Green's Function for an Oblate Hyperboloid PhL 1.27.10

Summary 2.28.10. This is perhaps my first "bloid attempt" dated 1.27.10. I use the complex P and Q functions, the "right" coordinate parameterization where an entire bloid has a single value of ξ (though I thought I was using the wrong one), and I use a hyperboloid instead of the later spheroid as the "dividing surface" between two regions. The Smythian form is this:

$$V(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_<) - Q_n^m(\xi_1) P_n^m(\xi_<)] P_n^m(\xi_>) Q_n^m(j\zeta) e^{im\varphi}$$
$$V(\zeta,\xi,\varphi) = \Sigma_{nm} F_{nm} [P_n^m(\xi_1) Q_n^m(\xi_<) - Q_n^m(\xi_1) P_n^m(\xi_<)] P_n^m(\xi_>) Q_n^m(j\zeta) e^{im\varphi}$$

This approach requires orthogonal Q functions to isolate the coefficients, something like this:

$$\int_0^\infty d\zeta \; Q_n^\mu(i\zeta) Q_{n'}^\mu(i\zeta)^* = H_n(\xi_1) \delta_{n,n'}$$

I was able to formulate a S-L problem in ξ with Q(0) = 0 and interval (0,i∞) which produced a spectrum for n and produced the above orthogonality. But then I found that the resulting n spectrum was forcing V = 0 in the neck [since Q(0) = 0] which I know is wrong (but good for the jigger problem). I tried some rescue attempts, but did not succeed. I tried (-i∞,i∞) as an alternate interval which gave just a continuous n spectrum, then I got side-tracked on those $Q_n^\mu(i\zeta)_{OR}$ type functions (" Idea #2"). I then gave up on this approach. In retrospect, my attempt #4 presumably-successful solution to this problem was

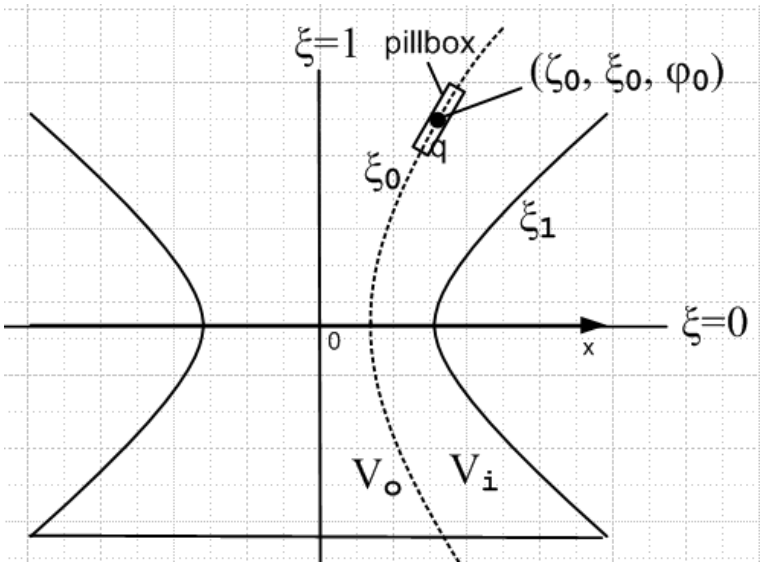
$$V(\zeta,\xi,\varphi) = \Sigma_{nm} C_{nm} [g_4(n,m) p_n^m(\zeta_<) - 2(-1)^m q_n^m(\zeta_<)] q_n^m(\zeta_>) P_n^m(\xi) \cos(m\varphi) \quad //\text{upper} \quad \zeta \geq 0$$
$$V(\zeta,\xi,\varphi) = \Sigma_{nm} C_{nm} [\quad 2(-1)^m q_n^m(\zeta_<)] q_n^m(\zeta_>) P_n^m(\xi) \cos(m\varphi) \quad //\text{lower} \quad \zeta \leq 0$$

I suspect now that the approach of this current doc might still be workable if you start with the real p and q functions, but it would be a very different *looking* solution since it would have $\xi_>$ instead of $\zeta_>$ etc. It would be difficult to verify this result in the cone limit, since the cone solution is based on the cone method rather than the method given here. But it still might be viable. The orthogonality of the Q would come out reminiscent of the messy orthogonality of P's in the cone method. At this point, I decided to use the cone method to get V = 0 on the bloid, then I went with the spheroid as the dividing Green's surface.

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Our Green's point charge sits inside the one-sheet hyperboloid:



V_i is on the "inside" meaning here only that it is the side close to the grounded surface. I choose this bad convention just because it matches the spheroid problem I just finished doing.

1. The Smythian Form. We seek a linear combination of oblate atoms that causes V = 0 on the bloid marked by parameter ξ_1

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi) Q_n^m(j\zeta) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) Q_n^m(j\zeta) e^{im\varphi}$$

These forms have these properties:

- when $\zeta \to \infty$, which can happen in either region, potential decays.
- when $\xi = \xi_1$, we have $V_i = 0$ as it must be on the bloid surface
- when $\xi = \xi_0$, $V_i = V_o$ everywhere away from the point charge.
- we don't get either V = 0 at ξ_0. (simpler forms violate this)

So, I think we have it! There are no other choices for the ζ function, or for the φ function.

We can write this Smythian form in the usual manner as a single expression

$$V(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_<) - Q_n^m(\xi_1) P_n^m(\xi_<)] P_n^m(\xi_>) Q_n^m(j\zeta) e^{im\varphi}$$

$$\xi_> = \max(\xi,\xi_0) \qquad \xi_< = \min(\xi,\xi_0)$$

Note added 2.14.10. When I was doing this, I had just come from the oblate spheroid problem, so I think I was assuming m = all integers, and n = 0,1,2... I am not doing the "cone method" here, but am causing V = 0 on the bloid using "the spheroid method", you might say, built into the Smythian form. Notice that the picture does not say which way I am parameterizing the coordinates in the lower region. At this point, my SL problems are in the φ and ξ spaces, causing m and n to quantize. That fact really says that I was assuming ξ goes to -1 in the lower region, and that in turn implies the parameterization type, which is that the entire spheroid gets the same ζ, and upper and lower have opposite ξ.

2. Compute Normal Derivatives and take difference, use Wronskian. Our derivative of interest is now ∂_ξ since this is normal to our new pillbox. So we have (note where the primes go)

$$\partial_\xi V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m{}'(\xi) Q_n^m(j\zeta) e^{im\varphi}$$
$$\partial_\xi V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m{}'(\xi) - Q_n^m(\xi_1) P_n^m{}'(\xi)] P_n^m(\xi_0) Q_n^m(j\zeta) e^{im\varphi}$$

Then we shall need the difference:

$$\partial_\xi V_i(\zeta,\xi) - \partial_\xi V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta) e^{im\varphi} *$$

$$\{ [P_n^m(\xi_1) Q_n^m{}'(\xi) - Q_n^m(\xi_1) P_n^m{}'(\xi)] P_n^m(\xi_0) - [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m{}'(\xi) \}$$

But we are only going to want this quantity at $\xi = \xi_0$ which marks the bloid holding our point charge, so set $\xi = \xi_0$

$$\partial_\xi V_i(\zeta,\xi) - \partial_\xi V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta) e^{im\varphi} *$$

$$\{ [P_n^m(\xi_1) Q_n^m{}'(\xi_0) - Q_n^m(\xi_1) P_n^m{}'(\xi_0)] P_n^m(\xi_0) - [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m{}'(\xi_0) \}$$

$$\partial_\xi V_i(\zeta,\xi) - \partial_\xi V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta) e^{im\varphi} *$$

$$\{ [P_n^m(\xi_1) Q_n^m{}'(\xi_0) P_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m{}'(\xi_0) P_n^m(\xi_0)]$$
$$- [P_n^m(\xi_1) Q_n^m(\xi_0) P_n^m{}'(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0) P_n^m{}'(\xi_0)] \}$$

$$\partial_\xi V_i(\zeta,\xi) - \partial_\xi V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta) e^{im\varphi} *$$

$$\{ P_n^m(\xi_1) Q_n^m{}'(\xi_0) P_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m{}'(\xi_0) P_n^m(\xi_0)$$
$$- P_n^m(\xi_1) Q_n^m(\xi_0) P_n^m{}'(\xi_0) + Q_n^m(\xi_1) P_n^m(\xi_0) P_n^m{}'(\xi_0) \}$$

