(z) + f_{odd}(z)]\, [f_{even}(-z) + f_{odd}(-z)] = 1$

$[f_{even}(z) + f_{odd}(z)]\, [f_{even}(z) - f_{odd}(z)] = 1$

$[f_{even}(z)]^2 - [f_{odd}(z)]^2 = 1$

$[f_{even}(z)]^2 = 1 + [f_{odd}(z)]^2$

$f_{even}(z) = \sqrt{1 + [f_{odd}(z)]^2}$

Generalization: To solve f(z)f(-z) = K, just replace 1 by K everywhere above.