Doing the a_n integral META notes PhL 3.4.11

Overview: The basic problem here is to do a certain integral. The major issue is that the integrand includes an arc trig function combined with a Legendre P function, all on a *partial* interval so that the normal properties of P_n orthogonal polys don't really help. I tried lots of different approaches:

- differentiation with respect to a parameter
- doing n = 0 and n=1 and then constructing an induction proof
- trying parts to get rid of the arc trig function
- power series for the arc trig function
- power series for the P_n function
- elevating P into an integral representation and then doing parts to remove the arc trig

It was the last method that finally worked.

In my notes "canonical -- Section 1_4_1 The Definition Method for the Bowl.doc" I describe the solution of the charged bowl problem in spherical coordinates. The solution potential is given in terms of certain coefficients called a_n which can be computed according to

$$a_n = (1/2) \int_0^{\theta_0} d\theta \sin\theta \, P_n(\cos\theta) \, g(\theta) \qquad (1)$$

where

$$g(\theta) = (2/\pi) \{\sqrt{1+\cos\theta_0} / \sqrt{\cos\theta - \cos\theta_0} + \pi/2 - \sin^{-1}[\sqrt{1+\cos\theta_0} / \sqrt{1+\cos\theta}\,] \} \qquad (2)$$

In my raw doc "doing the a_n integral" I show how , when (2) is inserted into (1), we obtain the result

$$a_n = (1/\pi) \{ \sin(na)/n + \sin[(n+1)a]/(n+1) \} \qquad (3)$$

which is "the classic result" for this problem, though I only know of its appearance in Sneddon and in Canonical. The sad fact is that it took me about 22 pages of fiddling to obtain this result, and that is why I am writing these meta notes. There surely is a faster way to do this integral, but I don't know what it is.

In **Section 1** I just restate the desired integral this way

$$a_n(b) = (1/\pi) \int_b^1 dz \, P_n(z) \{\sqrt{1+b} / \sqrt{z-b} + \cos^{-1}[\sqrt{1+b} / \sqrt{1+z}\,] \} \qquad b = \cos\theta_0$$

I refer to the integral above as I_n so that $a_n = (1/\pi)I_n$. It doesn't look like a 22-page problem, does it? I realized early-on that **the big issue** was going to be doing what is basically an indefinite integral of a Legendre polynomial against a messy arc trig function of a complicated argument. My first thought was trying to find a way to make the $\cos^{-1}$ function go away.

In **Section 2** I attempted to "make $\cos^{-1}$ go away" by **differentiating $a_n(b)$ with respect to b**. But since b is an integration endpoint, when I evaluate $\sqrt{1+b}\,/\sqrt{z-b}$ at z = b it diverges, and the resulting other terms also diverge. I comment about trying an ε-regulation approach, but does not seem viable.

In **Section 3** I flail a bit then list off a set of **possible methods** of proceeding, which I try in later sections.

In **Section 4** I try to **simplify the arc trig term as much as possible** and obtain

$$I_n(a)/(4\alpha^2) = \int_1^{1/\alpha} dy\; y\, P_n(2\alpha^2y^2-1)\; \{\; 1/\sqrt{y^2-1} + \sec^{-1}y \;\} \qquad \alpha = \cos(\theta_0/2)$$

But of course this complicates the P_n argument and we have no way to "look up" such an integral, at least not in GR7. I may look elsewhere at some point. But for **n = 0** I know $P_0 = 1$ and I find that in this case Maple or a lookup provides the arc trig integral and I get the right answer for a_0. At least this was a small confidence builder.

In **Section 5** I did the case **n = 1**, and again I got the right answer. The integrals were of the form $y^n \sec^{-1}y$ which are all doable in Maple or in lookup. So at this point, I knew that by brute force, I could compute any a_n I wanted by expanding $P_n(z)$ in powers of z, but I had no general formula for the result.

In **Section 6**, since I had the n=0 and n=1 results, I pondered an **induction proof** of the general formula for a_n. I tried doing this using the Legendre recursion relations, but I could not make it fly.

In **Section 7** I pondered my problem integral $J= \int_1^{1/\alpha} dy\; y\; P_n(2\alpha^2y^2-1)\; \sec^{-1}y$. The only sure-fire removal of the arc trig is to do **parts** integration, but I could not write $y\, P_n(2\alpha^2y^2-1)$ as $\partial_y f(y)$ for any simple f, so I gave up.

In **Section 8** I tried putting $P_n(2\alpha^2y^2-1)$ into an **integral representation**. This method required that I do internal integrals of the form $\int dy\; y\; \sec^{-1}y\,/\sqrt{y^2-c}$. This path might still work, but I decided not to pursue it. That is to say, I could have found f(y) such that $y/\sqrt{y^2-c} = \partial_y f$, then I could do parts integration to remove the arc trig. In fact I see now that $f = \sqrt{y^2-c}$ works. This is what I eventually did below.

In **Section 9** I ponder using a **power series expansion for the arc trig function**. This just led to more undoable integrals like $\int_{\cos a}^{1} dz\; P_n(z)\; (z+1)^{-n-1/2}$ n = 0,1,2,3..., so again I gave up.

In **Section 10** I instead try a **power series for $P_n(z)$,** since this is how I did my n = 0 and n = 1 cases. The problem here is that I am then faced with $\int_1^{1/\alpha} dy\; y^n \sec^{-1}(y)$ for arbitrary integer n, but Maple could not do such integrals in general only specific n cases, maybe GR7 could.