This time, it is the second and fourth terms which cancel, leaving us with

$$\partial_\xi V_i(\zeta,\xi) - \partial_\xi V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta)\ e^{im\varphi} * P_n^m(\xi_1)\ [Q_n^m{}'(\xi_0) P_n^m(\xi_0) - Q_n^m(\xi_0) P_n^m{}'(\xi_0)]$$

$$= \Sigma_{nm} E_{nm} Q_n^m(j\zeta)\ e^{im\varphi} * P_n^m(\xi_1)\ W[P_n^m(\xi_0), Q_n^m(\xi_0)]$$

Bateman tells us that

$$W[P_n^m(z), Q_n^m(z)] = (-1)^m (n+m)! / (n-m)! * 1/(1-z^2)$$

So we then have

$$\partial_\xi V_i(\zeta,\xi) - \partial_\xi V_o(\zeta,\xi) = (1-\xi_0^2)^{-1}\ \Sigma_{nm} (-1)^m [(n+m)! / (n-m)!]\ E_{nm}\ Q_n^m(j\zeta)\ P_n^m(\xi_1)\ e^{im\varphi} \qquad (*)$$

which I would again say is "not too bad".

3. Apply Gauss's Law to pillbox around point charge q.

In the Exterior Green's case, we got this result:

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\ (h_2/h_3h_1)\ \delta(\varphi-\varphi_0)\ \delta(\xi - \xi_0) \qquad q_1, q_2, q_3 = \xi, \zeta, \varphi$$

We now want to swap $1 \leftrightarrow 2$ in order to get [we maintain $q_1, q_2, q_3 = \xi, \zeta, \varphi$]

$$\partial_\xi V_i - \partial_\xi V_o = (q/\varepsilon)\ (h_1/h_3h_2)\ \delta(\varphi-\varphi_0)\ \delta(\zeta - \zeta_0)$$

where

$$h_1 = c_1 \sqrt{\xi^2+\zeta^2} / \sqrt{1-\xi^2}$$
$$h_2 = c_1 \sqrt{\xi^2+\zeta^2} / \sqrt{1+\zeta^2}$$
$$h_3 = c_1 \sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}$$

$$h_2h_3 = c_1 \sqrt{\xi^2+\zeta^2} / \sqrt{1+\zeta^2} * c_1 \sqrt{1+\zeta^2}\ \sqrt{1-\xi^2} = c_1^2 \sqrt{\xi^2+\zeta^2}\ \sqrt{1-\xi^2}$$
$$h_1 = c_1 \sqrt{\xi^2+\zeta^2} / \sqrt{1-\xi^2}$$

$$(1-\xi^2)\ c_1\ h_1 = h_2h_3 \qquad \Rightarrow \qquad (h_1/h_3h_2) = [(1-\xi^2)\ c_1]^{-1}$$

The charge density on the bloid surface is given by [this is Gauss's law for the pillbox]

$$\sigma/\varepsilon = \partial_n V_i - \partial_n V_o \qquad \partial_n = (1/h_1)\ \partial_\xi \quad \Rightarrow \qquad \sigma/\varepsilon = (1/h_1)[\ \partial_\xi V_i - \partial_\xi V_o\ ]$$

Therefore, the h_1 cancels when we write σ. Here are our two main results:

$$\partial_\xi V_i - \partial_\xi V_o = (q/\varepsilon)\ (h_1/h_3h_2)\ \delta(\varphi-\varphi_0)\ \delta(\zeta - \zeta_0) \qquad = q/[\varepsilon c_1(1-\xi^2)]\ \delta(\varphi-\varphi_0)\ \delta(\zeta - \zeta_0) \qquad (**)$$

$\sigma/\varepsilon = (1/h_1)[\ \partial_\xi V_i - \partial_\xi V_o\ ] = (q/\varepsilon)\,(1/h_3h_2)\,\delta(\varphi-\varphi_0)\,\delta(\varphi-\varphi_0)\,\delta(\zeta-\zeta_0)$

$dA_2 = h_3h_2dq_3dq_2 = h_3h_2d\varphi\,d\zeta$ // area of pillbox side

This then is the effective surface charge distribution "of the point charge q".

So we now know from (*) and (**) that

$$(q/\varepsilon)\,(h_1/h_3h_2)\,\delta(\varphi-\varphi_0)\,\delta(\zeta-\zeta_0) = (1-\xi_0^2)^{-1}\ \Sigma_{nm}\,(-1)^m\,[(n+m)!\,/\,(n-m)!]\,E_{nm}\ Q_n^m(j\zeta)\,P_n^m(\xi_1)\,e^{im\varphi}$$

To review: we computed the normal difference across the bloid at the point charge location and set this ~ equal first to the surface charge density that charge represents, and second to our Smythian form derivative difference, and we got the above! Now it should be an easy matter to computer the E_{nm}. In my form, we have both positive and negative m integers in the m sum, whereas Smythe uses $\cos(m[\varphi-\varphi_m])$.

4. Use orthogonality twice to compute the coefficients E_{nm} : first the φ step.

We first apply $\int d\varphi\ e^{-im\varphi}$ to both sides of the above and use the azimuthal orthogonality relation,

$$\int d\varphi\ e^{-im\varphi'}\ e^{im\varphi} = 2\pi\ \delta_{m',m}$$

to get

$$(q/\varepsilon)\,(h_1/h_3h_2)\,e^{-im'\varphi_0}\,\delta(\zeta-\zeta_0) = (1-\xi_0^2)^{-1}\ \Sigma_{nm}\,(-1)^m\,[(n+m)!\,/\,(n-m)!]\,E_{nm}\ Q_n^m(j\zeta)\,P_n^m(\xi_1)\,2\pi\,\delta_{m',m}$$

$$= [\ 2\pi\,/\,(1-\xi_0^2)]\ \Sigma_n\,(-1)^{m'}\,[(n+m')!\,/\,(n-m')!]\,E_{nm'}\ P_n^{m'}(\xi_1)\,Q_n^{m'}(j\zeta)$$

Now suddenly things are different! <u>We need some kind of Q orthogonality in order to continue</u>! So we pause for a digression, and hopefully after that we can continue here where we left off.

5. The ζ direction as a Sturm Liouville Problem: asking the questions

The ODE of interest is the Legendre equation, our solutions are $P_\nu^\mu(i\zeta)$ and $Q_\nu^\mu(i\zeta)$. Because we have a full azimuth in our Green's problem, we know μ = m (integers either sign including 0). In the "polar" ξ department we have ξ = 1 and ξ = -1 both included in our "range" of solution, so I think that forces ν = n (positive integers including 0). [This is wrong: my ξ range is only (0,1) in this problem's set up].

In the Smythe cone problem, which we did in sphericals, we had ν = the zeros of P. We could consider doing that here in ξ, but that would seem to violate our rule that we need n = integer in ξ to stay finite at ξ = -1, which is a very regular location it seems in our solution. So let's for the moment continue to assume $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$.

