Canonical -- correction to Equation 1_111 PhL 3.1.11

In Section 1.4, canonical makes this definition of g(θ)

$$\sum_{n=0}^{\infty}(2n+1)a_n P_n(\cos\theta) = \left\{ \begin{array}{ll} g(\theta), & \theta \in [0,\theta_0) \\ 0, & \theta \in (\theta_0,\pi] \end{array} \right\}. \qquad (1.\ 103)$$

and then on the next page they claim to arrive at this solution for g(θ)

$$g(\vartheta) = \frac{\sqrt{2}}{\pi}\left\{ \frac{2\cos\frac{1}{2}\theta_0}{\sqrt{\cos\vartheta - \cos\theta_0}} + \frac{\pi}{2} - \arcsin\left(\frac{\cos\frac{1}{2}\theta_0}{\cos\frac{1}{2}\theta}\right)\right\}. \qquad (1.\ 111)$$

I think this result is incorrect.

1. More official Edition or Errata for Canonical ?

My PDF has no page numbers and is only have the series, so I think it may be a preliminary document of some sort. A version of the document exists in Google books which has page numbers. It says 2001 Chapman/Hall CRC so it seems to be the same date. The above equation 1.111 is exactly as you see it above, and it is on page 32. No obvious errata page appears on the web. So I have done due diligence, looking to see if this 1.111 has been corrected, and it has not.

2. How do we relate g(θ) to something we know about?

The function g(θ) is defined in 1.103 as

$$\sum_{n=0}^{\infty}(2n+1)a_n P_n(\cos\theta) = \left\{ \begin{array}{ll} g(\theta), & \theta \in [0,\theta_0) \\ 0, & \theta \in (\theta_0,\pi] \end{array} \right\}. \qquad (1.\ 103)$$

I think g(θ) is related to $\sigma_{out} - \sigma_{in}$, let's see. In my canonical notes I started off using

$V_i(r,z) = \Sigma_{n=0}^{\infty}\, a_n\, (r/R)^n P_n(z)$
$V_o(r,z) = \Sigma_{n=0}^{\infty}\, a_n\, (r/R)^{-n-1} P_n(z)$

as a Smythian form for a bowl or radius R. The built-in BC here is that at the surface of the bowl, both on and off the bowl, these two forms match. We could take R = 1 if we wanted, but let's not. I then compute

$$(\partial_r V_o - \partial_r V_i)|^{r=R} \;=\; -\Sigma_{n=0}^{\infty}\, a_n (2n+1)\,(1/R) P_n(z)$$

But we know from flux = $4\pi Q_{enclosed}$ that above a metal surface $E = 4\pi\sigma = -\partial_r V_o$ for the outside surface. So we can say that $\sigma_o = -\partial_r V_o/4\pi$ and $\sigma_i = +\partial_r V_i/4\pi$ so we have

$$(\partial_r V_o - \partial_r V_i)|^{r=R} = -4\pi\sigma_o - 4\pi\sigma_i = -(1/R)\Sigma_{n=0}^{\infty} a_n(2n+1)\,P_n(z)$$

which says

$$\sigma_o + \sigma_i = (1/4\pi R)\,\Sigma_{n=0}^{\infty} a_n(2n+1)\,P_n(z) = (1/4\pi R)\,g(\theta)$$

Therefore the connection in cgs units is this:

$$g(\theta) = 4\pi R\,(\sigma_o + \sigma_i)$$

Now we go look up my Kelvin result in toroidal doc:

$$\sigma_i = (V_0/4\pi^2 R)\,\{\ \sqrt{1-\cos\alpha}\,/\,\sqrt{\cos\alpha - \cos\theta}\ -\ \tan^{-1}\,[\sqrt{1-\cos\alpha}\,/\sqrt{\cos\alpha - \cos\theta}\,]\,\}$$

$$\sigma_o = \sigma_i + V_0/(4\pi R)$$

$$\Rightarrow\ \sigma_o+\sigma_i = (V_0/2\pi^2 R)\,\{\ \sqrt{1-\cos\alpha}\,/\,\sqrt{\cos\alpha - \cos\theta}\ -\ \tan^{-1}\,[\sqrt{1-\cos\alpha}\,/\sqrt{\cos\alpha - \cos\theta}\,]\,\} + V_0/(4\pi R)$$

$$= (V_0/2\pi^2 R)\,\{\ \sqrt{1-\cos\alpha}\,/\,\sqrt{\cos\alpha - \cos\theta}\ -\ \tan^{-1}\,[\sqrt{1-\cos\alpha}\,/\sqrt{\cos\alpha - \cos\theta}\,]\ +\pi/2\ \}$$