In **Section 11** I decided to ignore the arc trig term for a while and **integrate the first term**. I was able to look this integral up in GR7 and got

$$I_n(a)_{\texttt{first_term}} = 2\cot(a/2)\,\{\cos(na) - \cos[(n+1)a]\}\,/\,(2n+1)$$

Knowing the correct answer for a_n, at this point I knew what the arc trig term had to be

$$S = \int_{\cos a}^{1} dz\, P_n(z)\, \cos^{-1}[\sqrt{1+\cos a}\,/\sqrt{1+z}\,]$$

$$= \{\, \sin(na)/n + \sin[(n+1)a]/(n+1)\,\} - 2\cot(a/2)\,\{\cos(na) - \cos[(n+1)a]\}\,/\,(2n+1) \equiv SS$$

In **Section 12** I returned to the arc trig integral which is the first term of I, which I called S. I went back to the Section 8 idea of installing an **integral representation for P**, and got

$$S = (\sqrt{2}\,/\pi) \int_{\cos a}^{1} dz\, \{ \int_{0}^{\cos^{-1} z} d\varphi\, \cos[(n+1/2)\varphi]/\sqrt{\cos\varphi - z}\,\}\, \cos^{-1}[\sqrt{1+\cos a}\,/\sqrt{1+z}\,]$$

With the aid of a picture, I could switch the order of integration to get

$$= (\sqrt{2}\,/\pi) \int_{0}^{a} d\varphi\, \cos[(n+1/2)\varphi]\ \{\ \int_{\varphi}^{a} d\theta\, \sin\theta/\sqrt{\cos\varphi - \cos\theta}\, *\, \cos^{-1}[\sqrt{1+\cos a}\,/\sqrt{1+\cos\theta}\,]\ \}$$

where I called the integral appearing here R. I first showed that

$$R = \int_{a}^{b} dz\, (b-z)^{-1/2}\, \cos^{-1}[\sqrt{1+a}\,/\sqrt{1+z}\,] \qquad b = \cos\varphi \qquad a = \cos\theta_0$$

So here was a chance to dump the arc trig by doing parts. I did these steps

$$R = -2 \int_{a}^{b} dz\, [\partial_z (b-z)^{1/2}]\, \cos^{-1}[\sqrt{1+a}\,/\sqrt{1+z}\,]$$

$$= 2 \int_{a}^{b} dz\, (b-z)^{1/2}\, \partial_z \sec^{-1}[\sqrt{1+z}\,/\sqrt{1+a}\,]$$

since both parts terms vanished (supporting this path of action). Doing the ∂_z shown I got

$$R = \sqrt{1+a} \int_{a}^{b} dz\, (b-z)^{1/2}\,/\,[(1+z)\sqrt{z-a}\,]$$

I was able to recast this into my corrected GR7 F integral form as follows:

$$R = \sqrt{1+a}\ (b-a)^{1/2}\,(1+a)^{-1} \int_{1/(b-a)}^{\infty} ds\, s^{-1}\, (\, s - [b-a]^{-1})^{1/2} (s + [1+a]^{-1})^{-1}$$

I then used that corrected GR7 form

$$\int_{u}^{\infty} ds\, s^{-\lambda} (s-u)^{\mu-1} (s+\beta)^{\nu} = u^{-\lambda} (\beta+u)^{\mu+\nu} B(\lambda-\mu-\nu,\mu)\, F(\lambda,\mu;\, \lambda-\nu;\, -\beta/u)$$

to show that

$$R = (\pi/2)\,(b-a)\,(b+1)^{1/2}\,(a+1)^{-1}\,F(1,3/2;\, 2;\, -(b-a)/(1+a))$$

But then Maple told me that

$$F(1,3/2;\, 2;\, -(b-a)/(1+a)) = = 2(a+1)\,[\sqrt{b+1} - \sqrt{a+1}\,] / [\sqrt{b+1}\,(b-a)\,]$$

and I ended up then with this very encouraging result

$$R = \pi\,[\sqrt{\cos\varphi+1} - \sqrt{\cos a+1}\,]$$

Note: In some table there must be sitting these definite integrals

$$R = \int_{a}^{b} dz\,(b-z)^{-1/2} \cos^{-1}[\sqrt{1+a}\,/\sqrt{1+z}\,]$$

$$= \sqrt{1+a} \int_{a}^{b} dz\,(b-z)^{1/2} / [(1+z)\sqrt{z-a}\,]$$

$$= \pi\,(\sqrt{b+1} - \sqrt{a+1}\,)$$

Probably it is right there in the GR7 section on algebraic forms, you would shift to get 0 as one endpoint. Well, Section 13 of the raw notes says no gain here.

and therefore I knew that my arc trig term S was given by

$$S = \sqrt{2} \int_{0}^{a} d\varphi\, \cos[(n+1/2)\varphi]\, [\sqrt{\cos\varphi+1} - \sqrt{\cos a+1}\,]$$

$$= 2 \int_{0}^{a} d\varphi\, \cos[(n+1/2)\varphi]\, [\cos(\varphi/2) - \cos(a/2)\,]$$

Maple was able to do this integral as follows:

$$n(n+1)(2n+1)\,S = [n + 1 - \cos(a)\, n]\, \sin(an) - n \sin a \cos(an)$$

I knew what my "desired result" for SS was and I entered it into Maple and I found that S = SS, so finally I had the result. This took me about 3 days!