Our next item is that the "interval" for our ODE is in z = i0 to z = i∞. We know (finally) that both P and Q are finite at the 0 end of things, and only Q is finite at the ∞ end of things. So if we consider this interval (i0,i∞), we might be able to argue that the $Q_n^m(i\zeta)$ are not only solutions, but are orthogonal

eigenfunctions somehow. Q is certainly oscillatory in this range (example below). So let's just approach this as an isolated SL problem and ask about such orthogonal solutions. Knowing nothing about the other dimensions, we would be looking at the general solution $Q_\nu^\mu(i\zeta)$ with no quantization. If we think of μ as a fixed number, maybe we expect to see something *like* this:

$$\int_0^\infty d\zeta \; Q_\nu^\mu(i\zeta)\, Q_{\nu'}^\mu(i\zeta)^* = K_{\nu\mu}\, \delta(\nu-\nu')$$

Firstly, I want to know: what is the "general theory" on this type of question?

Is the ODE still self-adjoint when we replace $z = i\zeta$? Well, here is the normal Legendre ODE

$L = (1-z^2)\, D^2 - 2z\, D + [\, n(n+1) - m^2/(1-z^2)]$ $\qquad$ $Lu = 0$ $\qquad$ Bateman p 121

Let's rewrite this to show it has the Stak self-adjoint form $L = -D(pD) + q$

$$L_z = \partial_z[(1-z^2)\partial_z] + [\, n(n+1) - m^2/(1-z^2)] \quad \Rightarrow \quad p = -(1-z^2) \quad q = [\, n(n+1) - m^2/(1-z^2)]$$

Now if we take $z = i\zeta$, then $dz = i\, d\zeta$ and $\partial_z = -i\partial_\zeta$ and it all becomes

$$L_\zeta = (-i\partial_\zeta)[(1+\zeta^2)(-i\partial_\zeta)] + [\, n(n+1) - m^2/(1+\xi^2)]$$

$$= -\partial_\zeta[(1+\zeta^2)\partial_\zeta] + [\, n(n+1) - m^2/(1+\xi^2)] \quad \Rightarrow \quad p = +(1+\zeta^2) \quad q = [\, n(n+1) - m^2/(1+\xi^2)]$$

So this is manifestly formally self-adjoint.

What about the boundary conditions at the endpoints. They are supposed to be homogeneous or finite in a SL problem, I think. At the 0 end, something like $Au(0) + Bu'(0) = 0$ is required in an ODE. This is where my understanding is lacking I think.

6. The ζ direction as a Sturm Liouville Problem: answering the questions!

I had to go off on a *long* side trip which involved these documents:

Properties of Legendre functions.doc	math / ODE / Legendre
When are ODE solutions orthogonal.doc	math / ODE
Attempting the oblate Q integral.doc	math / Integrals * GR

Here is what I learned:

(a) the orthogonality relation conjectured above does in fact exist and is valid, but in this form:

$$(\, u^\mu_\nu,\, u^\mu_{\nu'}) = \int_0^\infty d\zeta \; Q_\nu^\mu(i\zeta)^*\, Q_{\nu'}^\mu(i\zeta) = K_\nu^\mu\, \delta_{\nu,\nu'}$$

(b) for a given value of μ, there is a finite discrete spectrum with these eigenvalues

$\nu = \mu - N \quad N = 2,4,6... \quad$ but stops at last N such that $\nu > -1/2$

(c) the spectrum is created by our imposition of the homogeneous BC at $i\zeta = 0+$:

$$u^{\mu}_{\nu}(0+) = Q_{\nu}^{\mu}(0+) = 0$$

I showed that there was no other homo BC that is different than this. It is the only one. [What about using the derivative Q' instead?]

(d) the truncation of the spectrum at $\nu > -1/2$ is caused by the requirement that u^{μ}_{ν} be of finite s norm at the high endpoint.

(e) An exact and general formula for these eigenfunctions is this:

$$Q_{\mu-N}^{\mu}(z) = 2^{\mu-N}\,[\,\Gamma(\kappa)\,\Gamma(\kappa+\mu)/\,\Gamma(2\kappa)\,]\,(z+1)^{-\mu/2+N-1}\,(z-1)^{-\mu/2}\;F(1-N,\,\kappa,\,2\kappa;\;2/(1+z))$$

where $\kappa = \kappa \equiv 1+\mu-N = 1+\nu \qquad N = 2,4,6...$

The F function is a finite series in $2/(1+z)$ having N terms, since its "a" parameter is a negative integer. So the general form of the eigenfunction is this:

$$Q_{\mu-N}^{\mu}(z) = A_{\mu N}\;(z+1)^{-\mu/2+N-1}\,(z-1)^{-\mu/2}\,\Sigma_{n=0}^{N-1}\,a_{n\mu}\;[2/(1+z)]^{n} \qquad a_{0\mu} = 1$$

As you see, there are no logs. We just have some power factors and a finite inverse series.

In the special case that μ = integer, one can use the following easy formula to compute the functions:

$$Q_{m-N}^{m}(z) = (z^2-1)^{m/2}\partial_m Q_{m-N}(z)$$

One can obtain the $Q_i(z)$ functions from Maple or Maxima, respectively:

LegendreQ(ν,μ,z) assoc_legendre_q(n,m,z)

(f) In order to compute the normalization constants K_{ν}^{μ}, you have to write out the function explicitly as shown above in (e) and then *do* the integral.

$$\int_0^{\infty} d\zeta\;|\,Q_{\nu}^{\mu}(i\zeta)|^2 = K_{\nu}^{\mu}$$

The integrals involve nothing but elementary functions. I found, for example, that $K_{2}^{4} = 72\pi$.

(g) In our application here we have μ = integer m. Although all integers m are part of the φ spectrum, only certain m are allowed in order to even have a ζ spectrum! And for each allowed m, there are only m/2 allowed values of n. Here is a list of the eigenfunctions, ordered by increasing integer m. Notice there are no eigenfunctions for negative m or even for m = 0 or m = 1. This does seem unusual, but it is true.

$m \leq 1$	none!
m=2	$Q_0^2(i\xi)$
m=3	$Q_1^3(i\xi)$
m=4	$Q_2^4(i\xi)$, $Q_0^4(i\xi)$
m=5	$Q_3^5(i\xi)$, $Q_1^5(i\xi)$
m=6	$Q_4^6(i\xi)$, $Q_2^6(i\xi)$, $Q_0^6(i\xi)$
m=7	$Q_5^7(i\xi)$, $Q_3^7(i\xi)$, $Q_1^7(i\xi)$
etc.	// in general, we have n = m-2, m-4.... (1 or 0)

(h) Two of these functions that I calculated using the methods above:

$$Q_2^4(z) = 48z/(z^2-1)^2$$
$$Q_0^4(z) = 24z(z^2+1)/(z^2-1)^2$$

Notice that for $z = i\xi$, these particular functions are pure imaginary.

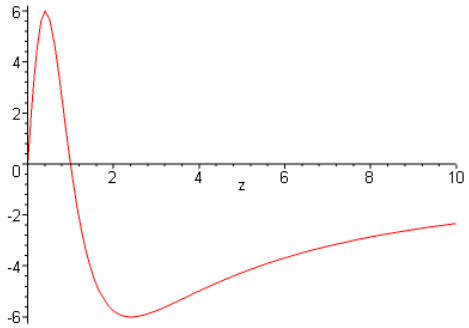
(i) An easy way to the functions *when μ is an integer m* is to use the formula

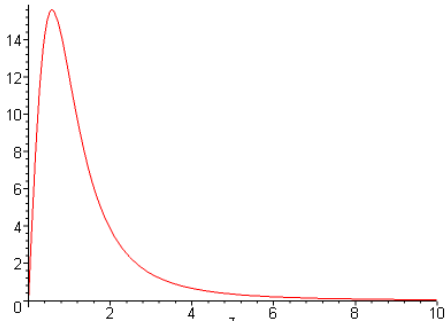
$$Q_{m-N}^m(z) = (z^2-1)^{m/2}\partial_m Q_{m-N}(z)$$

For general μ, one must use the formula shown in (e) above.

