Hilbert Space PhL 12.1.04

read 11.28.24

(Dave Hilbert, 1862-1943, *in honorem*)

You are in your kitchen having a Scotch. Grab some **vectors**, and toss some of them around.

Figure out how to add and subtract them, a+b a-b

Figure out how to have more than one of the same vector, 5a 3a + 7b

Congratulations, you have a **vector space** !

Figure out how to multiply them, you have a **scalar product**. a*b

Use that to determine the length of a vector, you have a **norm**: $||a|| = \sqrt{a*a}$

Use that to determine the distance between two vectors, you have a **metric**: $d(a,b) = || a - b ||$

Exercise #1: $|| a - b ||^2 = (a-b)*(a-b) = ||a||^2 + ||b||^2 - a*b - b*a$ Good work!

Make sure you have not left out any vectors from your space, it must be **complete**.

Congratulations, you have a **Hilbert Space**.

Example #1: Real numbers form a Hilbert Space. You know how to do +, - and *.

Example #2: Rational numbers almost form a Hilbert Space but some numbers are missing, so not complete. An example is π. Cannot write π as ratio of two integers like 368/117, even if each integer is allowed to have four hundred billion digits. You can get arbitrarily close though. Because you can get arbitrarily close but the limit is missing from your space, your space is not complete, so sorry. If you add all the missing points, you get the reals.

Example #3: Your normal physical world with position vectors $\mathbf{x} = (x_1,x_2,x_3)$ is a Hilbert Space. To add two such vectors, add the components. The * rule is that $\mathbf{y}*\mathbf{x} = y_1x_1 + y_2x_2 + y_3x_3$, known as the dot product. The norm (length of a vector) is then $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ which hopefully sounds familiar.

Example #4: Complex numbers form a Hilbert Space.

$z = a+ib$, $w = c+id$.

$z*w = (a-ib)(c+id) = (ac + bd) - i (bc - ad) = [w*z]^*$

$||z||^2 = z*z = (a-ib)(a+ib) = a^2 + b^2$ which is > 0 the way the length of something should be.

Example #5: In quantum mechanics, the "state" of some physical system is a vector in a Hilbert Space. It is usually written $|\psi\rangle = |$ a,b,c...$\rangle$ where a,b,c are properties of the system of interest. For example, a "particle" sitting at position x is in a state $|x\rangle$. The scalar product has the form $\langle$state 1 | state 2$\rangle$ and is a complex number. An example is $\langle x | \psi \rangle$ which is usually written $\psi(x)$ and is called a wave-function. It is very strange because it is a complex number, but if you follow Example 4 and compute $\psi(x)^*\psi(x)$, you get a real positive number which is the probability that your object is located at position x. You can only know probabilities, which is why Einstein objected that God does not throw dice.