Therefore, I claim that g(θ) as defined above must be equal to this:

$$g(\theta) = 4\pi R\,(\sigma_o+\sigma_i)$$

$$= V_0(2/\pi)\,\{\ \sqrt{1-\cos\alpha}\,/\,\sqrt{\cos\alpha - \cos\theta}\ -\ \tan^{-1}\,[\sqrt{1-\cos\alpha}\,/\sqrt{\cos\alpha - \cos\theta}\,]\ +\pi/2\ \}$$

In the VSV world (ie, Canonical three authors) angles are measured from the bowl pole, but in my toroidal Kelvin stuff, angles are measured from the cap pole, so this means $\psi = \pi - \psi$ for all angles (these are all spherical polar angles). Doing this translation on the above gives

$$g(\theta) = V_0(2/\pi)\,\{\ \sqrt{1+\cos\alpha}\,/\,\sqrt{\cos\theta - \cos\alpha}\ +\pi/2 -\ \tan^{-1}\,[\sqrt{1+\cos\alpha}\,/\sqrt{\cos\theta - \cos\alpha}\,]\ \}$$

and in VSV we are using α = bowl edge angle = θ_0 so write again as

$$g(\theta) = V_0(2/\pi)\,\{\ \sqrt{1+\cos\theta_0}\,/\,\sqrt{\cos\theta - \cos\theta_0}\ +\pi/2 -\ \tan^{-1}\,[\sqrt{1+\cos\theta_0}\,/\sqrt{\cos\theta - \cos\theta_0}\,]\ \}$$

Now I draw my usual little triangle for the $\tan^{-1}$ and I find that

$$\tan^{-1}\,[\sqrt{1+\cos\theta_0}\,/\sqrt{\cos\theta - \cos\theta_0}\,] = \sin^{-1}[\sqrt{1+\cos\theta_0}\,/\sqrt{1+\cos\theta}\,] = \sin^{-1}[\,\cos(\theta_0/2)\,/\,\cos(\theta/2)]$$

Although VSV don't say it explicitly, they assume $V_0 = 1$ in other equations, inserting this we have

$$g(\theta) = (2/\pi)\,\{\ \sqrt{1+\cos\theta_0}\,/\,\sqrt{\cos\theta - \cos\theta_0}\ +\pi/2 - \sin^{-1}[\,\cos(\theta_0/2)\,/\,\cos(\theta/2)\,\}$$

One last step is to replace $\sqrt{1+\cos\theta_0} = \sqrt{2}\ \cos(\theta_0/2)$ so *my* final result is this (theirs is underneath):

$$g(\theta) = (2/\pi)\ \{\ \sqrt{2}\ \cos(\theta_0/2)/\sqrt{\cos\theta - \cos\theta_0}\ + \pi/2 - \sin^{-1}[\ \cos(\theta_0/2) / \cos(\theta/2)\ \}$$

$$g(\vartheta) = \frac{\sqrt{2}}{\pi}\left\{\frac{2\cos\frac{1}{2}\theta_0}{\sqrt{\cos\vartheta - \cos\theta_0}} + \frac{\pi}{2} - \arcsin\left(\frac{\cos\frac{1}{2}\theta_0}{\cos\frac{1}{2}\theta}\right)\right\}. \qquad (1.\ 111)$$

// wrong!

So we see two relatively small typo errors here:

(1) the 2 and $\sqrt{2}$ should be swapped.
(2) the θ in their arcsin denominator should be their script θ variable.

They are thus "pretty close". So I will be using this

$g(\theta) = (2/\pi)\ \{\sqrt{2}\ \cos(\theta_0/2)/\sqrt{\cos\theta - \cos\theta_0}\ + \pi/2 - \sin^{-1}[\ \cos(\theta_0/2) / \cos(\theta/2)\ \}$ // 1.111 corrected