I conjectured that these functions were "oscillatory" above, and here is a plot of the two functions shown above:

plot(Im(LegendreQ(0,4,I*z)),z=0..10); left
plot(Im(LegendreQ(2,4,I*z)),z=0..10); right





You can see how they might be orthogonal. I verified this analytically *and* numerically. I computed the norm factor K_2^4 and got 72π, but analytically and numerically !

7. Continuing with our oblate bloid solution : the final answer.

In section 4 above, we left off with this result:

$$(q/\varepsilon)\,(h_1/h_3h_2)\,e^{-im'\varphi_0}\,\delta(\zeta-\zeta_0)$$
$$= [\,2\pi / (1-\xi_0^2)]\;\Sigma_n\,(-1)^{m'}\;[(n+m')!\,/\,(n-m')!]\;E_{nm'}\;P_n^{m'}(\xi_1)\;Q_n^{m'}(j\zeta)$$

We now multiply both sides by $Q_{n'}^{m'}(j\zeta)^*$ and integrate $\int_0^\infty d\zeta$. But first we have a new observation: the values of n which appear in the n sum on the right are not "all positive integers", but are only those integers that are "in the spectrum of the ζ problem". Only then can we do our "inversion". If we find a solution, since it is unique, we have it, and then this assumption is justified de facto. So we use the integral result shown above, which we restate here as

$$\int_0^\infty d\zeta\;\; Q_{n'}^{m'}(i\zeta)^*\;Q_n^{m'}(i\zeta)\;=K_{n'}^{m'}\;\delta_{n',n}$$

We then find that

$$(q/\varepsilon)\,(h_1/h_3h_2)\,e^{-im'\varphi_0}\,Q_{n'}^{m'}(j\zeta_0)^*$$

$$= [\,2\pi / (1-\xi_0^2)]\;\Sigma_n\,(-1)^{m'}\;[(n+m')!\,/\,(n-m')!]\;E_{nm'}\;P_n^{m'}(\xi_1)\;K_{n'}^{m}\;\delta_{n',n}$$

$$= [\,2\pi / (1-\xi_0^2)]\;(-1)^{m'}\;[(n'+m')!\,/\,(n'-m')!]\;E_{n'm'}\;P_n^{m'}(\xi_1)\;K_{n'}^{m'}$$

and we now remove all primes to get

$$(q/\varepsilon)\,(h_1/h_3h_2)\,e^{-im\varphi_0}\,Q_n^m(j\zeta_0)^*\;=[\,2\pi / (1-\xi_0^2)]\;(-1)^m\,[(n+m)!\,/(n-m)!]\;E_{nm}\;P_n^m(\xi_1)\;K_n^m$$

The scale factors on the LHS are evaluated at $\xi, \zeta, \varphi = \xi_0, \zeta_0, \varphi_0$ which is the location of our point charge around which we put the pillbox. In fact, we have

$$(h_1/h_3h_2)\;=\;[(1-\xi_0^2)\,c_1]^{-1}$$

and the factor $(1-\xi_0^2)^{-1}$ pleasantly cancels on both side so we get:

$$(q/c_1\varepsilon)e^{-im\varphi_0}\,Q_n^m(j\zeta_0)^*\;=2\pi\,(-1)^m\,[(n+m)!\,/(n-m)!]\;E_{nm}\;P_n^m(\xi_1)\;K_n^m$$

We now know our coefficients E_{nm} (this is the whole point of this exercise !)

$$E_{nm}\;=(q/[2\pi c_1\varepsilon\,K_n^m])\;e^{-im\varphi_0}(-1)^m\;\;[(n-m)!\,/(n+m)!]\;Q_n^m(j\zeta_0)^*\,/\,P_n^m(\xi_1)$$

We can now copy down our Smythian form

$$V(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_<) - Q_n^m(\xi_1) P_n^m(\xi_<)] P_n^m(\xi_>) Q_n^m(j\zeta) e^{im\varphi}$$

$$\xi_> = \max(\xi,\xi_0) \qquad \xi_< = \min(\xi,\xi_0)$$

and we then install our hard-won coefficients:

$$V(\zeta,\xi,\varphi) = (q/[2\pi c_1 \varepsilon]) \Sigma_{nm} (1/K_n^m] (-1)^m [(n-m)! /(n+m)!] e^{im(\varphi-\varphi 0)}$$

$$Q_n^m(j\zeta_0)^* / P_n^m(\xi_1) [P_n^m(\xi_1) Q_n^m(\xi_<) - Q_n^m(\xi_1) P_n^m(\xi_<)] P_n^m(\xi_>) Q_n^m(j\zeta)$$

$$V(\zeta,\xi,\varphi) = (q/[2\pi c_1 \varepsilon]) \Sigma_{nm} (1/K_n^m) (-1)^m [(n-m)! /(n+m)!] e^{im(\varphi-\varphi 0)}$$

$$* [Q_n^m(\xi_<) - \{ Q_n^m(\xi_1)/P_n^m(\xi_1)\} P_n^m(\xi_<)] P_n^m(\xi_>) Q_n^m(j\zeta) [Q_n^m(j\zeta_0)]^*$$

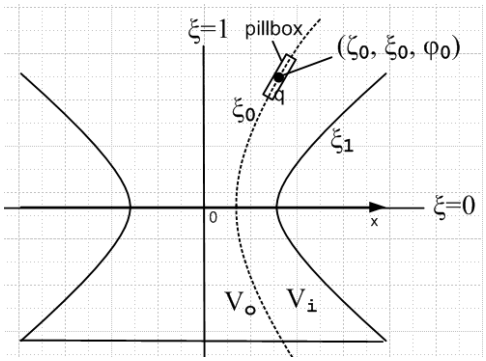
$$= g(\zeta,\xi,\varphi \mid \zeta_0,\xi_0,\varphi_0) \qquad \text{for hyperboloid}$$

$$\xi_> = \max(\xi,\xi_0) \qquad \xi_< = \min(\xi,\xi_0)$$

This is our **final answer** in closed form! The K_n^m are the Q function orthogonality constants that we have to go compute if we really want to use this formula. Note that the ζ_0 Q has a * on it. And note that the summation only includes spectral points. So we might write it as

$$\Sigma_{nm} = \Sigma_{m=2}^{\infty} \Sigma_{n=(0/1)}^{m-2}$$

where the lower endpoint is 0 if m is even, and 1 if m is odd. And here again is our picture.



Comment: For me, this is an astounding result. You put your Green's point charge at an arbitrary location on the inside of this infinite grounded one-sheeted oblate hyperboloid, some complicated charge is induced on the bloid, and that charge plus the point charge produces the potential shown above!

As quick checks on our result, we let $\xi \rightarrow \xi_1$ and we expect to get V = 0. In this case,

$$\xi_> = \max(\xi,\xi_0) = \xi_0 \qquad \xi_< = \min(\xi,\xi_0) = \xi = \xi_1$$

$$V(\zeta,\xi,\varphi) = (q/[2\pi c_1\varepsilon]) \Sigma_{nm} (1/K_n^m) (-1)^m \ [(n-m)!/(n+m)!] \ e^{im(\varphi-\varphi 0)}$$

$$* \ [Q_n^m(\xi_1) - \{ Q_n^m(\xi_1)/P_n^m(\xi_1)\} P_n^m(\xi_1)] \ P_n^m(\xi_0) \ Q_n^m(j\zeta) \ [Q_n^m(j\zeta_0)]^* = 0$$

so at least that limit works.

I am now of course going to be very interested in taking limits of this result, maybe looking at the charge distribution, and so on.

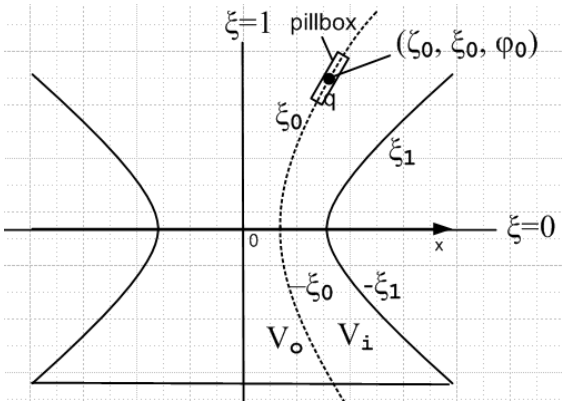
8. Physical Interpretation (looking for trouble).

Right off the bat, I see trouble. In the ζ SL problem, I required $Q_n^m(0+) = 0$ to get quantization of the n for given m. But, looking at the above solution, that implies that V = 0 when $\zeta = 0$. But $\zeta = 0$ is the super flat spheroid that is the interior disk in this limit. So my "solution" to the problem is producing V = 0 everywhere in the hole! I know that cannot be right. Just put the charge in the hole a sit one inch away from it, potential cannot possibly be zero in the hole.

This arose from the way I treated the ζ SL problem. I arbitrarily required $Q_n^m(0+) = 0$ just to get a spectrum, with no physical connection to things. So something has to be fixed!

Basically, I eliminated all (n,m) loci which could make V vary on the ζ=0 disk. If I don't do this, then we have full capability with the $P_n^m(\xi)\, e^{im\varphi}$ combination to obtain an arbitrary on-disk value.

As I examine my picture, I see another related issue. In my "choice" of oblate coordinates, the upper and lower bloid halves have different ξ labels! We run (1→0) on the top, then (0→-1) on the bottom. This reminds me of the Smythe hole in plate problem set up. If I keep it the way I have it, then I have not really met the V=0 BC on the entire bloid. AND, there is a discontinuity in the potential as you cross the center disk, which is just free space.



It certainly would seem better for THIS problem to split the spheroids into two halves and have $\zeta > 0$ on the upper half and $\zeta < 0$ on the lower half, as Smythe did for his hole in plate problem. Then ξ is the label for a full top + bottom bloid.

How do we justify these two possible "choices" of coordinates? Certainly if we keep $\xi > 0$, our P and Q functions are still oblate "atoms", and our products still solve Laplace. That is a key point. And making $\zeta < 0$ does not stop $Q_n^m(j\zeta)$ from being a solution of its separated equation. This alternate setup is MUCH more appropriate for the geometry of this problem! Notice that $Q_n^m(z)$ decays for large |z| in all directions, so going up and down would give decay, and in fact we need decay in a large solid angle both up and

down in this problem. So somehow, if we adopt this new coordinate sub-choice for oblate, we have these Smythian forms

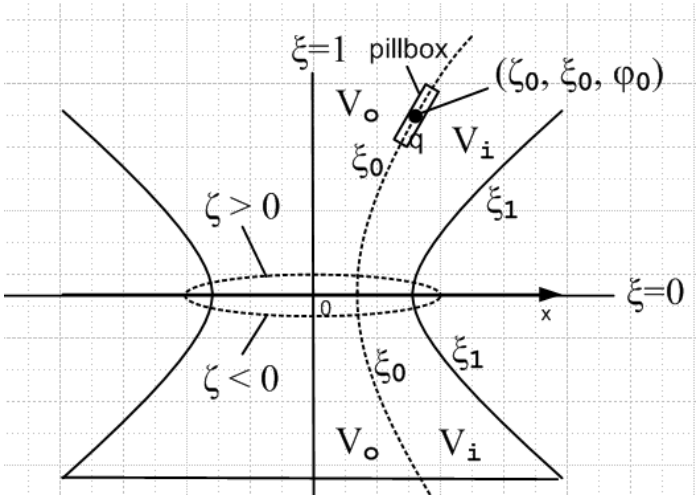
For $\zeta \geq 0$ and for ξ in (0,1) only

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi) Q_n^m(j\zeta) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) Q_n^m(j\zeta) e^{im\varphi}$$

For $\zeta \leq 0$ and for ξ in (0,1) only

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} F_{nm} [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi) Q_n^m(j\zeta) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} F_{nm} [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) Q_n^m(j\zeta) e^{im\varphi}$$

where we now have E and F coefficients to worry about. Certainly we would not claim the potential to be up/down symmetric by setting E = F. [But setting E = F does not make things symmetric if the Q(jζ) function is not symmetric.] Here is our corresponding picture:



So I am sort of repeating my own hole-in-*plate* approach here. We then have a new BC to impose, which is that the potential and its gradient be continuous going across the hole (assuming charge not in the hole). By using $Q_n^m(j\zeta)$ in both upper and lower regions, I have assured continuity of V since Q is continuous through the origin (I will discuss the cut there soon).

Matching of Potential and Its Gradients in the Hole.

Matching the potential gives $E_{nm} Q_n^m(j0) = F_{nm} Q_n^m(j0)$. So either $Q_n^m(j0) = 0$, or $E_{nm} = F_{nm}$.

So this brings us to the gradient. It has three directions of interest of course.

(1) The ξ direction gradient.

This is gradient in the $\hat{\boldsymbol{\rho}}$ direction in the hole. We have already computed things for this direction: So we now have: $\partial_n = (1/h_1)\,\partial_\xi$ and $h_1 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1-\xi^2} = c_1\xi\,/\sqrt{1-\xi^2}$ in the hole.

For $\zeta \geq 0$ and for ξ in (0,1) only

$$\partial_\xi V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, E_{nm}\,[P_n^m(\xi_1)\,Q_n^m(\xi_0) - Q_n^m(\xi_1)\,P_n^m(\xi_0)\,]\,P_n^m{}'(\xi)\,Q_n^m(j\zeta)\,e^{im\varphi}$$
$$\partial_\xi V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, E_{nm}\,[P_n^m(\xi_1)\,Q_n^m{}'(\xi) - Q_n^m(\xi_1)\,P_n^m{}'(\xi)\,]\,P_n^m(\xi_0)\,Q_n^m(j\zeta)e^{im\varphi}$$

For $\zeta \leq 0$ and for ξ in (0,1) only

$$\partial_\xi V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, F_{nm}\,[P_n^m(\xi_1)\,Q_n^m(\xi_0) - Q_n^m(\xi_1)\,P_n^m(\xi_0)\,]\,P_n^m{}'(\xi)\,Q_n^m(j\zeta)\,e^{im\varphi}$$
$$\partial_\xi V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, F_{nm}\,[P_n^m(\xi_1)\,Q_n^m{}'(\xi) - Q_n^m(\xi_1)\,P_n^m{}'(\xi)\,]\,P_n^m(\xi_0)\,Q_n^m(j\zeta)e^{im\varphi}$$

Now if we require that things match up in the hole, we set $\zeta = 0$ and repeat the above four lines:

For $\zeta \geq 0$ and for ξ in (0,1) only

$$\partial_\xi V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, E_{nm}\,[P_n^m(\xi_1)\,Q_n^m(\xi_0) - Q_n^m(\xi_1)\,P_n^m(\xi_0)\,]\,P_n^m{}'(\xi)\,Q_n^m(0)\,e^{im\varphi}$$
$$\partial_\xi V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, E_{nm}\,[P_n^m(\xi_1)\,Q_n^m{}'(\xi) - Q_n^m(\xi_1)\,P_n^m{}'(\xi)\,]\,P_n^m(\xi_0)\,Q_n^m(0)e^{im\varphi}$$

For $\zeta \leq 0$ and for ξ in (0,1) only

$$\partial_\xi V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, F_{nm}\,[P_n^m(\xi_1)\,Q_n^m(\xi_0) - Q_n^m(\xi_1)\,P_n^m(\xi_0)\,]\,P_n^m{}'(\xi)\,Q_n^m(0)\,e^{im\varphi}$$
$$\partial_\xi V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, F_{nm}\,[P_n^m(\xi_1)\,Q_n^m{}'(\xi) - Q_n^m(\xi_1)\,P_n^m{}'(\xi)\,]\,P_n^m(\xi_0)\,Q_n^m(0)e^{im\varphi}$$

where we are using the $Q_n^m(0)_{0,0R}$ (principle sheet, z=1 cut pulled right) as discussed in a separate doc. In order to have things match for all φ and for all ξ, it certainly seems that we will need:

$$E_{nm}\,Q_n^m(0) = F_{nm}\,Q_n^m(0)$$

which we write this way to allow for the possibility that $Q_n^m(0)$ might = 0 for certain m,n values. In fact we know that

$$Q_n^m(0)_{0,0R} = \sqrt{\pi}\;2^{m-1}\,e^{+i\pi(-n-1)/2}\,[\,\Gamma(1/2 + n/2 + m/2)\,/\,\Gamma(1 + n/2 - m/2)]$$

We know what this means. For $n = m - N$, $N = 2,4,6...$ we will have $Q_n^m(0) = 0$, so for these values, our matching condition here tells us nothing about the relationship between E_{nm} and F_{nm}. For all other values, the two must be equal. Let's put this on hold and consider the other gradients.

(2) The φ direction gradient. Let's just do the derivative on all four of our Smythian form lines, AND lets set $\zeta = 0$ for being in the hole:

For $\zeta \geq 0$ and for ξ in (0,1) only

$$\partial_\varphi V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, im\, E_{nm}\,[P_n^m(\xi_1)\, Q_n^m(\xi_0) - Q_n^m(\xi_1)\, P_n^m(\xi_0)\,]\, P_n^m(\xi)\; Q_n^m(0)\, e^{im\varphi}$$
$$\partial_\varphi V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, im\, E_{nm}\,[P_n^m(\xi_1)\, Q_n^m(\xi) - Q_n^m(\xi_1)\, P_n^m(\xi)\,]\, P_n^m(\xi_0)\, Q_n^m(0) e^{im\varphi}$$

For $\zeta \le 0$ and for ξ in (0,1) only

$$\partial_\varphi V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, im\, F_{nm}\,[P_n^m(\xi_1)\, Q_n^m(\xi_0) - Q_n^m(\xi_1)\, P_n^m(\xi_0)\,]\, P_n^m(\xi)\; Q_n^m(0)\, e^{im\varphi}$$
$$\partial_\varphi V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, im\, F_{nm}\,[P_n^m(\xi_1)\, Q_n^m(\xi) - Q_n^m(\xi_1)\, P_n^m(\xi)\,]\, P_n^m(\xi_0)\, Q_n^m(0) e^{im\varphi}$$

The related scale factor is $h_\varphi = h_3 = c_1 \sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}$ and $\partial_n = (1/h_3)\,\partial_\varphi$. In the hole, $\zeta = 0$ and this just becomes $h_\varphi = h_3 = c_1 \sqrt{1-\xi^2}$ which is thankfully not discontinuous. It is the same both above and below the hole plane. So, if we require that $\partial_\varphi V_o$ and be the same on both sides of the hole plane (which makes $\partial_\varphi V_o$ be the same, where **n** points in the φ direction), for all values of ξ and φ, it seems that we need to have

$$E_{nm}\, m\, Q_n^m(0) = F_{nm}\, m Q_n^m(0)$$

which is quite similar to our last result, except here we have an added bit of information. If m = 0, we know for sure there is no relationship forced here between the E_{nm} and the F_{nm} (that is to say, no relation between the E_{n0} and the F_{n0} coefficients, at least no relation forced by this gradient condition.)

So this too we shall now put on hold and do the one remaining gradient:

(3) The ζ direction gradient. First, here is the action of derivative ∂_ζ on things

For $\zeta \ge 0$ and for ξ in (0,1) only

$$\partial_\zeta V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, j\, E_{nm}\,[P_n^m(\xi_1)\, Q_n^m(\xi_0) - Q_n^m(\xi_1)\, P_n^m(\xi_0)\,]\, P_n^m(\xi)\; Q_n^m{}'(0)\, e^{im\varphi}$$
$$\partial_\zeta V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, j\, E_{nm}\,[P_n^m(\xi_1)\, Q_n^m(\xi) - Q_n^m(\xi_1)\, P_n^m(\xi)\,]\, P_n^m(\xi_0)\, Q_n^m{}'(j\zeta) e^{im\varphi}$$

For $\zeta \le 0$ and for ξ in (0,1) only

$$\partial_\zeta V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, j\, F_{nm}\,[P_n^m(\xi_1)\, Q_n^m(\xi_0) - Q_n^m(\xi_1)\, P_n^m(\xi_0)\,]\, P_n^m(\xi)\; Q_n^m{}'(0)\, e^{im\varphi}$$
$$\partial_\zeta V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, j\, F_{nm}\,[P_n^m(\xi_1)\, Q_n^m(\xi) - Q_n^m(\xi_1)\, P_n^m(\xi)\,]\, P_n^m(\xi_0)\, Q_n^m{}'(0) e^{im\varphi}$$

Again, for Q on the 0,0R sheet, we know $Q_n^m(z)$ is continuous as we move across the location the cut used to be in the 0,0L analysis. Nothing special happens at z = 0, so the function and all derivatives are going to be continuous there, so it is the same $Q_n^m{}'(0)$ we have in the above and below formulas.

Again doing the comparison, we conclude that

$$E_{nm}\, Q_n^m{}'(0) = F_{nm}\, Q_n^m{}'(0)$$

Again we can write $\partial_n = (1/h_2)\,\partial_\zeta$ where $h_2 = c_1 \sqrt{\xi^2+\zeta^2} / \sqrt{1+\zeta^2} = c_1\, \xi$ in the hole.

(4) <u>Summary of gradient matching in the hole</u>:

potential match: $E_{nm}\, Q_n^m(j0) = F_{nm}\, Q_n^m(j0)$

ξ gradient match: $E_{nm}\, Q_n^m(0) = F_{nm}\, Q_n^m(0)$ $\quad \partial_n = (1/h_1)\,\partial_\xi$ $\quad h_1 = c_1\xi / \sqrt{1-\xi^2}$

φ gradient match: $E_{nm}\, m\, Q_n^m(0) = F_{nm}\, mQ_n^m(0)$ $\quad \partial_n = (1/h_3)\,\partial_\varphi$ $\quad h_3 = c_1 \sqrt{1-\xi^2}$

ζ gradient match: $E_{nm}\, Q_n^{m\,\prime}(0) = F_{nm}\, Q_n^{m\,\prime}(0)$ $\quad \partial_n = (1/h_2)\,\partial_\zeta$ $\quad h_2 = c_1\,\xi$

Where do we go from here?

At this point, my inclination is to just have one set of constants E_{nm} and go back to the original Smythian form:

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, E_{nm}\, [P_n^m(\xi_1)\, Q_n^m(\xi_0) - Q_n^m(\xi_1)\, P_n^m(\xi_0)\,]\, P_n^m(\xi)\; Q_n^m(j\zeta)\, e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm}\, E_{nm}\, [P_n^m(\xi_1)\, Q_n^m(\xi) - Q_n^m(\xi_1)\, P_n^m(\xi)\,]\, P_n^m(\xi_0)\, Q_n^m(j\zeta) e^{im\varphi}$$

and now we have our upper/lower asymmetry handled by the innate asymmetry of the $Q_n^m(j\zeta)_{0,0R}$ functions, my prototype being $Q_1(i\zeta) = -1 + \zeta \cot^{-1}(\zeta)$. Now at least I don't have any mystery about this asymmetry.

What about quantum numbers? The φ gives all integer m as usual. Then we get all "the usual" integer n from the ξ dimension. And in the ζ dimension, I do not require $Q_n^m(0) = 0$, so I get no quantization there, and then any $Q_n^m(i\zeta)$ is viable as long as Re(n) > -1/2, [wrong] and this includes everything in my spectrum.

So the entire presentation above goes through unaltered until we reach this point, which I will now call section 7A:

7A. Continuing with our oblate bloid solution.

In section 4 above, we left off with this result:

$$(q/\varepsilon)\,(h_1/h_3h_2)\, e^{-im'\varphi_0}\, \delta(\zeta-\zeta_0)$$
$$= [\,2\pi / (1-\xi_0^2)]\; \Sigma_n\, (-1)^{m'}\; [(n+m')!\,/\,(n-m')!]\; E_{nm'}\; P_n^{m'}(\xi_1)\, Q_n^{m'}(j\zeta)$$

I am just dead in the water unless I can come up with some new orthogonality stuff! This solution method is now on permanent hold until this barrier can be overcome.

Idea #1: Suppose we try the SL problem in ζ with the range $(-i\infty, i\infty)$. In this SL problem, both endpoints are singular and both are limit point, and we know that $Q_\nu^\mu(j\zeta)$ are the only possible solutions. We no longer care anything about $Q_\nu^\mu(0)$ in particular. The spectrum for ν is then the entire continuous real axis from $(-1/2,\infty)$. We then expect or at least hope for a result I mentioned long ago, but which takes this new form:

$$\int_{-\infty}^{\infty} d\zeta \;\; Q_n^{\mu}(i\zeta)\, Q_{n'}^{\mu}(i\zeta)^* \;\; = C_n^{\mu}\, \delta(n-n') \qquad\qquad n \text{ in } (-1/2,\infty)$$

Suppose this were true, then we have some trouble as follows. Here is our form

$$(q/\varepsilon)\,(h_1/h_3h_2)\, e^{-im'\varphi_0}\, \delta(\zeta-\zeta_0)$$
$$= [\, 2\pi / (1-\xi_0^2)]\;\; \Sigma_n\, (-1)^{m'}\; [(n+m')!\,/\,(n-m')!]\; E_{nm'}\;\; P_n^{m'}(\xi_1)\; Q_n^{m'}(j\zeta)$$

and we want to apply $\int_{-\infty}^{\infty} d\zeta\, Q_{n'}^{m'}(i\zeta)^*$ to both sides and we get

$$(q/\varepsilon)\,(h_1/h_3h_2)\, e^{-im'\varphi_0}\, Q_{n'}^{m'}(i\zeta)^*$$
$$= [\, 2\pi / (1-\xi_0^2)]\;\; \Sigma_n\, (-1)^{m'}\; [(n+m')!\,/\,(n-m')!]\; E_{nm'}\;\; P_n^{m'}(\xi_1)\; C_n^{m'}\; \delta(n-n')$$

and then we have a "measure problem" on the RHS. We would then have to assume that Σ_n is really an integral over n, but then that violates the $P_n^m(\xi)$ rule which said we need n = integer to keep this from blowing up at $\xi = -1$ which is a loci inside our solution space.

So, as usual, I am between a rock and a hard place, with no one do advise me but myself and the web and the few books I have.

Idea #2. Go through the cut

$$Q_\nu^{\mu}(z)_{0,0L} \;=\; 2^{\nu}\;\; [\,\Gamma(1+\nu)\Gamma(1+\nu+\mu)/\,\Gamma(2+2\nu)]\;\; z^{-\nu-1} \qquad\qquad \text{// Bateman p 132 (36)}$$

I also fiddled around and found that

$$Q_1(z)_{0,1L} = Q_1(z)_{0,0L} \;- i\pi z$$

which makes me think that we are NOT going to get $z^{-\nu-1}$ going for $Q_n^{m'}(j\zeta)_{0,0R}$. So how should I go after the large z asymptotic behavior going south? All my Bateman results are north only. Let's look at a simple case:

$$Q_1(z)_{0,1L} = 2^{-2}\,\sqrt{\pi}\;\Gamma(2)/\Gamma(5/2)\;\; z^{-2}\; F(3/2,\, 1;\, 5/2;\;\; 1/z^2) \qquad\qquad \text{// Bateman p 122 (5)}$$

Well, I went off and studied this whole subject in the Legendre section, see Maze doc for a summary. It did not give me a way out.

At this point I wrote the Maze document, and I am now resuming here for a while.

Idea #3: The odd-ν idea to get V continuity at ζ = 0

This during a 4 AM Smith's walk today Feb 1. For the ζ SL homo BC, instead of taking Q(j0+) = 0 [which is the jigger problem], do this instead:

(1) Considering [Note that we are talking about the $Q_\nu^\mu(z=0\pm)_{0,0L}$ function here , and we are talking the parameterization where each spheroid has two ζ values, so $\zeta < 0$ in the lower bloid region. This is the preferred way to go. $Q_n^m(j\zeta)$ decays on all rays.]

$$Q_\nu^\mu(z=0\pm) = \sqrt{\pi}\, 2^{\mu-1}\, e^{\pm i\pi(-\nu-1)/2}\, [\, \Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)] \quad // \text{ p 134 (40)}$$

limit the spectrum to ν = values such that $e^{\pm i\pi(-\nu-1)/2}$ = the same for + and -. This would cause $Q_n^m(j\zeta)$ to be continuous at $\zeta = 0$, a condition we certainly want to have. This despite the presence of the cut. This seems to limit us to the following values for ν: $(\nu+1)/2$ = integer N. Then $e^{\pm i\pi(-\nu-1)/2} = e^{\pm i\pi N} = (-1)^N$ which is clearly the same for + and -. So that says $\nu = 2N+1$ N = 0,1,2... or ν = 1,3,5,7 or ν = odd. So then we are not forcing $Q_\nu^\mu(0) = 0$, so we are staying away from the jigger problem.

(2) Then require that $Q_\nu^\mu\,'(z=0\pm) = 0$ as the homo BC condition. This means the slope is zero coming in from both + and - directions, so the slope of Q is then continuous as well as Q being continuous. Having the slope continuous is consistent with there being no charge lying on the hole disk! I only set the slope to 0 because then we have a homo BC for the SL problem. Otherwise I would just say $Q_\nu^\mu\,'(z=0\pm)$ = same for + and - and does not have to be 0.

But you have to ask, in an asymmetric problem like this, why would you expect to have $Q_\nu^\mu\,'(z=0\pm) = 0$? Looking at

$$\partial_\zeta V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, j\, E_{nm}\, [P_n^m(\xi_1)\, Q_n^m(\xi_0) - Q_n^m(\xi_1)\, P_n^m(\xi_0)\,]\, P_n^m(\xi)\; Q_n^m\,'(j\zeta)\, e^{im\varphi}$$

we would conclude that $\partial_\zeta V_o(\zeta,\xi,\varphi) = 0$ at the hole plane. But for the Green's charge just above the hole plane, we know this cannot be true on the entire V_o part of the hole plane. It is true that we are getting into the region of how we parameterize the coordinates, but if we stay at $\zeta = +i\varepsilon$ and wander just above and over the hole disk, it does seem this is forcing $\partial_\zeta V_o(\zeta=0+,\xi,\varphi) = 0$. The close point charge shows this cannot be right, so this idea is no good. "Wrong Way, Dead End" .

But maybe we can take idea (1) without idea (2).

(2') Want to have $Q_\nu^\mu\,'(z=0+) = Q_\nu^\mu\,'(z=0-)$. Could figure out what this implies spectrally from Legendre properties. But how would we bring this into the SL problem as a homo BC?

Comments: In this idea, we let go of $Q_\nu^\mu(0) = 0$ as a homo BC and let $Q_\nu^\mu(0)$ come out whatever it is. It is the same for $\pm$ because we select odd ν only. But for gradient we need $Q_\nu^\mu\,'(z=0+) = Q_\nu^\mu\,'(z=0-)$ but this is not in the form of a homo BC for the ζ SL problem. I later learned that

$$Q_\nu^m\,'(0\pm) = \mp\, i\, Q_\nu^{m+1}(0\pm) \qquad m \geq 0 \text{ and integer}$$

$$Q_\nu^{-m}\,'(0\pm) = \mp\, i\, Q_\nu^{m+1}(0\pm)\, \Gamma(\nu-m+1)/\Gamma(\nu+m+1) \qquad m \geq 0 \text{ and integer}$$

With ν = odd only, these would say

$$Q_\nu^m\,'(0\pm) = \mp\, i\, Q_\nu^{m+1}(0) \qquad m \geq 0 \text{ and integer}$$

$$Q_\nu^{-m}{}'(0\pm) = \mp i\, Q_\nu^{m+1}(0)\, \Gamma(\nu-m+1)/\Gamma(\nu+m+1) \qquad m \geq 0 \text{ and integer}$$

and then our (2') condition becomes (for μ=m>0)

$$Q_\nu^{\mu}{}'(z=0+) = Q_\nu^{\mu}{}'(z=0-)$$

$$-\, i\, Q_\nu^{m+1}(0) = +\, iQ_\nu^{m+1}(0)$$

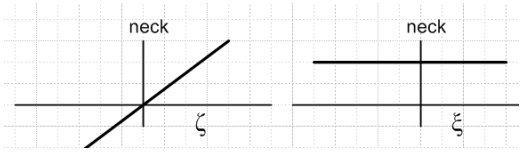
and we are then forced back to $Q_\nu^{m+1}(0) = 0$ again, which is what we were trying to "get away from", since this forces $V_o = 0$ on the hole plane (ie, this is the jigger problem). So Idea #4 fails.

Idea #4 Why not use the following Q function: $Q_n^m(j|\zeta|)$?

Above the hole this is $Q_n^m(j\zeta)$, and below the hole it is $Q_n^m(-j\zeta)$. In both regions this is a Laplace atom (see Bateman p 122 (6) area). Granted, one must be careful at ζ = 0. But this lets us use $Q_n^m(j\zeta)_{0,0L}$ on both top and bottom, so we get decay of the Q function both ways. The cost is the extra burden of requiring continuity at ζ = 0. We have seen Jackson use $e^{-k|z|}$ in his cylindrical coordinate work for example. Let's track through what this would do. We start with our Smythian form. (where I have now added abs value on ζ)

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi) Q_n^m(j|\zeta|) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) Q_n^m(j|\zeta|) e^{im\varphi}$$

Now, which coordinates are we using? If we use "spheroid has two labels ±ζ", then the ξ is (0,1) only. Our $Q_n^m(j|\zeta|)$ decays whether ζ is + or -, that is, whether we are above or below the hole plane. In this system, a point in the hole plane has an unambiguous single value of ξ. This is our system:



In this scheme, as we approach the hole ζ = 0 from either above or below, we have $Q_n^m(j0+)$ appearing in our formulas for both V_o and V_i. Therefore, we have achieved continuity of V in the hole.

But we also need to have continuity of $Q_n^m{}'(j|\zeta|)$ at the hole plane so that we don't have any charge appearing on that plane. Now the only way that f(|ζ|) can have a continuous slope at ζ = 0 is if that slope is zero (draw a picture), because this f is a mirror function. Thus, we need to have $Q_n^m{}'(j0+) = 0$. According to our Leg props doc we have [I show both signs here, but we are only interested in the upper sign]

$$Q_\nu^m{}'(0\pm) = \mp i\, Q_\nu^{m+1}(0\pm) \qquad m \geq 0 \text{ and integer}$$
$$Q_\nu^{-m}{}'(0\pm) = \mp i\, Q_\nu^{m+1}(0\pm)\, \Gamma(\nu-m+1)/\Gamma(\nu+m+1) \qquad m \geq 0 \text{ and integer}$$
$$Q_\nu^m{}'(0\pm) = \mp i\, Q_\nu^{-m+1}(0\pm)\, \Gamma(\nu+m+1)/\Gamma(\nu-m+1) \qquad m < 0 \text{ and integer}$$

Let's first look at the m > 0 case. In order to have $Q_n^m\,'(j0+) = 0$, we need $Q_\nu^{m+1}(0+) = 0$. But our Theorem 1 says $Q_\nu^{m+1}(0+) = 0$ only when m+1>0 and n = m-1,m-3.... -m-1. So these are the allowed n values when $m \geq 0$ which make $Q_n^m\,'(j0+) = 0$.

Now consider m < 0. Let's define $m_1 = -m$. Then we have m1 > 0 and from the last line above,

$$Q_\nu^{-m1}\,'(0\pm) = \mp i\, Q_\nu^{m1+1}(0\pm)\, \Gamma(\nu-m1+1)/\Gamma(\nu+m1+1) \qquad m1 \geq 0 \text{ and integer}$$

$$= \mp i\, Q_\nu^{m1+1}(0\pm)\, f(\nu,-m1) = \mp i\, Q_\nu^{m1+1}(0\pm)\, f(\nu,m)$$

There are now two possible ways to have a zero. The first is when $Q_\nu^{m1+1}(0\pm)$ has a zero, and this occurs when ν = m1-1,m1-3.... -m1-1. The second is when f(ν,m) has a zero. But Theorem 3 says for m<0, f(ν,m) has no zeros. So the only zeros when m < 0 are for ν = m1-1,m1-3.... -m1-1. BUT, be careful, because f(ν,m) has poles when ν = |m|-1, |m|-2...- |m| = m1-1,m1-2.... -m1. Only ν = -m1-1 avoids these poles. So we have to eliminate Q functions which blow up, and only ν = -m1-1 = m-1 survives for m < 0.

We can combine our results to say this:

$Q_n^m\,'(j0+) = 0$ and is finite when:

m > 0: n = m-1,m-3.... -m-1
m < 0: n = m-1

Now let's impose our requirement that our Q function decay for large ξ. Since $Q \sim |\zeta|^{-n-1}$ we have to rule out all n values which are < 0. This kills off the second line above, and leaves only part of the first line. So here is our new spectrum

$Q_n^m\,'(j0+) = 0$ and is finite, and $Q_n^m(j\xi)$ is finite s-norm, when:

m > 0: n = m-1,m-3.... 0/1 whichever one you end on

We now observe that V_o contains the product $P_n^m(\xi)\, Q_n^m(j|\zeta|)$. We now call upon,

Theorem 2: If m is an integer,
(a) $P_n^m(\xi)$ = finite for all complex n, including all integer n.
(b) $P_n^m(\xi) = 0$ only for m > 0 *and* n = m-1,m-2, ... -m. [n integer and $-m \leq n < m$]

We see that the set of n values that make P = 0 *includes all* of our allowed Q spectrum!!! This means that every single one of our "allowed" Q functions is killed off by its partner P function. We now have

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm}\, E_{nm}\, [P_n^m(\xi_1)\, Q_n^m(\xi_0) - Q_n^m(\xi_1)\, P_n^m(\xi_0)\,]\, P_n^m(\xi)\, Q_n^m(j|\zeta|)\, e^{im\varphi}$$
$$= 0 \text{ !!!}$$

Therefore, our whole plan crashes to the ground (as I expected and almost hoped it would).

So I now return to the Maze document to summarize this last finding.